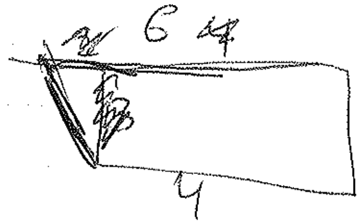
Lecture 1

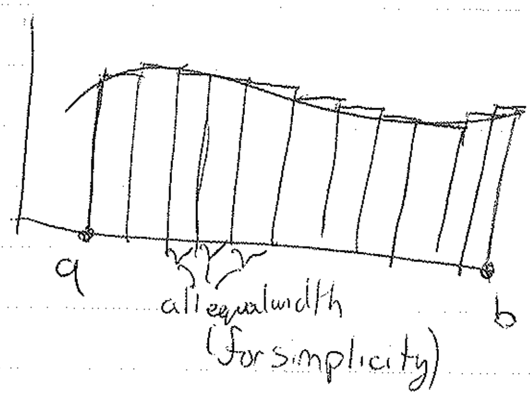
Recall that in Math 100, we spent much of our time computing derivatives (tangent lines)

In Math 101, we will spend most of our time studying areas and integrals.

For nice shapes □, △, etc. we have a notion of area. When curves are involved, it's trickier. For something like



we could break it up into shapes we know. For us,



First, some notation

Sigma notation

"Σ" - Sigma

we want to write sums with patterns succinctly.

Stop at term $n > m$

addition symbol $\rightarrow$ $$\sum_{i=m}^{n} i = m+(m+1)+\dots+n$$

index $i = m$

Start adding at term m.

n & m are integers. i, j, k are usually indices

Ex: First 10 numbers

$$\sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10 = 55$$

$$= \sum_{j=2}^{11} j-1 \qquad \boxed{j = i+1} \qquad \begin{array}{l} i=1 \Rightarrow j=2 \\ i=10 \Rightarrow j=11 \end{array}$$

First 3 powers of 2

$$\sum_{i=1}^{3} 2^i = 2^1+2^2+2^3 = 2+4+8 = 14$$

Sum of constants

$$\sum_{i=3}^{7} 4 = \underset{i=3}{4}+\underset{i=4}{4}+\underset{i=5}{4}+\underset{i=6}{4}+\underset{i=7}{4} = 20 = 4(5) = 4(7-3+1)$$

Sum of function values

$$\sum_{k=1}^{3} f(k) = f(1)+f(2)+f(3)$$

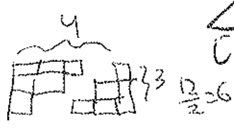
Properties:

$$\left.\begin{array}{l} \displaystyle\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \\ \displaystyle\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i \end{array}\right\} \text{Linear}$$

$$\sum_{i=m}^{n} c = c\sum_{i=m}^{n} 1 = c(n-m+1)$$

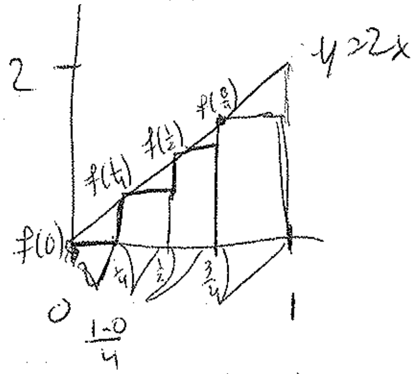
Other important Sums: (Proofs are online)

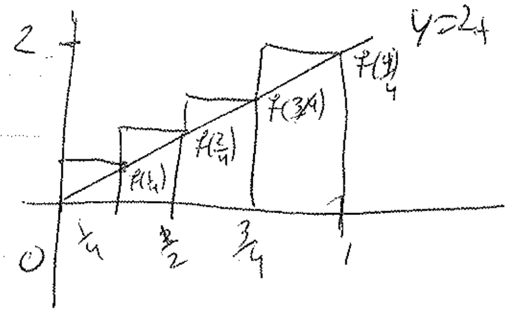
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$



Back to area...

Let's try the rectangle technique with $f(x)=2x$. from 0 to 1





With 4 rectangles of equal length

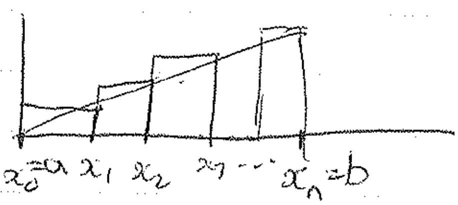
$\frac{1}{4} f(0)+\frac{1}{4} f(\frac{1}{4}) + \frac{1}{4} f(\frac{1}{2}) + \frac{1}{4} f(\frac{3}{4}) = \sum_{i=0}^{3} \frac{1}{4} f(\frac{i}{4})$ Left endpoint.

$= \frac{1}{4}(0+\frac{1}{2}+1+\frac{3}{2}) = \frac{3}{4} < 1$

$\frac{1}{4} f(\frac{1}{4}) + \frac{1}{4} f(\frac{2}{4}) + \frac{1}{4} f(\frac{3}{4}) + \frac{1}{4} f(\frac{4}{4}) = \sum_{i=1}^{4} \frac{1}{4} f(\frac{i}{4})$ Right endpoint.

$= \frac{1}{4}(\frac{1}{2}+1+\frac{3}{2}+2) = \frac{5}{4} > 1$

In general, for right endpoints and n-rectangles.



$a=0$
$b=1$

as all ~~distances~~ widths equal:

$x_0=a=1$ $\quad x_1=\frac{1}{n} \quad x_2=\frac{2}{n} \ldots x_i=\frac{i}{n} \ldots x_n=\frac{n}{n}=1=b$

Delta x or change in x. $\Rightarrow \Delta x = x_i - x_{i-1} = \frac{i}{n} - \frac{(i-1)}{n} = \frac{1}{n}$ = width.

Heights are $f(x_i) = 2x_i = \frac{2i}{n}$

Each rectangle has ~~height~~ area $f(x_i)\Delta x$ so

total area of rectangles $R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \frac{2i}{n}\left(\frac{1}{n}\right)$

Riemann Sums. $= \frac{2}{n^2}\sum_{i=1}^{n} i = \frac{2}{n^2}\frac{(n)(n+1)}{2}$

$= \frac{n^2+n}{n^2} = 1+\frac{1}{n} \xrightarrow{n\to\infty} 1$.

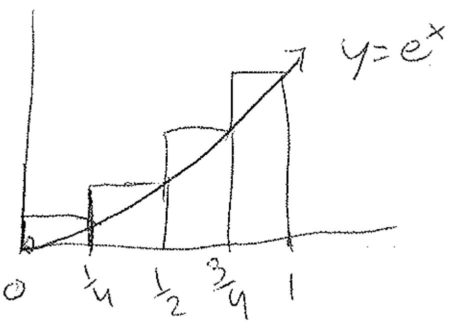
If we used the left end point

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \cancel{\frac{n(n-1)}{n^2} \xrightarrow{n\to\infty} 1}$$

$$= \sum \frac{2}{n^2}(i-1)\frac{1}{n} = \frac{2}{n^3}\sum_{i=1}^{n}(i-1) \overset{\boxed{j=i-1}}{=} \frac{2}{n^2}\sum_{j=0}^{n-1} j = \cdots = \frac{n(n-1)}{n^2} \xrightarrow{n\to\infty} 1$$

This technique can also give us an approximation of the area

Eg:



Using 4 rectangles, approximate the area of $y=e^x$ from 0 to 1 using the right endpoints.

$\Delta x = \frac{1}{4}$ $\quad x_i = \frac{i}{4}$ for $i=1,\dots 4$.

$$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x_i = f(x_1)\tfrac{1}{4} + f(x_2)\tfrac{1}{4} + f(x_3)\tfrac{1}{4} + f(x_4)\tfrac{1}{4}$$

$$= f(\tfrac{1}{4})\tfrac{1}{4} + f(\tfrac{2}{4})\tfrac{1}{4} + f(\tfrac{3}{4})\tfrac{1}{4} + f(\tfrac{4}{4})\tfrac{1}{4}$$

$$= \frac{e^{1/4}}{4} + \frac{e^{2/4}}{4} + \frac{e^{3/4}}{4} + \frac{e^{4/4}}{4}$$

$$\approx 1.94.$$

Close estimate.

