MATHEMATICS 101 Section 211
Quiz #5, February 27, 2012
Show all your work. Use back of page if necessary. Calculators are not allowed.

Last Name: **First Name:** **UBC Stud. No.:**

1) Write down the Simpson's rule expansion for the integral $\int_0^{1.5} \cos(x^2)\,dx$ explicitly in terms of the function when $n = 6$ (so S_6 in the textbook notation). Do not evaluate. (1 point)

Solution: Since $\Delta x = \frac{1.5-0}{6} = \frac{3}{12} = \frac{1}{4}$, $\Delta x/3 = \frac{1}{12}$ and $x_i = a + i\Delta x = 0 + \frac{i}{4}$ we have

$$S_6 = \frac{1}{12}\left(\cos(0) + 4\cos(\tfrac{1}{4}) + 2\cos(\tfrac{1}{2}) + 4\cos(\tfrac{3}{4}) + 2\cos(1) + 4\cos(\tfrac{5}{4}) + \cos(\tfrac{3}{2})\right)$$

Marking Scheme: Answers can be in unsimplified fraction and unsimplified decimal format. -0.5 if the general shape is right but the values are incorrect.

2) For $\int_0^1 \sin(x)\,dx$, how large does n have to be in order to guarantee that the exact value of the aforementioned integral differs from S_n, the approximation given by Simpson's rule, by at most 0.0001? Estimate your answer so that your final answer gives an explicit integer value for n. You may use the fact that the error made in using the Simpson's rule approximation to approximate $\int_a^b f(x)\,dx$ satisfies

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

where $|f^{(4)}(x)| \le K$ for $a \le x \le b$. (Recall $f^{(4)}(x) = f''''(x)$) (2 points)

HINT: Some or all of the following values may be useful in your estimation

$$2^4 = 16 \qquad 3^4 = 81 \qquad 4^4 = 256 \qquad 5^4 = 625 \qquad 6^4 = 1296 \qquad 7^4 = 2401$$

Solution: Notice that the derivatives of $\sin(x)$ are just $\pm\sin(x)$ and $\pm\cos(x)$ and so are all bounded above by 1. Hence, we may choose $K = 1$. Now $b = 1$ and $a = 0$. We want $|E_S| \le 0.0001$ and so to achieve this, we can simply require that

$$\frac{K(b-a)^5}{180n^4} = \frac{1(1-0)^5}{180n^4} = \frac{1}{180n^4} \le 0.0001 = \frac{1}{10000}$$

Isolating for n, we have

$$\frac{10000}{180} \le n^4 \quad \Rightarrow \quad \sqrt[4]{\frac{10000}{180}} \le n.$$

Lastly, we estimate n. To be safe in our estimation, we can raise the lower bound and require that

$$\sqrt[4]{100} \le \sqrt[4]{\frac{10000}{100}} \le n$$

For if this estimation is satisfied, then the required one is also satisfied. Raising the bound once again, we see that its enough to require that

$$4 = \sqrt[4]{4^4} = \sqrt[4]{256} \le n$$

and as 4 is even, we have that $n \ge 4$ and even will work. ■

Marking Scheme: 1 mark for plugging in the values correctly. 1 mark for a nice correct integer answer with justification. Please highlight answers that are not even or that do not mention the condition of evenness (only remove half a mark if the final answer is something along the lines of choose $n = 5$).

3) Factor $x^3 - x^2 - x - 2$ into irreducibles and justify that the terms you are left with are irreducible . (2 point)

Solution: For this question to work, this polynomial should have a root that is a factor of 2 so one of $\pm 1, \pm 2$. Trying each one in order gives

$$(1)^3 - (1)^2 - (1) - 2 = -3$$
$$(-1)^3 - (-1)^2 - (-1) - 2 = -1$$
$$(2)^3 - (2)^2 - (2) - 2 = 0$$

So the third equation, by the factor theorem, says $(x-2)$ is a root. Long division gives

$$\begin{array}{r|llll} & x^2 & +x & +1 & \\ \hline x-2 & x^3 & -x^2 & -x & -2 \\ & x^3 & -2x^2 & & \\ \hline & & x^2 & -x & \\ & & x^2 & -2x & \\ \hline & & & x & -2 \\ & & & x & -2 \\ \hline & & & & 0 \end{array}$$

The quotient is $x^2 + x + 1$.
The remainder is 0.

