MATHEMATICS 101 Section 201
Quiz #1, January 16, 2012
Show all your work. Use back of page if necessary. Calculators are not allowed.

Last Name: **First Name:** **UBC Stud. No.:**

For reference, you may need some or all of the following formulas

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

1) (5 marks) By evaluating the limit of Riemann sums obtained by using n equal-length subintervals and right endpoints (i.e. R_n), evaluate $\int_1^3 (x^2+3x)\,dx$. *No credit will be given for a solution that uses antidifferentiation.*

Solution: By definition, we see that

$$\Delta x = \frac{3-1}{n} = \frac{2}{n} \qquad x_i = 1 + \frac{2i}{n} \qquad f(x) = x^2 + 3x$$

and so we have

$$\begin{aligned}
\int_1^3 (x^2+3x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(\left(1+\frac{2i}{n}\right)^2 + 3\left(1+\frac{2i}{n}\right)\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}+3+\frac{6i}{n}\right)\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(4+\frac{10i}{n}+\frac{4i^2}{n^2}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8}{n}+\frac{20i}{n^2}+\frac{8i^2}{n^3}\right).
\end{aligned}$$

Now, by the linearity of summations,

$$\int_1^3 (x^2+3x)\,dx = \lim_{n\to\infty} \frac{8}{n}\sum_{i=1}^{n} 1 + \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

Using the formulas above,

$$\begin{aligned}
\int_1^3 (x^2+3x)\,dx &= \lim_{n\to\infty} \frac{8}{n}(n) + \frac{20}{n^2}\frac{n(n+1)}{2} + \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= 8 + 10 + \frac{8}{3} = \frac{62}{3}
\end{aligned}$$

completing the question. ■

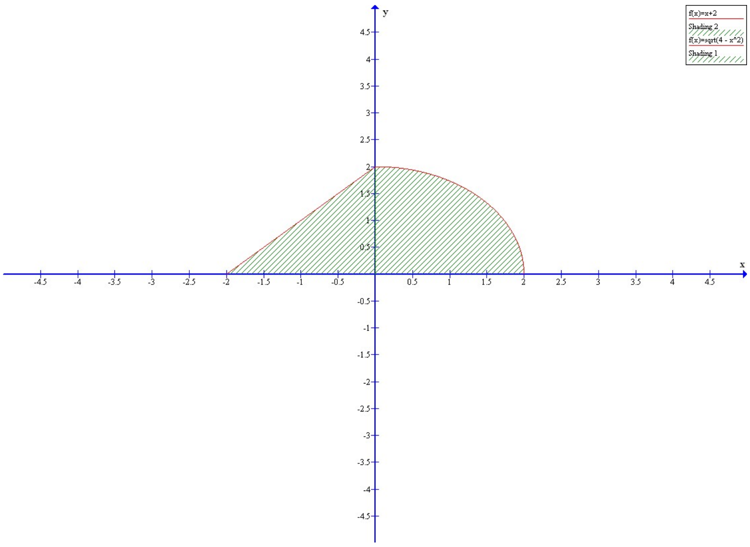
Marking scheme: 1 point for identifying the variables, 1 point for writing the sum, one point for breaking it down and correctly applying the formulas, 1 point for the limit, 1 point for the final arithmetic.

2) (3 marks) Evaluate

$$I = \int_{-2}^{0} (x+2)\,dx + \int_0^2 \sqrt{4-x^2}\,dx$$

by interpreting it in terms of areas. In other words, draw a picture (both integrals on one diagram) of the region the integrals represent, and find the area using high-school geometry. *No credit will be given for a solution that uses antidifferentiation.*

Solution: By the diagram



we see that we have a triangle of area $0.5(2)(2) = 2$ and a quarter circle of area $0.25\pi(2)^2 = \pi$ and so the total area is $2 + \pi$ as required. ■

Marking scheme: 1 point for a complete diagram with shading, 1 point for the area of the triangle, 1 point for the area of the quarter circle and summarizing as a total area.

3) (2 marks) Suppose that $f(x)$ is continuous. Write the following integrals as a single integral of the form $\int_a^b f(x)\,dx$

$$\int_{-4}^{2} f(x)\,dx + \int_{2}^{7} f(x)\,dx - \int_{-4}^{-1} f(x)\,dx$$

Solution: Notice that

$$\int_{-4}^{2} f(x)\,dx = \int_{-4}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx$$

and hence the above sum reduces to

$$\begin{aligned}\int_{-4}^{2} f(x)\,dx + \int_{2}^{7} f(x)\,dx - \int_{-4}^{-1} f(x)\,dx \\ &= \int_{-4}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx + \int_{2}^{7} f(x)\,dx - \int_{-4}^{-1} f(x)\,dx \\ &= \int_{-1}^{7} f(x)\,dx\end{aligned}$$

completing the proof. ■

Marking scheme: 2 points for a complete solution (partial credit can be given if reasonable progress is made).