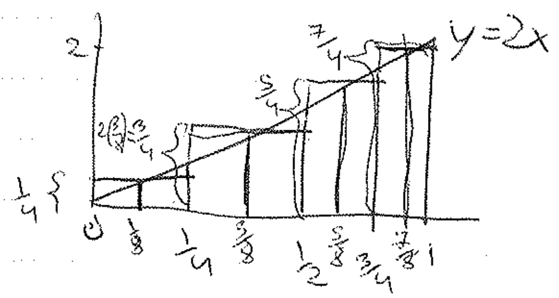
Later, we will see that the actual area is $e-1 \approx 1.72$

Lecture 2

Recap: • Sigma Notation
• Right Endpoint rule
• Left Endpoint rule

Now:
• Midpoint Rule
• General Setup
• Distance Problem

Recall our example



$n = 4$ rectangles

$\Delta x = \text{width} = \frac{1-0}{4} = \frac{1}{4}$

$x_0 = 0, \quad x_1 = x_0 + \Delta x = 0 + \frac{1}{4} = \frac{1}{4}$

$x_2 = x_0 + 2\Delta x = \frac{2}{4}, \ldots, x_4 = 1$

Area of rectangles $= \cancel{\sum_{i=1}^{4} \frac{1}{4}\left(\frac{2i-1}{4}\right)} = \frac{1}{16}\sum_{i=1}^{4}(2i-1)$

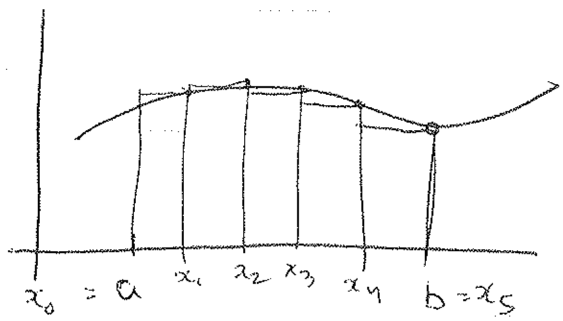
$= \cancel{\frac{1}{16}\left(\sum_{i=1}^{4}(2i) - \sum_{i=1}^{4} 1\right)} = \frac{1}{16}(20-4) = 1$

$= \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{5}{4}\right) + \frac{1}{4}\left(\frac{7}{4}\right)$

$= \frac{1}{16} + \frac{3}{16} + \frac{5}{16} + \frac{7}{16} = 1$

(2)

General Setup:



$n =$ 5 rectangles in the example

For n ~~rectangles~~ subintervals

$\Delta x = \frac{b-a}{n}$ = width of rectangle
$x_i = a + i\Delta x = x_0 + i\Delta x$
$f(x_i)$ = height of rectangle

Setting $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$ ← Riemann Sum, we see

$$\text{Area} = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

When $f(x)$ is continuous, this limit always exists (though computing can be hard)

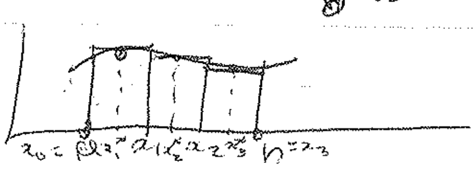
We could have defined area using the left end point.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

(or midpoints $M_n = \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$).

In fact, we could define this using any point in $[x_{i-1}, x_i]$.
Let $x_i^* \in [x_{i-1}, x_i]$. Then

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$



Example

$y = 2x$ from 0 to 1

$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$x_i = a + i\Delta x = 0 + \frac{i}{n} = \frac{i}{n}$

$f(x_i) = \frac{2i}{n}$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{2}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty} \frac{n^2+n}{n^2}$$

$$= \lim_{n\to\infty} 1 + \frac{1}{n} = 1$$

Exercise: This is equal to the left and midpoint rule limits.

Q: ~~What function and region~~

Determine a region whose area is the Riemann Sum below

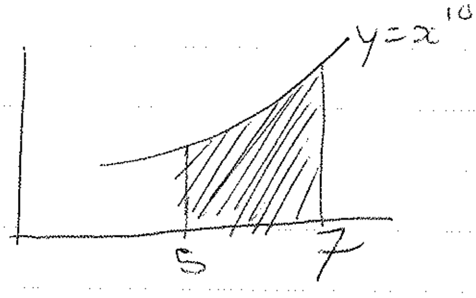
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}$$

A: Let $x_i = 5 + \frac{2i}{n}$ so $a = x_0 = 5$ $b = x_n = 5 + \frac{2n}{n} = 7$

$\Delta x = \frac{2}{n}$

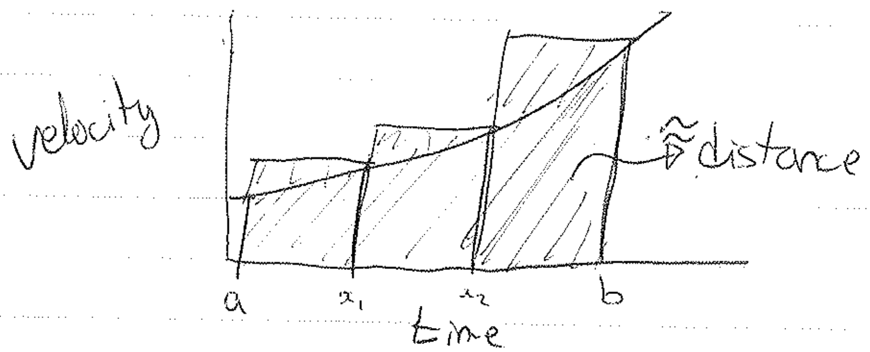
$f(x_i) = \left(5 + \frac{2i}{n}\right)^{10} = x_i^{10}$ so $f(x) = x^{10}$

Hence,



Distance Problem

distance = ~~[illegible]~~ velocity × time
area = height × width.



Ex: A car travels ~~[illegible]~~

	$i=0$	$i=1$	2	3	4	5	6
t_i = time (s)	0	10	20	30	40	50	60
v_i = velocity (m/s)	3	7	8	11	13	12	11

We can approximate distance using the right end points.

$\Delta x = 10 - 0 = 10$ ~~$v_1 = 7$ $v_2 = 8$ $v_3 = 11$ $v_4 = 13$ $v_5 = 12$ $v_6 = 11$~~

So distance $= \sum_{i=1}^{6} \Delta x \, v_i = \Delta x \sum_{i=1}^{6} v_i = 10(7+8+11+13+12+11) = 620\text{m}$.

Definite Integrals:

Def'n: Let $f(x)$ be a function defined for $a \le x \le b$ (a, b are real numbers). Divide $[a,b]$ into n equal subintervals of width $\Delta x = \frac{b-a}{n}$. Let $x_0 = a, x_1, \dots x_n = b$ be the end points of these subintervals and let $x_i^* \in [x_{i-1}, x_i]$ be sample point.

The <u>definite integral of f from a to b</u> is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Provided the limit exists. If it does, we say <u>f is integrable on $[a,b]$</u>.

Notes:

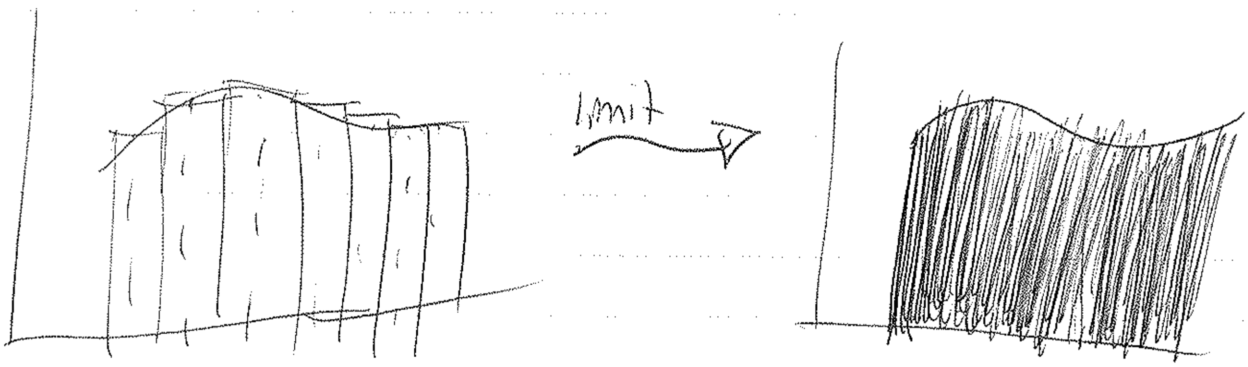
- $\int$ is called an <u>integral sign</u> (\int in Piazza). Liebniz introduced it
- $f(x)$ is the <u>integrand</u>.
- a & b are <u>limits of integration</u>
- dx on its own has no meaning.
- Integrals are ~~[illegible]~~ sums so $\int_0^b f(x)\,dx$ is a number (doesn't depend on x).
- In fact $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(z)\,dz$ and x, t, z in this context are <u>dummy variables</u>.

Q: When is f integrable?

Theorem: If f is continuous on $[a,b]$ OR has only finitely many jump discontinuities, then f is integrable on $[a,b]$

⑩

Recap:
- Distance Problem
- Riemann Sums



Today:
Fun with integrals.

Ex: Compute $\int_0^1 2x\,dx$

Example: Compute $\int_0^4 (x^2-3x)$

A: Turn into a Riemann Sum

$a=0 \qquad b=4 \qquad \Delta x = \frac{b-a}{n} = \frac{4}{n} \qquad x_i = a+i\Delta x = \frac{4i}{n}$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}(x_i^2-3x_i)\frac{4}{n} = \sum_{i=1}^{n}\left(\left(\frac{4i}{n}\right)^2 - 3\left(\frac{4i}{n}\right)\right)\frac{4}{n}$$

$$= \sum_{i=1}^{n}\left(\frac{16i^2}{n^2} - \frac{12i}{n}\right)\frac{4}{n} = \sum_{i=1}^{n}\frac{64i^2}{n^3} - \frac{48i}{n^2} = \frac{64}{n^3}\sum_{i=1}^{n} i^2 - \frac{48}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{48}{n^2}\left(\frac{n(n+1)}{2}\right)$$

Taking limits,

$(n^2+n)(2n+1)$

$$\int_0^3 (x^2-3x)\,dx = \lim_{n\to\infty} R_n$$

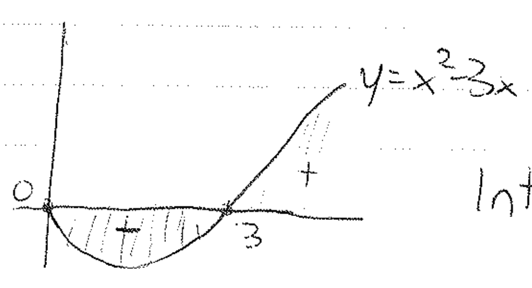
$$= \lim_{n\to\infty}\frac{64}{6}\left(\frac{n(n+1)(2n+1)}{n^3}\right) - \lim_{n\to\infty}\frac{48}{2}\left(\frac{n(n+1)}{n^2}\right)$$

$$= \frac{64}{6}(2) - \frac{48}{2}$$

$$= \frac{64}{3} - 24 = \boxed{-\frac{8}{3}}$$

Notes: (1) This is not so easy to compute ← Tedius computation

(2) The answer is negative!