This gives

$$x^3 - x^2 - x - 2 = (x-2)(x^2 + x + 1)$$

Notice that the linear term is irreducible and the quadratic term is irreducible since its discriminant is $1^2 - 4(1)(1) = -3$. Thus completes the proof. Fun fact - the term when you take the positive square root of the root of the quadratic factor above is called the golden ratio. ■

Marking Scheme: One mark for finding the root. One mark for factoring (ie the long division). Give -0.5 if justification for why the quadratic term is irreducible is missing.

4) Evaluate $\displaystyle\int \frac{3x^2 - 4x + 9}{(x+1)(x^2 - 2x + 5)}\,dx$. (5 points)

Solution: Notice that $x^2 - 2x + 5$ has discriminant $(-2)^2 - 4(1)(5) = -16 < 0$ and hence this factor is irreducible. Now it's time for partial fractions! Let's break this up into two fractions

$$\frac{3x^2 - 4x + 9}{(x+1)(x^2 - 2x + 5)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 - 2x + 5} = \frac{A(x^2 - 2x + 5) + (Bx + C)(x+1)}{(x+1)(x^2 - 2x + 5)}$$

As the denominators are the same, we have that the numerators must be the same and hence

$$3x^2 - 4x + 9 = A(x^2 - 2x + 5) + (Bx + C)(x+1)$$

Plugging in values, we see

$$\begin{aligned} x = -1 \quad & 16 = 3(-1)^2 - 4(-1) + 9 = A((-1)^2 - 2(-1) + 5) + (B(-1) + C)((-1) + 1) = 8A \\ x = 0 \quad & 9 = 3(0)^2 - 4(0) + 9 = A((0)^2 - 2(0) + 5) + (B(0) + C)((0) + 1) = 5A + C \\ x = 1 \quad & 8 = 3(1)^2 - 4(1) + 9 = A((1)^2 - 2(1) + 5) + (B(1) + C)((1) + 1) = 4A + 2B + 2C \end{aligned}$$

The first equation tells us that $A = 2$. The second tells us that $9 - 5A = C$ and since $A = 2$, we have $C = -1$. The final equation tells us that $8 - 4A - 2C = 2B$ equivalently that $4 - 2A - C = B$ and plugging in the values of A and C yields $B = 1$. Hence

$$\frac{3x^2 - 4x + 9}{(x+1)(x^2 - 2x + 5)} = \frac{2}{x+1} + \frac{x - 1}{x^2 - 2x + 5}$$

Integrating the first term gives $2\ln(x+1)$. The second term is more complicated. We complete the square of the denominator to get that

$$x^2 - 2x + 5 = x^2 - 2x + 1 - 1 + 5 = (x-1)^2 + 4$$

This gives

$$\int \frac{(x-1)dx}{x^2 - 2x + 5} = \int \frac{(x-1)dx}{(x-1)^2 + 4} = \frac{\ln((x-1)^2 + 4)}{2} + C$$

Thus, combining the above information yields

$$\begin{aligned}\int \frac{3x^2 - 4x + 9}{(x+1)(x^2 - 2x + 5)}\,dx &= \int \left(\frac{2}{x+1} + \frac{x-1}{x^2 - 2x + 5}\right) dx \\ &= \int \left(\frac{2}{x+1} + \frac{x-1}{(x-1)^2 + 4}\right) dx \\ &= 2\ln(x+1) + \frac{\ln((x-1)^2 + 4)}{2} + C\end{aligned}$$

completing the question. ■

Marking Scheme: Half a mark for justifying that the quadratic factor is irreducible. Half mark for setting up the partial fraction decomposition, one mark for the values of A,B,C, one mark for deducing the $2\ln(x+1)$ integral, one mark for completing the square, one mark for the final answer. As usual, -0.5 for forgetting $+C$.