**MATHEMATICS 101 Section 201**
**Quiz #3, January 30, 2012**
*Show all your work. Use back of page if necessary. Calculators are not allowed.*

**Last Name:** **First Name:** **UBC Stud. No.:**

1) Evaluate and simplify $\int_0^{\frac{\pi}{2}} 15\sin^2(x)\cos^3(x)\,dx$. (3 points)

**Solution:** We proceed mechanically,

$$\begin{aligned} 15\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^3(x)\,dx &= 15\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^2(x)\cos(x)\,dx \\ &= 15\int_0^{\frac{\pi}{2}} \sin^2(x)(1-\sin^2(x))\cos(x)\,dx \end{aligned}$$

Now, let $u = \sin(x)$ so that $du = \cos(x)dx$, $u(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$ and $u(0) = \sin(0) = 0$ and so

$$\begin{aligned} 15\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^3(x)\,dx = 15\int_0^{\frac{\pi}{2}} \sin^2(x)(1-\sin^2(x))\cos(x)\,dx \quad &= 15\int_0^1 u^2(1-u^2)\,du \\ &= 15\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^1 \\ &= (5u^3 - 3u^5)\big|_0^1 \\ &= 5(1)^3 - 3(1)^5 - (5(0)^3 - 3(0)^5) \\ &= 2 \end{aligned}$$

**Marking Scheme:** 1 point for the correct substitution. 1 point for the integral. 1 point for the solution.

2) Evaluate $\int \frac{\tan^3(x)}{\cos^5(x)}\,dx$. (3 points)

**Solution:** We proceed mechanically,

$$\begin{aligned} \int \frac{\tan^3(x)}{\cos^5(x)}\,dx &= \int \tan^3(x)\sec^5(x)\,dx \\ &= \int \tan^2(x)\sec^4(x)\tan(x)\sec(x)\,dx \\ &= \int (\sec^2(x)-1)\sec^4(x)\tan(x)\sec(x)\,dx \end{aligned}$$

Now, let $u = \sec(x)$ so that $du = \tan(x)\sec(x)dx$. Hence

$$\begin{aligned} \int \frac{\tan^3(x)}{\cos^5(x)}\,dx &= \int (\sec^2(x)-1)\sec^4(x)\tan(x)\sec(x)\,dx \\ &= \int (u^2-1)u^4\,du \\ &= \frac{u^7}{7} - \frac{u^5}{5} + C \\ &= \frac{\sec^7(x)}{7} - \frac{\sec^5(x)}{5} + C \end{aligned}$$

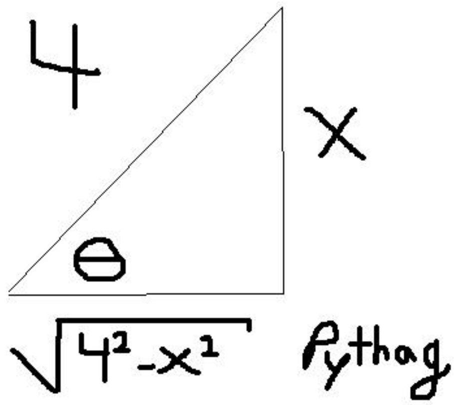
**Marking Scheme:** 1 point for the correct substitution. 1 point for the integral. 1 point for back substituting the $x$ and the $+C$.

3) Evaluate and simplify completely $\int \frac{x^2dx}{\sqrt{16-x^2}}$. (4 points)

**Solution:** We use a trig substitution. Let $x = 4\sin(\theta)$ so $dx = 4\cos(\theta)$. Then

$$\begin{aligned}
\int \frac{x^2dx}{\sqrt{16-x^2}} &= \int \frac{(4\sin(\theta))^2(4\cos(\theta))d\theta}{\sqrt{16-(4\sin(\theta))^2}} \\
&= \int \frac{64\sin^2(\theta)\cos(\theta)d\theta}{4\sqrt{1-\sin^2(\theta)}} \\
&= \int \frac{64\sin^2(\theta)\cos(\theta)d\theta}{4\sqrt{\cos^2(\theta)}} \\
&= \int \frac{64\sin^2(\theta)\cos(\theta)d\theta}{4\cos(\theta)} \\
&= \int 16\sin^2(\theta)d\theta \\
&= 16\int \frac{1-\cos(2\theta)}{2}d\theta \\
&= 16\int \frac{1}{2}d\theta - 16\int \frac{\cos(2\theta)}{2}d\theta \\
&= 8\theta - 8\frac{\sin(2\theta)}{2} + C \\
&= 8\theta - 8\sin(\theta)\cos(\theta) + C
\end{aligned}$$

Now, as $x = 4\sin(\theta)$, we have that $\sin(\theta) = \frac{x}{4}$ and $\theta = \arcsin(\frac{x}{4})$. For the last piece, we refer to




and notice that $\cos(\theta) = \frac{\sqrt{16-x^2}}{4}$. Plugging in yields

$$\begin{aligned}
\int \frac{x^2dx}{\sqrt{16-x^2}} &= 8\theta - 8\sin(\theta)\cos(\theta) + C \\
&= 8\arcsin\left(\frac{x}{4}\right) - 8\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + C \\
&= 8\arcsin\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C
\end{aligned}$$

**Marking Scheme:** 1 point for the correct substitution. 1 point for getting to the $16\sin^2(2\theta)$ step. 1 point for the integral with $+C$ (the usual $-0.5$ if no $C$). 1 point for the triangle and back substituting in the $x$.