Integrals compute $\boxed{\text{signed areas.}}$

Example: Express as a definite integral

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1-\left(1+\frac{i}{n}\right)^3}$$

A: $\Delta x = \frac{1}{n}$

$x_i = a + i\Delta x = a + \frac{i}{n}$ $\quad a = 1. \quad b = x_n = 1 + \frac{n}{n} = 2$

$f(x_i) = \frac{1}{1-(1+\frac{i}{n})^3}$ so $f(x) = \frac{1}{1-x^3}$

<u>So</u> $\quad L = \int_1^2 \frac{1}{1-x^3}\,dx$

Ex: (Class attempt) $\int_0^1 (x^3 + 2x)\,dx$

$a = 0 \quad b = 1 \quad \Delta x = \frac{1}{n} \quad x_i = \frac{i}{n}$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(\frac{i^3}{n^3} + \frac{2i}{n}\right)\frac{1}{n}$$

$$= \frac{1}{n^4}\sum_{i=1}^{n} i^3 + \frac{2}{n^2}\sum_{i=1}^{n} i = \frac{n^2(n+1)^2}{4n^4} + \frac{2n(n+1)}{2n^2}$$

Taking limits

$$A = \lim_{n\to\infty} R_n = \frac{1}{4} + 1 = \frac{5}{4}$$

②

Properties of integrals. Let a, b, c ~~real numbers~~ $\in \mathbb{R}$ (so a,b,c are real.) and f, g integrable on $[a,b]$. Then

0.

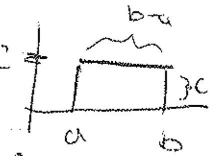
2. $$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

1. $$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

Linear.

$$\int_a^a f(x)\,dx = 0$$ (Integral of a trivial interval)

3. $$\int_a^b c\,dx = c(b-a)$$



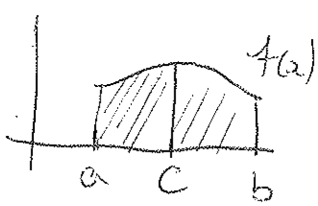
4. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$ ie $\Delta x = \frac{b-a}{n}$ becomes $\frac{a-b}{n} = \frac{-(b-a)}{n}$

Even more properties... Pictures first.

$a, b \in \mathbb{R}$, f & g are integrable on $[a,b]$

5. If $c \in (a,b)$ then

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

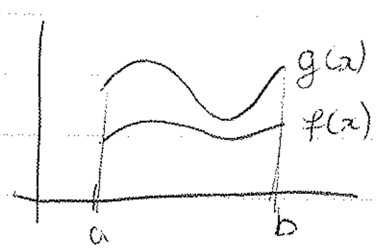


6. If $f(x) \geq 0$ for all $x \in [a,b]$ then $\int_a^b f(x)\,dx \geq 0$

(Nonnegative functions have nonnegative integrals/signed areas)

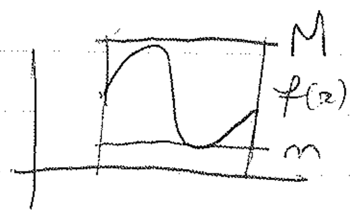
7. If $f(x) \leq g(x)$ for all $x \in [a,b]$ then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$



8. If $m \leq f(x) \leq M$ for all $x \in [a,b]$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$



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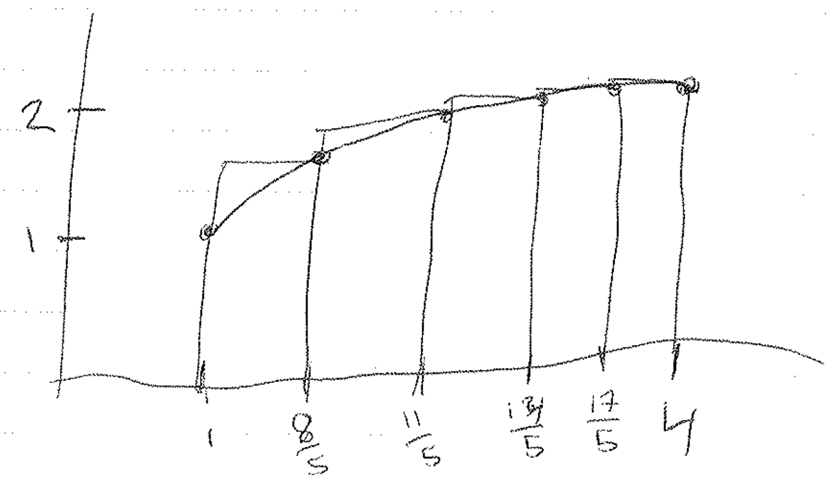
Eg. Estimate $\int_1^4 \sqrt{x}\,dx$

A. $\sqrt{x}$ is an increasing function so $\sqrt{1}=1$ is the min and $\sqrt{4}=2$ is the maximum. Thus, by Property 4,

$$1\cdot(4-1) \le \int_1^4 \sqrt{x}\,dx \le 2\cdot(4-1)$$

$$3 \le \int_1^4 \sqrt{x}\,dx \le 6 \quad \text{(actual value } \tfrac{14}{3}\text{).}$$

Eg. Use the right endpoint rule with $n=5$ rectangles to approximate the above.



$\Delta x = \frac{4-1}{5} = \frac{3}{5}$

$$\text{Area} = \frac{3}{5}\sqrt{\frac{8}{5}} + \frac{3}{5}\sqrt{\frac{11}{5}} + \frac{3}{5}\sqrt{\frac{14}{5}} + \frac{3}{5}\sqrt{\frac{17}{5}} + \frac{3}{5}\sqrt{4}$$
$$\approx 4.9592$$

Actual is $\frac{14}{3} \approx 4.6667$

Notation: $f(x)\Big|_a^b = f(b) - f(a)$ ②

~~An~~ FTC II: Let f be continuous on $[a,b]$ and let F be an antiderivative of f. Then

$$\int_a^b f(t)\,dt = F(b) - F(a) = F(x)\Big|_a^b$$

ie $$\int_a^b F'(t)\,dt = F(b) - F(a).$$

Ex: $\int_0^1 2x\,dx = x^2\Big|_0^1 = (1)^2 - (0)^2 = 1$.

$$\int_1^3 e^x\,dx = e^x\Big|_1^3 = e^3 - e^1.$$

$$\int_0^4 (x^2 - 3x)\,dx = \frac{x^3}{3} - \frac{3x^2}{2}\Big|_0^4 = \frac{(4)^3}{3} - \frac{3(4)^2}{2} - \left(\frac{(0)^3}{3} - \frac{3(0)^2}{2}\right)$$

$$= \frac{64}{3} - \frac{48}{2} = -\frac{8}{3}$$

Mini table (Chp 5.4 ~~[illegible]~~ Look!)

f (function)	F (antiderivatives)	
x^n	$\frac{1}{n+1} x^{n+1} + C$	
$1/x$	$\log\lvert x\rvert + C$	$\lvert\cdot\rvert$ since x can be negative.
e^{ax}	$\frac{1}{a} e^{ax} + C$	
a^x	$\frac{1}{\log(a)} a^x + C$	
$\cos(ax)$	$\frac{1}{a}\sin(ax) + C$	
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + C$	
$\sec^2(ax)$	$\frac{1}{a}\tan(ax) + C$	
$\tan(ax)$	? (later)	

FTC §5.3

Fundemental Theorem of Calculus I (FTC): Let f be cts on $[a,b]$ and Let $F(x) = \int_a^x f(t)\,dt$. Then F is differentiable on (a,b) and $F'(x) = f(x)$.

Ex: If $F(x) = \int_0^x \cos(x)\,dx$ then $F'(x) = \cos(x)$.
If $F(x) = \int_{10}^x t^2 + \sqrt{\cos(t)+1}\,dt$ then $F'(x) = x^2 + \sqrt{\cos(x)+1}$
If $F(x) = \int_1^{x^2} \cos(t)\,dt$, then use chain rule

Let $u = x^2$ and $G(u) = \int_1^u \cos(t)\,dt$. Then
$$\frac{dF}{dx} = \frac{dG}{du}\frac{du}{dx} = \cos(u)\,(2x) = 2x\cos(x^2)$$

Main idea: Differentiating undoes integrating.

Def'n: Given a function f, if F is another function such that $F' = f$, then F is an <u>antiderivative</u> of f.

Ex: x^3 is an antiderivative of $3x^2$. So is $x^3 + 7$ and $x^3 + \pi$.

How many antiderivatives are there?

Theorem: If F is an antiderivative of f, then all other antiderivatives are of the form $F(x) + c$.

Pf: Let F & G be two antiderivatives of f. Let $H = F - G$. Then $H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$. So $H(x)$ is constant (only constants have 0 derivative by a corollary of Mean Value Theorem). Thus $F(x) - G(x) = C$ so $F(x) = G(x) + C$ ∎

A bad example:

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$ why is this bad?

