MATHEMATICS 101 Section 201
Quiz #3, January 30, 2012
Show all your work. Use back of page if necessary. Calculators are not allowed.

Last Name: **First Name:** **UBC Stud. No.:**

1) Find the area of the region R enclosed by $y = \sqrt{x} + 1$ and $y = 2x + 1$. (4 points)

Solution: We find the points of intersection. Equating the two curves gives

$$\sqrt{x} + 1 = 2x + 1 \quad \Rightarrow \quad \sqrt{x} = 2x \quad \Rightarrow \quad x = 4x^2$$

and solving the last equation gives $0 = 4x^2 - x = x(4x - 1)$ and so $x = 0$ or $x = \frac{1}{4}$. The curve $\sqrt{x}+1$ is the top curve since if we choose a point between 0 and $\frac{1}{4}$, say $\frac{1}{9}$, we have that

$$\sqrt{\tfrac{1}{9}} + 1 = \tfrac{1}{3} + 1 = \tfrac{4}{3} = \tfrac{12}{9}$$

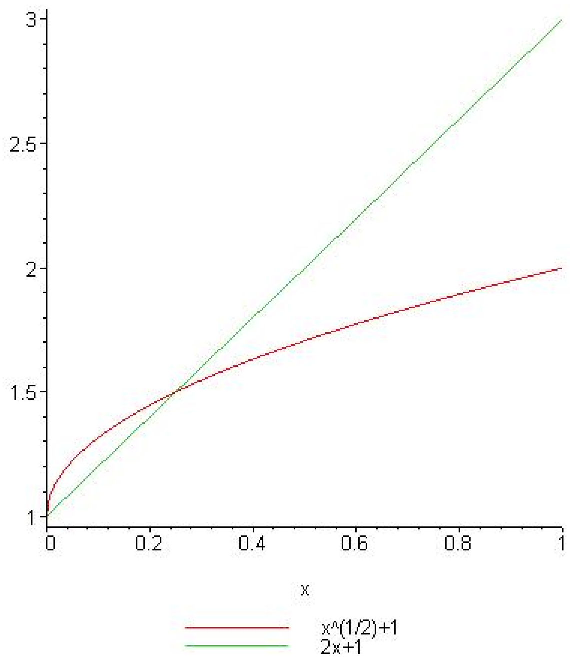
and

$$2(\tfrac{1}{9}) + 1 = \tfrac{11}{9} < \tfrac{12}{9}$$

and so the square root function is greater than the line on the entire interval. Hence the area is

$$\int_0^{\frac{1}{4}} ((\sqrt{x} + 1) - (2x + 1))dx = \int_0^{\frac{1}{4}} (\sqrt{x} - 2x)dx = \frac{2x^{\frac{3}{2}}}{3} - x^2 \Bigg|_0^{\frac{1}{4}}$$
$$= \frac{2(\frac{1}{4})^{\frac{3}{2}}}{3} - (\tfrac{1}{4})^2 - \frac{2(0)^{\frac{3}{2}}}{3} - (0)^2 = \frac{2}{3} \cdot \frac{1}{8} - \frac{1}{16}$$
$$= \frac{1}{12} - \frac{1}{16} = \frac{4}{48} - \frac{3}{48} = \frac{1}{48}$$

A sketch is drawn below (notice that its not necessary to draw a sketch to solve the problem, but it does help). ■



Marking Scheme: 1 point for the points of intersection. 1 point for figuring out which curve is on top (needn't be made explicit to earn the point - essentially the first integral must be correct). 1 point for correct integrating (students may earn this point even if the integral they construct is incorrect). 1 point for final answer (-0.5 if answer is correct but not simplified).

2) The curves $y = x^4 - 2x^2 + 2x + 1$, $y = x^4 - 2x^2 + 1$ and $x = c$ where $c \in \mathbb{R}$, $c > 0$ bound a region of area of 36. Find c. (3 points)

Solution: We find the point(s) of intersection. Equating the two curves gives

$$x^4 - 2x^2 + 2x + 1 = x^4 - 2x^2 + 1 \quad \Rightarrow \quad 2x = 0$$

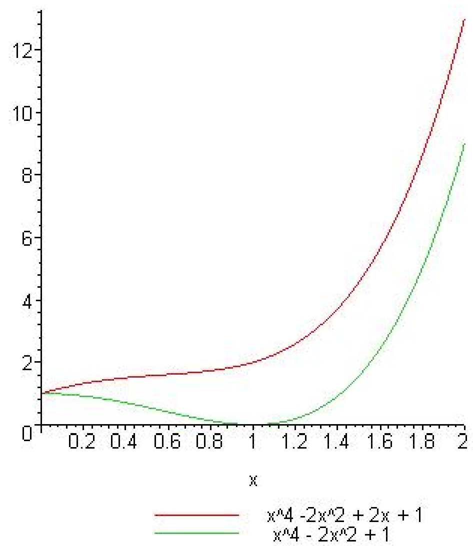
and so $x = 0$. The curve $x^4 - 2x^2 + 2x + 1$ is the top curve when $x > 0$ since on this region, we know that $2x > 0$ and so, adding $x^4 - 2x^2 + 1$ to both sides of this equality gives us that