Not Continuous at $0 \in [-1,3]$!

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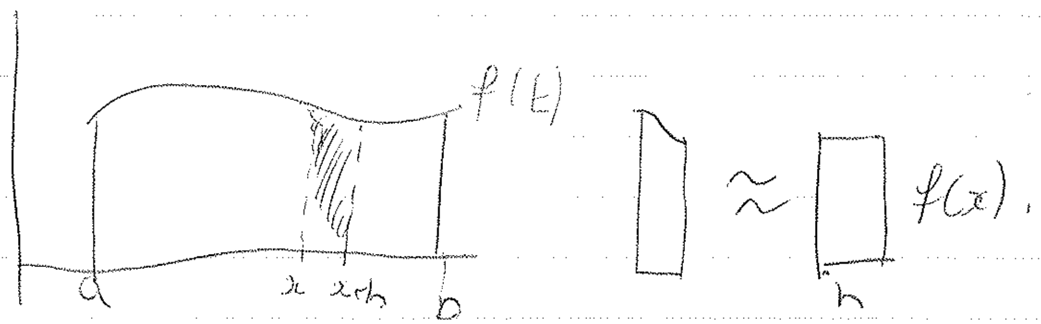
Today's take home lesson:

Differentiating undoes integrating

Integrating undoes differentiating (up to a constant).

Proof sketch of FTC I. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Consider



Move x to $x+h$...

$$\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\boxed{\int_x^{x+h} f(t)\,dt.}$$

"Shaded rectangle"

We expect $\int_x^{x+h} f(t)\,dt$ gets closer and closer to $f(x)h$.

(needs to be proven).

<u>So</u> $\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{1}{h}f(x)h = f(x)$.

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Proof of FTC II. ~~Plotson~~ F an antiderivative of continuous f

Notice that $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

Thus, $\int_a^x f(t)\,dt = F(x) + C$. Note that at $x = a$

$$\int_a^a f(t)\,dt = 0 \quad \text{so } F(a) + C = 0 \Rightarrow C = -F(a)$$

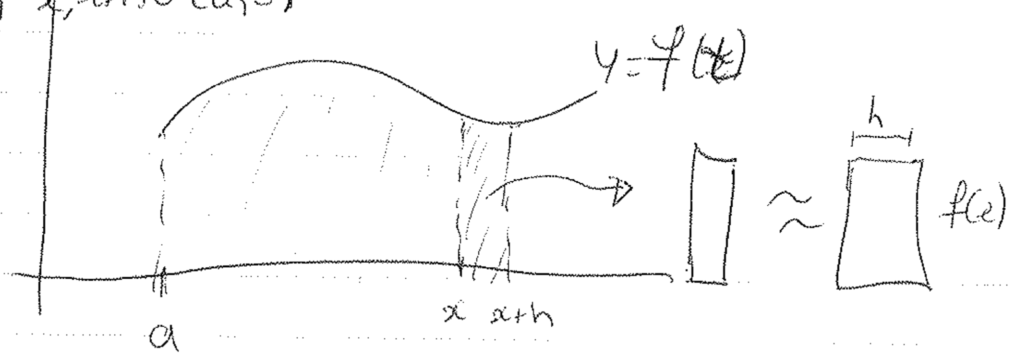
At $x = b$,

$$\int_a^b f(t)\,dt = F(b) + C = F(b) - F(a) \quad \blacksquare$$

Fundamental Theorem of Calculus

Theorem: Let f be continuous on $[a,b]$. Let
$$F(x) = \int_a^x f(t)\,dt \quad \text{for } x \in [a,b].$$
Then $\frac{d}{dx}F(x) = f(x)$.

Pf: Let $x, x+h \in (a,b)$.



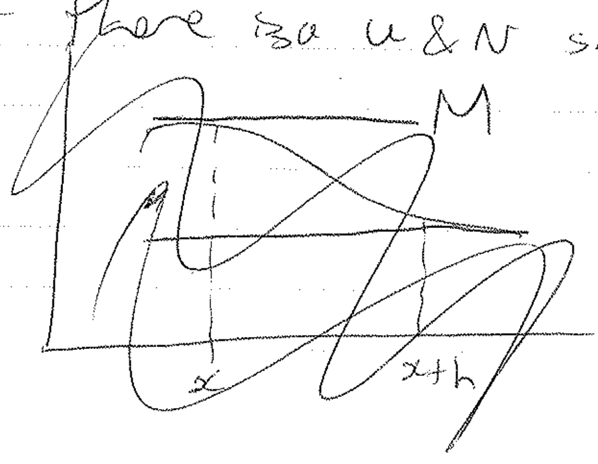
Note: $\cancel{F}$ $\frac{d}{dx}F(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$

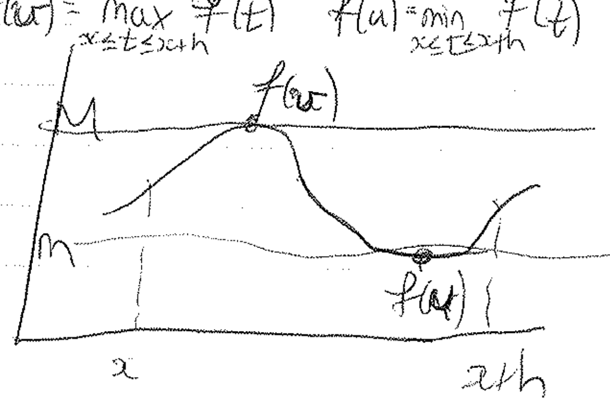
$$= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_x^{x+h} f(t)\,dt\right)$$

Now, assume $h>0$. Extreme Value Theorem says on $[x, x+h]$ that there is a u & w s.t. $f(w) = \max_{x\le t\le x+h} f(t)$ $\quad f(u) = \min_{x \le t \le x+h} f(t)$





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So, $f(u)h = mh \le \int_x^{x+h} f(t)\,dt \le Mh \le f(v)h$

$$\Rightarrow \quad f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)$$

as $h \to 0$, $u \to x$ and $v \to x$ since $u, v \in [x, x+h]$.

& $f(u) \xrightarrow{h\to 0} f(x)$ $\quad f(v) \xrightarrow{h\to 0} f(x)$

So taking limits, the squeeze theorem says

$$\cancel{f(x) \le \frac{d}{dx}F(x) \le f(x)}$$

$$\frac{d}{dx}F(x) = f(x).$$

The other cases can be found in the textbook ∎.

§5.4

1

Before

$$\int_a^b f(x)\,dx = \text{signed area} = \text{number}$$

↖ DEFINITE INTEGRAL.

Now,

$$\int f(x)\,dx = \text{Antiderivative of } f(x)$$

↖ INDEFINITE INTEGRAL.

Ex: $\int x^2\,dx = \frac{x^3}{3} + C$ $\qquad \int \frac{dx}{x^2+1} = \arctan(x) + C$

~~$\int \sin(x)\,dx = -\cos x + C$~~

LEARN TABLE IN 5.4!!! (No sinh or cosh)

Ex (For class) $\int (3x^3 - \csc^2 x)\,dx$

$$= 3\int x^3\,dx - \int \csc^2 x\,dx$$

$$= \frac{3}{4}x^4 - (-\cot(x)) + C = \frac{3}{4}x^4 + \cot(x) + C.$$

Recall that FTC II says $\int_a^b F'(x)\,dx = F(b) - F(a)$

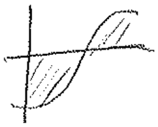
$F'(x)$ is the rate of change of F.

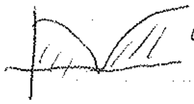
Net change Theorem: The integral of a rate of change is the net change

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Examples

- Velocity is rate of change of position $v(t) = \frac{dx}{dt}$ ← x is distance function.
  - Hence $\int_{t_1}^{t_2} v(t)\,dt = x(t_2) - x(t_1)$ = displacement
    NOT distance


displacement


distance

$$\int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance travelled.}$$

- Ex: If rate of growth of population is $\frac{dn}{dt}$, then

$$\int_{t_1}^{t_2} \frac{dn}{dt} = n(t_2) - n(t_1) = \text{net change of population.}$$

Ex: Water flows from the bottom of a tank at a rate of $r(t) = 200 - 4t$ L/min, where $0 \le t \le 50$. Find the amount of water that flows from the tank during the first 10 min.

Ans: $\int_0^{10} 200 - 4t\,dt = \cancel{\int_0^{10} 200 - 4\int_0^{10} t}$

SEE 2.5

$$= \left(200t - \frac{4}{2}t^2\right)\Big|_0^{10}$$
$$= 200(10) - 2(10)^2 - (200(0) - 2(0)^2)$$
$$= 2000 - 200 = 1800.$$

One last BAD example. Why is the following wrong?

$$\int_{-1}^{3} \frac{1}{x^2} = \int_{-1}^{3} x^{-2} = -x^{-1}\Big|_{-1}^{3} = -\frac{1}{3} - \left(\frac{-1}{-1}\right) = -\frac{4}{3}$$

Ans: $\frac{1}{x^2}$ is NOT continuous on $[-1, 3]$
FTC II DOES NOT APPLY!

Example:

A particle moves along a straight line with acceleration $a(s) = 3$ in m/s^2. The initial velocity is -6 m/s. Find the velocity at time t. Find the displacement ~~at time~~ from 0 to 5. Find the distance travelled from 0 to 5.