$$2x + (x^4 - 2x^2 + 1) > (x^4 - 2x^2 + 1)$$

which when rearranging is precisely the two curves in question. Hence $x^4 - 2x^2 + 2x + 1$ is bigger whenever x is positive. Hence the area is

$$36 = \int_0^c ((x^4 - 2x^2 + 2x + 1) - (x^4 - 2x^2 + 1))dx = \int_0^c 2x dx = x^2\Big|_0^c = c^2$$

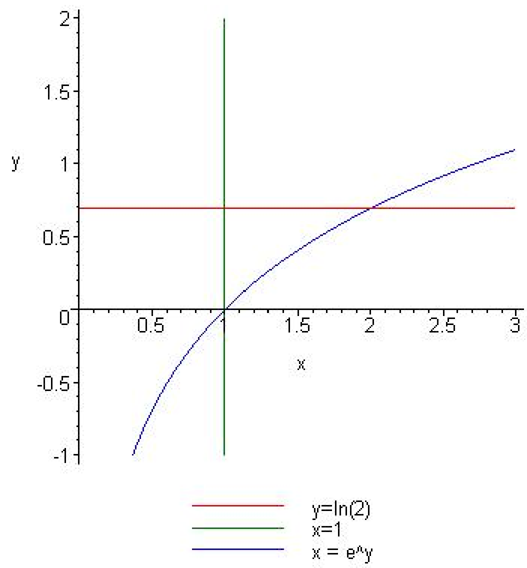
and hence, $c = 6$ (since $c > 0$ was given in the question) as required. A sketch is drawn below (notice that its not necessary to draw a sketch to solve the problem). ■



Marking Scheme: 1 point for the point of intersection. 1 point for setting up the right integral (so figuring out the bounds, which curve is on top, etc. basically the first integral must be correct). 1 point for the answer (maybe split it up half for the integral and half for the correct c value).

3) Find the volume of revolution of the region bounded by $x = e^y$, $y = \ln(2)$ and $x = 1$ when revolved around the line $x = -1$. (3 points)

Solution: For this problem, it is paramount that we draw a picture. It might help to draw the first curve in its equivalent form $y = \ln(x)$.



The points of intersection are seen to be $(1,0)$,$(1,\ln(2))$ and $(2,\ln(2))$ (these can also be found by pairwise equating the terms). Since we rotating about a line parallel to the y-axis, we want to use the x_L and x_R method. Notice that we are an additional unit away from the y-axis and so the formula becomes

$$\begin{aligned}
V &= \pi \int_0^{\ln(2)} ((x_R+1)^2 - (x_L+1)^2)dy = \pi \int_0^{\ln(2)} ((e^y+1)^2 - (1+1)^2)dy \\
&= \pi \int_0^{\ln(2)} (e^{2y} + 2e^y + 1 - 4)dy = \pi \int_0^{\ln(2)} (e^{2y} + 2e^y - 3)dy \\
&= \pi \left(\frac{e^{2y}}{2} + 2e^y - 3y \right)\Bigg|_0^{\ln(2)} = \pi \left(\left(\frac{e^{2\ln(2)}}{2} + 2e^{\ln(2)} - 3\ln(2) \right) - \left(\frac{e^{2(0)}}{2} + 2e^0 - 3(0) \right) \right) \\
&= \pi \left(\left(\frac{e^{\ln(2^2)}}{2} + 2(2) - 3\ln(2) \right) - (\frac{1}{2} + 2) \right) \\
&= \pi \left(\left(\frac{4}{2} + 4 - 3\ln(2) \right) - \frac{5}{2} \right) \\
&= \pi \left(6 - 3\ln(2) - \frac{5}{2} \right) \\
&= \pi \left(-3\ln(2) + \frac{7}{2} \right)
\end{aligned}$$

and this completes the proof. ■

Marking Scheme: 1 point for the diagram and or points of intersection. 1 point for setting up the right integral (so figuring out the bounds, which curve is on top, especially adding the extra 1 in the equation). 1 point for the answer (maybe split it up half for the integral and half for the correct numerical answer. The answer must be simplified to look something like mine, the π term might be expanded, the fraction might be mixed or written as a sum those are fine but the final answer should have no exponential terms).