Ans: let $v(t)$ be the velocity, ~~$d(t)$ the displacement~~ and $D(t)$ the distance

$$v(t) - v(0) = \int_0^t a(s)\,ds = \int_0^t 3\,ds = 3s\Big|_0^t = 3t - 3(0) = 3t.$$

$$v(t) = 3t + v(0) = 3t - 6$$

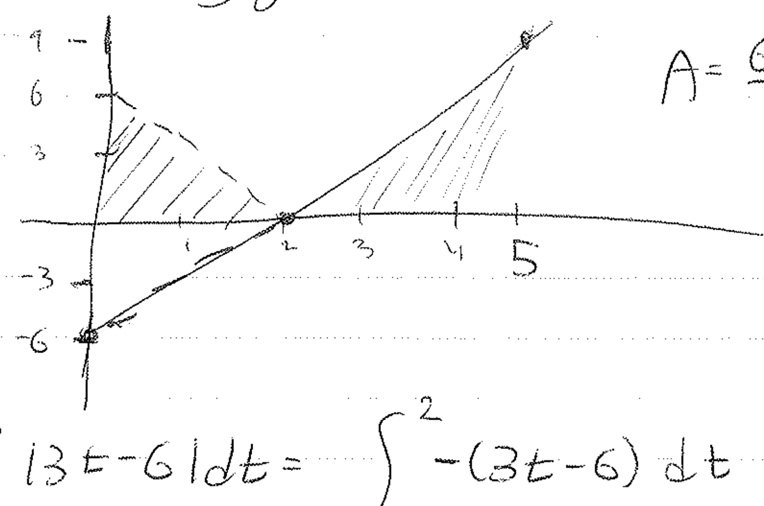
Total displacement

$$\cancel{d(5) - d(0)} = \int_0^5 v(t)\,dt = \int_0^5 (3t - 6)\,dt = \left(\frac{3t^2}{2} - 6t\right)\Big|_0^5$$

$$= \frac{3(5)^2}{2} - 6(5) - \left(\frac{3(0)^2}{2} + 6(0)\right)$$

$$= \frac{75}{2} - 30 = \frac{15}{2}$$

Total Distance $= \int_0^5 |3t - 6|\,dt$



$$A = \frac{6 \cdot 2}{2} + \frac{9 \cdot 3}{2} = 6 + \frac{27}{2} = \frac{39}{2}$$

$$D = -6 + 12 + \frac{75}{2} - 30 - 6 + 12 = \frac{39}{2}$$

OR

$$\int_0^5 |3t - 6|\,dt = \int_0^2 -(3t - 6)\,dt + \int_2^5 (3t - 6)\,dt$$

$$= -\frac{3t^2}{2} + 6t\Big|_0^2 + \frac{3t^2}{2} - 6t\Big|_2^5$$

$$= -\frac{3(2)^2}{2} + 6(2) - \left(\frac{3(0)^2}{2} + 6(0)\right) + \frac{3(5)^2}{2} - 6(5) - \left(\frac{3(2)^2}{2} - 6(2)\right)$$

3

Substitution Rule

Let's try to undo the chain rule

$$\frac{d}{dx} F(g(x)) = F'(g(x))\, g'(x)$$

Let $u = g(x)$ (often u is the complicated part)

Integrating

$$\int F'(g(x))\, g'(x)\, dx = \int \frac{d}{dx} F(g(x))\, dx = F(g(x)) + C$$

$$= F(u) + C = \int \frac{d}{du} F(u)\, du$$

$$= \int F'(u)\, du.$$

Theorem: If $u = g(x)$ is differentiable with range I (an interval) ~~where~~ and f is a continuous function on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Ex $\int \frac{x}{1+x^2}\,dx$

Let $u = 1+x^2$ then $du = 2x\,dx$. so $\frac{du}{2} = x\,dx$ and

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{u}\left(\frac{du}{2}\right) = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|1+x^2|.$$

Ex: $\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx$

Look for one function to be the derivative of the other.

Let $u = \sin(x)$ so $du = \cos(x)\,dx$

$$\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx = \int \frac{du}{u} = \ln|u| + C$$
$$= \ln|\sin(x)| + C.$$

If you choose $\cos(x)$,

$u = \cos(x)$ $\quad du = -\sin(x)\,dx$

No want $\frac{dx}{\sin(x)}$

Exercise: $\int \tan(x)\,dx$.

Point: We can treat du and dx as "differentials" & manipulate them like variables.

§ 5.5

Ex: $\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx$

Let $u = \sin(x)$ so $du = \cos(x)\,dx$

$$\int \cot(x)\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin(x)| + C$$

What about for definite integrals?

Substitution Rule for Definite Integrals

Let $u = g(x)$ be differentiable with continuous derivative range I and let $f(x)$ be continuous on I. Then if

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Ex: $\int_0^{\pi} \sin(x)\,e^{\cos(x)}\,dx$

Let $u(x) = \cos(x)$ so $du = -\sin(x)\,dx$ so $-du = \sin(x)\,dx$
$u(0) = \cos(0) = 1$ and $u(\pi) = \cos(\pi) = -1$

$$\int_0^{\pi} \sin(x)\,e^{\cos(x)}\,dx = \int_1^{-1} e^u(-du) = -\int_1^{-1} e^u\,du$$
$$= \int_{-1}^{1} e^u\,du = e^u\Big|_{-1}^{1} = e^1 - e^{-1}.$$

Examples to try:

$$\int \tan(x)\,dx$$

$= \int \frac{\sin(x)}{\cos(x)}\,dx$

Let $u = \cos(x)$ so $du = -\sin(x)\,dx$

$= \int \frac{-du}{u}$

$= -\ln|u| + C$

$= -\ln|\sin(x)| + C$

$$\int_0^1 \frac{x^2}{x^2+1}\,dx$$

$= \int_0^1 \frac{x^2+1-1}{x^2+1}\,dx$

$= \int_0^1 \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right)dx$

$= \int_0^1 dx - \int_0^1 \frac{1}{x^2+1}\,dx$

$= 1 - \arctan(x)\Big|_0^1$

$= 1 - (\arctan(1) - \arctan(0))$

$= 1 - \left(\frac{\pi}{4} - 0\right)$

$= 1 - \frac{\pi}{4}$

$$\int \sec(x)\,dx$$

$= \int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)dx$

$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$

Let $u = \sec(x) + \tan(x)$

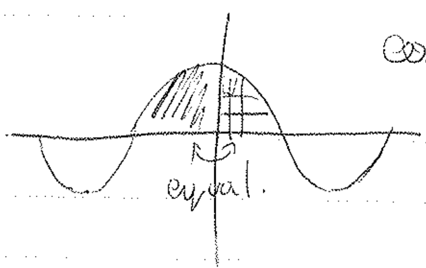
$du = (\sec(x)\tan(x) + \sec^2(x))\,dx$

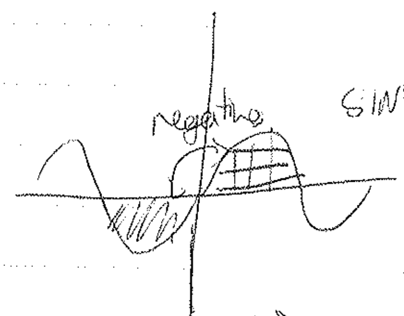
$= \int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|\sec(x) + \tan(x)| + C.$

Symmetry:





Note $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$

Def'n: A function is even if $f(-x) = f(x)$ eg. x^2, $1+3x^4+x^6$, etc

A function is odd if $f(-x) = -f(x)$ eg. x, $1+x^3+x^{17}$,

Note: odd fcn × odd fcn = even fcn = even fcn + even fcn

odd fcn + even fcn = odd fcn.

Theorem Let $f(x)$ be continuous on $[-a,a]$ for $a\in\mathbb{R}$.

- If f is even then $\int_{-a}^{a} f(t)\,dt = 2\int_0^a f(t)\,dt$
- If f is odd then $\int_{-a}^{a} f(t)\,dt = 0$

Pf: $\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_0^a f(t)\,dt$

$= \boxed{-\int_0^{-a} f(t)\,dt} + \int_0^a f(t)\,dt.$

Let $u=-t$ $\quad du=-dt$

$u(0)=0,\ u(-a)=a$

$= -\int_0^a f(u)(-du) + \int_0^a f(t)\,dt$

$= \int_0^a f(-u)\,du + \int_0^a f(u)\,du$

odd $f(-u)=-f(u)$ $\qquad$ $f(-u)=f(u)$ even

$= -\int_0^a f(u)\,du + \int_0^a f(u)\,du \qquad = \int_0^a f(u)\,du + \int_0^a f(u)\,du$

$=0 \qquad\qquad = 2\int_0^a f(u)\,du.$

Ex: $\int_{-\pi}^{\pi} \sin(x)\,dx = 0.$

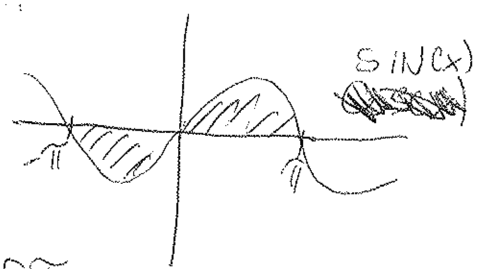
even even odd = odd.

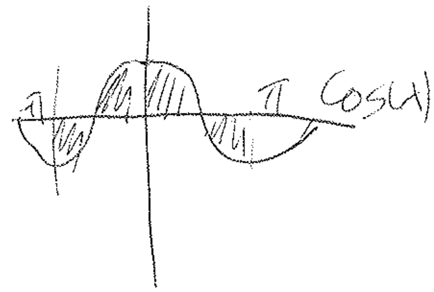
Ex: $\int_{-1}^{1} |x| \cos^{10}(x)\sin(x)\,dx = 0$

$\frac{\text{odd}}{\text{even}}$ = odd

Ex: $\int_{-100}^{100} \frac{\sin(x)}{1+x^2+x^{100}} = 0.$

Symmetry





$$\int_{-\pi}^{\pi} \sin(x)\,dx = 0$$

$\sin(-x) = -\sin(x)$

$$\int_{-\pi}^{\pi} \cos(x)\,dx = \int_0^{\pi} \cos(x)\,dx$$

$\cos(-x) = \cos(x)$

Def'n: A function $f(x)$ is odd if $f(-x) = -f(x)$
" " even if $f(-x) = f(x)$

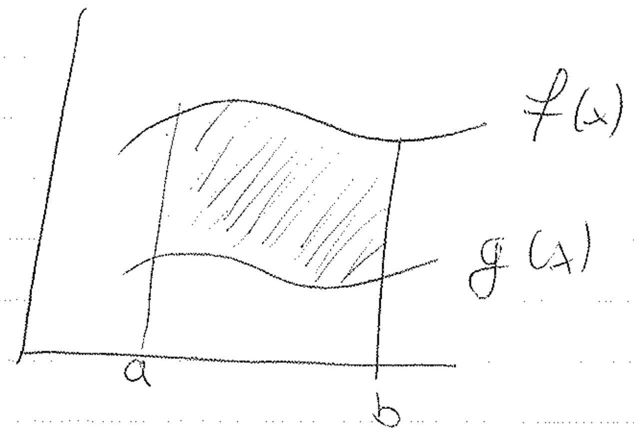
Theorem Suppose $f(x)$ is continuous on $[-a, a]$. Then
- If f is even, $\int_{-a}^{a} f(t)\,dt = 2\int_0^a f(t)\,dt$
- If f is odd, $\int_{-a}^{a} f(t)\,dt = 0$

Ex: $$\int_{-2\pi}^{2\pi} \frac{\sin(x)}{1+x^2+x^4}\,dx = 0$$ $\frac{\text{odd}}{\text{even}}$ = odd function

$$\int_{-\pi}^{\pi} |x| \cos^{10}(x) \sin(x) = 0$$ even * even * odd = odd function

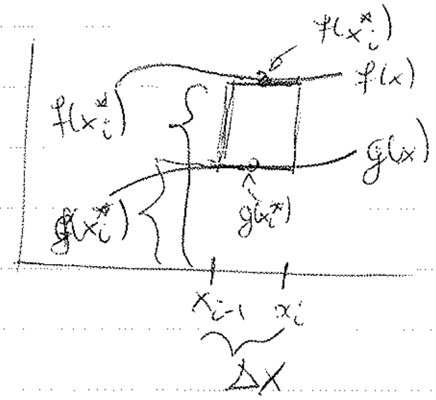
6.1 Area b/w Curves.

Setup: f, g continuous on $[a,b]$ with $f(x) \geq g(x)$ for all $x \in [a,b]$.



$$\int_a^b (f(x)-g(x))\,dx = \text{area b/w } f \text{ and } g.$$

Using rectangles



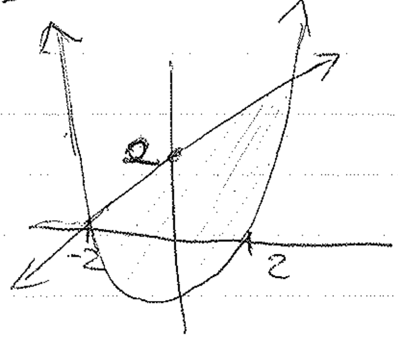
Area between two curves $y=f(x)$ and $y=g(x)$ when $f(x) \geq g(x)$ is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

$$= \int_a^b (f(x)-g(x))\,dx.$$

Note: The usual notion of integration is recovered by setting $g(x) = 0$.

Ex: Find the area of the region enclosed by $f(x) = x+2$ and $g(x) = x^2-4$.



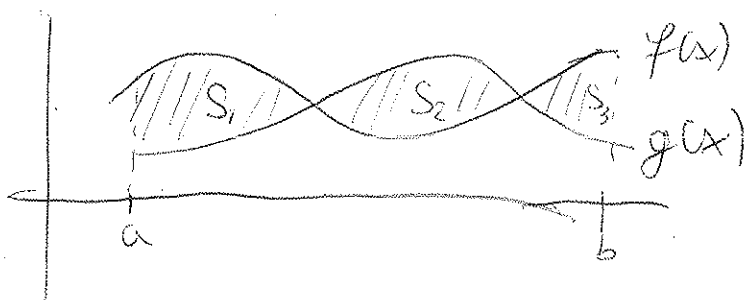
f and g intersect at
$$x^2-4 = x+2$$
$$x^2-x-6=0$$
$$(x+2)(x-3)=0$$
so at $x = -2, 3$

③

Hence, the area is

$$A = \int_{-2}^{3} (x+2-(x^2-4))\,dx$$
$$= \int_{-2}^{3} (-x^2+x+6)\,dx$$
$$= -\frac{x^3}{3} + \frac{x^2}{2} + 6x \Big|_{-2}^{3}$$
$$= -\frac{(3)^3}{3} + \frac{(3)^2}{2} + 6(3) - \frac{(-2)^3}{3} + \frac{(-2)^2}{2} + 6(-2)$$
$$= \frac{125}{6}$$

Problem: What if $f(x) < g(x)$?



Idea: Split up into regions when $f(x) \geq g(x)$ and $g(x) \geq f(x)$.

Note that: $|f(x)-g(x)| = \begin{cases} f(x)-g(x) & \text{when } f(x) \geq g(x) \\ g(x)-f(x) & \text{when } g(x) \geq f(x) \end{cases}$

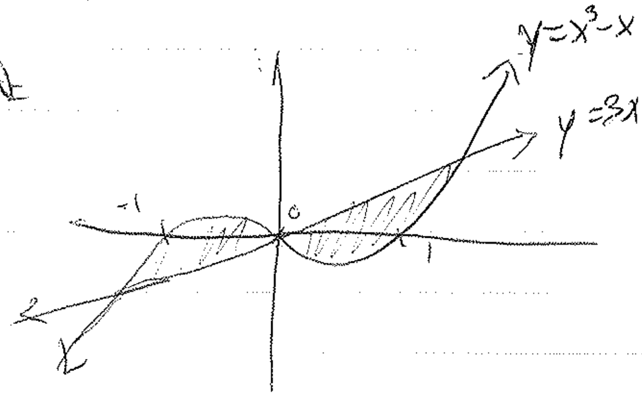
Hence:

Def'n: The area between two ~~function~~ curves $y=f(x)$ and $y=g(x)$ ~~continuous functions $f(x)$ and $g(x)$~~ between $[a,b]$ is

$$A = \int_a^b |f(x)-g(x)|\,dx$$

Ex: Compute the total area (NOT signed area) between $y = x^3 - x$ and $y = 3x$.

Want:



Interseet when $x^3 - x = 3x$

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$$x(x-2)(x+2) = 0 \qquad x = -2, 0, 2.$$

$x^3 - 4x$

$+ \quad - \quad +$

$-2 \quad 0 \quad 2$

$\boxed{x^3 - 4x} \quad \boxed{4x - x^3}$

So $\displaystyle\int_{-2}^{2} |x^3 - x - 3x|\,dx$

$\displaystyle\int_{-2}^{0}(x^3 - x - 3x)\,dx + \int_0^2 (3x - (x^3 - x))\,dx$

OR $\displaystyle = \int_{-2}^{0} (x^3 - 4x)\,dx + \int_0^2 (4x - x^3)\,dx$

$$= \left(\frac{x^4}{4} - 2x^2\right)\Big|_{-2}^{0} + \left(2x^2 - \frac{x^4}{4}\right)\Big|_{0}^{2}$$

$$= \frac{(0)^4}{4} - 2(0)^2 - \left(\frac{(-2)^4}{4} - 2(-2)^2\right) + 2(2)^2 - \frac{(2)^4}{4} - \left(2(0)^2 - \frac{(0)^4}{4}\right)$$

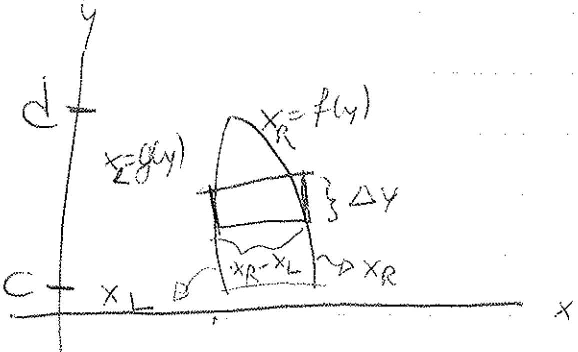
$$= -\left(\frac{16}{4} - 2(4)\right) + 8 - \frac{16}{4}$$

$$= -(4 - 8) + 8 - 4$$

$$= 8$$

5

Sometimes, swapping ~~the~~ the roles of x & y makes the problem easier



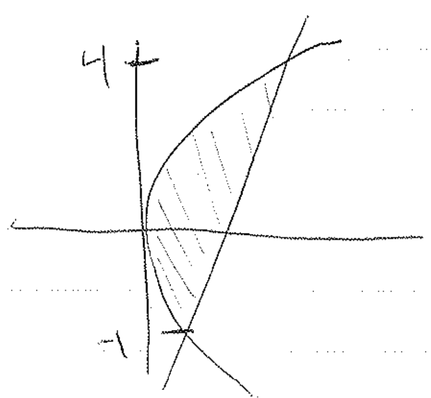
So the area between two curves $x = f(y)$ and $x = g(y)$ is

$$A = \int_c^d (x_R - x_L)\,dy$$

Ex: Find the area ~~btw~~ between the curves $y^2 = 4x$ $4x - 3y = 4$

NOTE Not a function!

Notice that in this case, x is easier to isolate than y. OR flip the roles!



$y^2 = 4x \longrightarrow 4x - 3y = 4$

$y^2 - 3y - 4 = 0$

$(y-4)(y+1) = 0$

$y = -1, 4$

$4x - 3y = 4 \rightsquigarrow x = \frac{3}{4}y + 1$

$y^2 = 4x \rightsquigarrow x = \frac{y^2}{4}$

~~Using the rule~~ Here.

$$\int_{-1}^{4} (x_R - x_L)\,dy = \int_{-1}^{4} \left(\frac{3}{4}y + 1 - \frac{y^2}{4}\right) dy$$

$$= \left(\frac{3y^2}{8} + y - \frac{y^3}{12}\right)\Big|_{-1}^{4}$$

$$= \frac{3}{8}(4)^2 + 4 - \frac{(4)^3}{12} - \left(\frac{3}{8}(-1)^2 + 1 - \frac{(-1)^3}{12}\right)$$

$$= \frac{125}{24}$$

DO WEBWORK! Quiz Monday up to ~~symmetry~~.

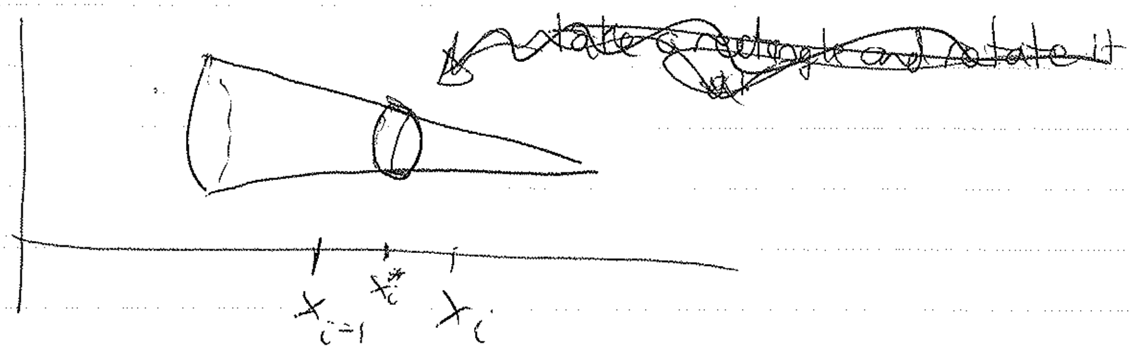
6.2 Volumes

Do flash stuff!

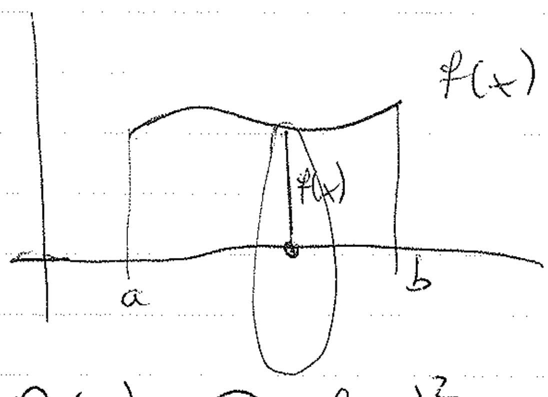
(1)

Def'n: Let S be a solid between $x=a$ and $x=b$. Let $A(x)$ be the cross sectional area of S in the plane through x and perpendicular to the x-axis. Assume $A(x)$ is continuous. Then

$$\mathrm{Vol}(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$



Solids of Revolution

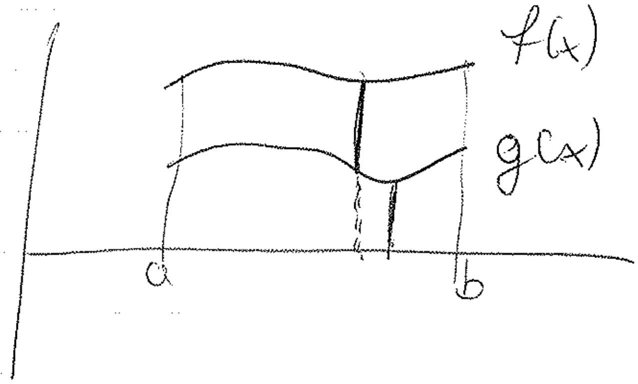


a rectangle (or $f(x)$)
Start with and rotate about x-axis.

$$A(x) = \pi f(x)^2$$

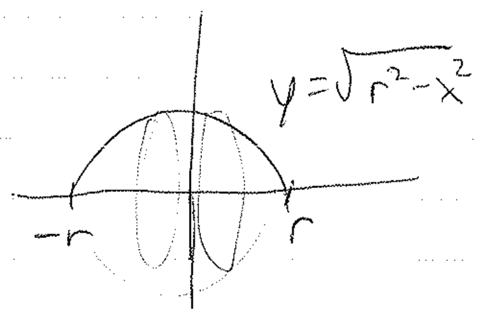
So $V = \int_a^b \pi f(x)^2\,dx$

Volumes of bowls



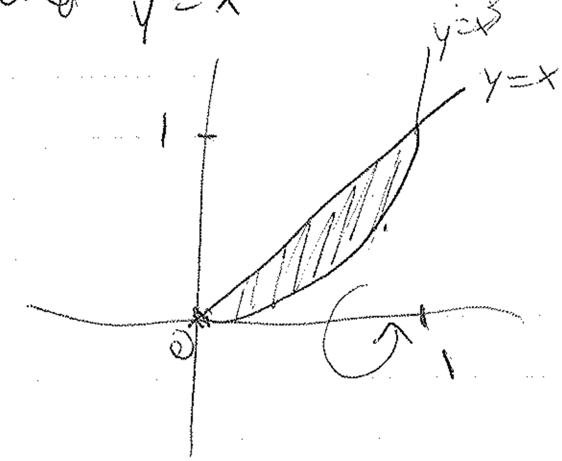
$$V = \int_a^b \pi f(x)^2 \, dx - \int_a^b \pi g(x)^2 \, dx$$

Ex: Volume of a sphere $x^2 + y^2 = r^2$ $r \in \mathbb{R}^+$ fixed.



$$\begin{aligned}
V &= \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx \\
&= \int_{-r}^{r} \pi (r^2 - x^2) \, dx \\
&= 2\int_0^r \pi (r^2 - x^2) \, dx \quad \text{Even function.} \\
&= 2\pi \left(r^2 x - \frac{x^3}{3} \right) \Bigg|_0^r \\
&= 2\pi \left(r^3 - \frac{r^3}{3} - \left(r^2(0) - \frac{0^3}{3}\right) \right) \\
&= \frac{4\pi}{3} r^3.
\end{aligned}$$

Ex: Find the ~~area~~ volume of the ~~volume~~ solid of revolution between the curves $y=x$ and $y=x^3$



$$V = \int_0^1 \pi\left(x^2 - (x^3)^2\right)dx$$

$$= \int_0^1 \pi x^2 dx - \int_0^1 \pi x^6 dx$$

$$= \pi \frac{x^3}{3}\Big|_0^1 - \pi \frac{x^7}{7}\Big|_0^1$$

$$= \frac{\pi}{3} - \frac{\pi}{7}$$

$$= \frac{4\pi}{21} \doteq 0.5984$$

About the y-axis

$x=y$ $\qquad$ $\boxed{x = y^{\frac{1}{3}}}$ is bigger

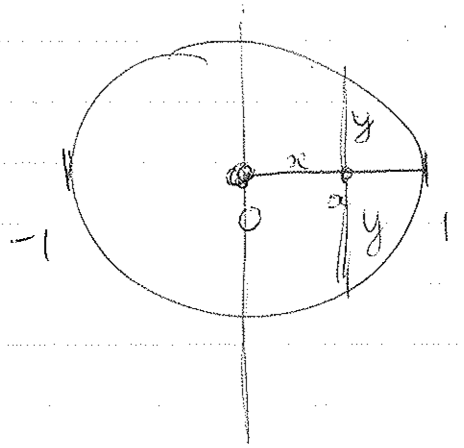
$$\int_c^d \pi(x_R^2 - x_L^2)\,dy = \int_0^1 \pi\left(y^{\frac{2}{3}} - y^2\right)dy = \pi\left(\frac{3y^{5/3}}{5} - \frac{y^3}{3}\right)\Big|_0^1$$

$$= \pi\left(\frac{3}{5} - \frac{1}{3}\right)$$

$$= \frac{4\pi}{15} \doteq 0.8378$$

Question: Compute the volume of a Solid with base a unit circle and cross sections are ~~as~~ squares (perpendicular to x-axis.

Sol'n:



$x^2+y^2=1$

$y=\sqrt{1-x^2}$

~~Are~~

Cross section area $=(2y)^2 = 4(\sqrt{1-x^2})^2 = 4(1-x^2) = A(x)$

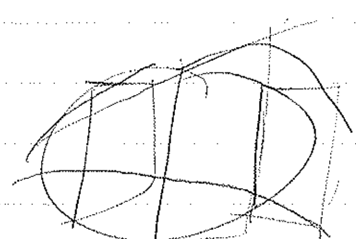
Volume is $\displaystyle\int_{-1}^{1} 4(1-x^2)\,dx$

$$= 2\int_0^1 4(1-x^2)\,dx$$

$$= \left(8x - \frac{8}{3}x^3\right)\Big|_0^1$$

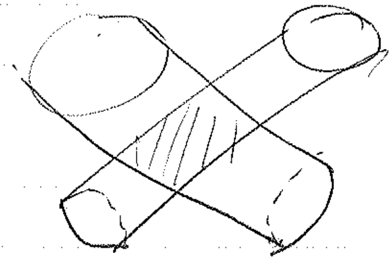
$$= 8 - \frac{8}{3}$$

$$= 16/3$$

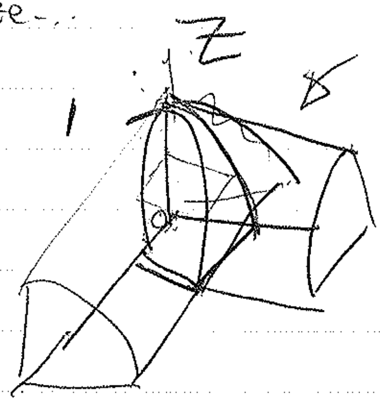


DO WEBWORK!   ①

Ex: Find the volume of the intersection of 2 perpendicular cylinders of unit radius



Tough to visualize...



$\frac{1}{8}$th the region

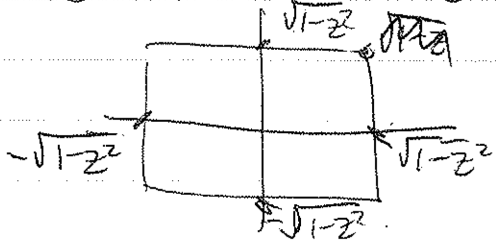
Try algebraic approach.

Cylinder 1 : $x^2 + z^2 \leq 1$
Cylinder 2 : $y^2 + z^2 \leq 1$

With the height z fixed, note

$$x^2 \leq 1 - z^2 \Rightarrow -\sqrt{1-z^2} \leq x \leq \sqrt{1-z^2}$$
$$y^2 \leq 1 - z^2 \Rightarrow -\sqrt{1-z^2} \leq y \leq \sqrt{1-z^2}$$

The slice of Area is a square!



↻

2 8

8. Area is $(2\sqrt{1-z^2})^2 = 4(1-z^2)$.

Volume side is $4(1-z^2)\Delta z$.

Hence, $V = \int_{-1}^{1} 4(1-z^2)\,dz$

$= \cancel{\left(4z - \frac{4z^3}{3}\right)\Big|_{-1}^{1}}$

$= 2\int_0^1 4(1-z^2)\,dz$

$= \left(8z - \frac{8z^3}{3}\right)\Big|_0^1$

$= 8 - \frac{8}{3}$

$= \frac{16}{3}$.

Work §6.4. ~~NO~~ 6.3 3

Notation: Time t in seconds
Position s in metres
Mass m in ~~Kg~~ kilograms

Newton's 2nd Law

$$F = \text{Force} = \text{mass} \times \text{acceleration} = m\frac{d^2s}{dt^2}$$

Force: In Newtons $N = kg \cdot m/s^2$

$W = \text{Work} = \text{Force} \times \text{displacement} \qquad W = Fd$ (energy to act on a force).

Work in ~~Newton metres~~ Joules $= N \cdot m$

If force is constant, this is easy.

Eg: How much work is done moving a 1kg book from the floor to a 2m high shelf?

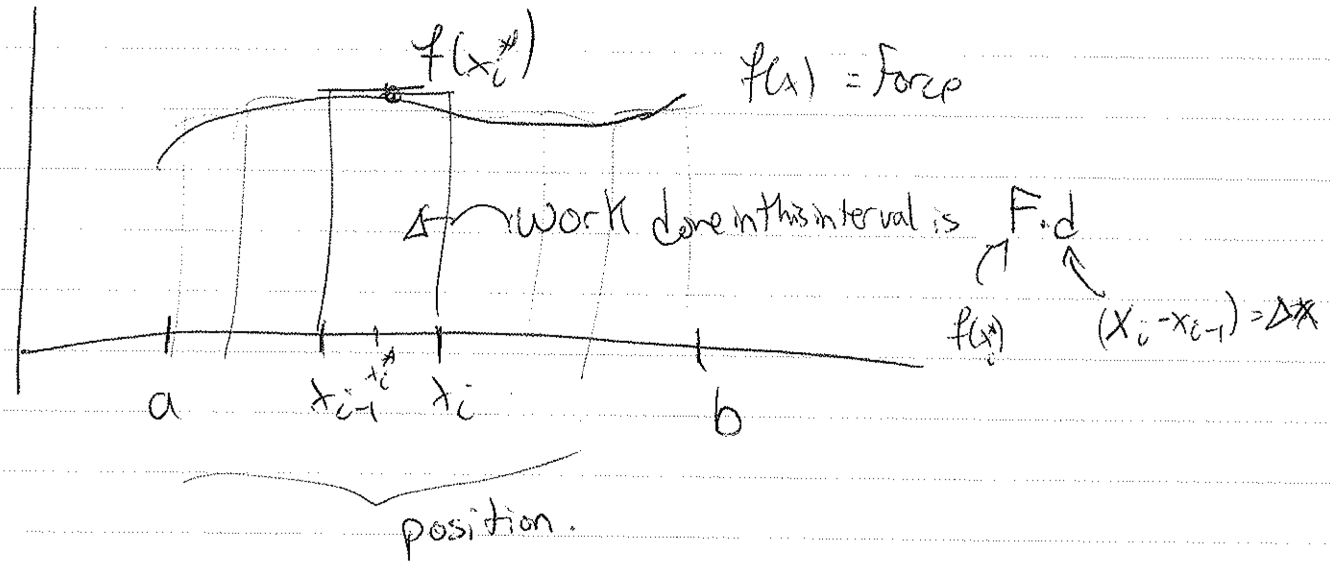
Ans: Acceleration due to gravity $= 9.8 m/s^2$
Force due to gravity $= ma = \cancel{(1)(9.8)} = (1kg)(9.8 m/s^2) = 9.8 N$
Work done against force $= (9.8N)(2m) = 19.6 J$.

If the force is not constant, then Riemann Sums!

§6.4

When work is not constant...

Let $f(x)$ be the force acting on an object at position x.



$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

Ex: When a particle is x metres from the origin, a force of x^3+3x kilograms acts on it. How much work is done moving it from $x=2$ to $x=4$

$$W = \int_2^4 (x^3+3x)\,dx = \left(\frac{x^4}{4} + \frac{3x^2}{2}\right)\Big|_2^4 = 64+24-4-6 = \boxed{78\,J}$$

Hooke's Law: ~~Relates force exerted~~

The force required to maintain a spring stretched x units beyond its natural length is proportional to x

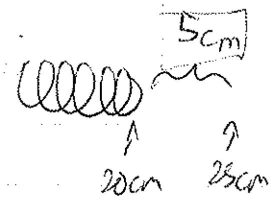
$F = kx$ some positive constant k in N/m

(Holds provided x is not too large.

Question: A spring has natural length of 20cm. If a 25N force is required to keep it stretched at a length of 30cm, How much work is required to stretch it from 20cm to 25cm?

- Units need to be in metres!

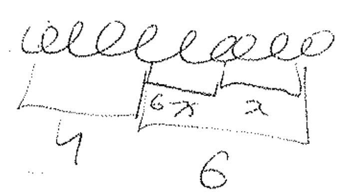
$$F = Kx$$
$$25 = K(0.3 - 0.2)$$
$$25 = 0.1K$$
$$\boxed{250 = K}$$



$$W = \int_0^{0.05} F(x)\,dx = \int_0^{0.05} 250x\,dx = 125x^2\Big|_0^{0.05} = 125 \cdot 0.0025$$
$$= 0.3125J.$$

Ex: A chain lying on the ground is 10m long with weight 80Kg.. How much work is required to raise the bottom end to a height of 6m?

Assumptions: • Assume lifting as an L-shape.
• No friction
• Weight is evenly distributed. Let x be the distance from the top.



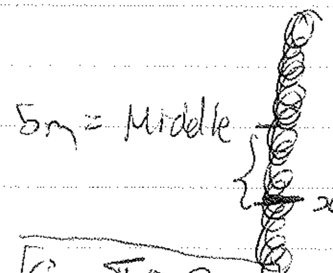
lifted $6 - x_i^*$ metres.

Each segment weighs $80\,\text{kg}/10\text{m} \cdot \Delta x = 8\Delta x$. ~~Force 8Δx~~

Force $= ma = 8\Delta x(9.8) = 78.4\Delta x$

$$\text{Work} = \lim_{n\to\infty} \sum_{i=1}^{n} (6 - x_i^*)\,78.4\Delta x = \int_0^6 (6-x)\,78.4$$
$$= 78.4\left(6x - \frac{x^2}{2}\right)\Big|_0^6 = 78.4(36-18) = 1411.2J.$$

Ex: The chain from before is dangling from a roof.



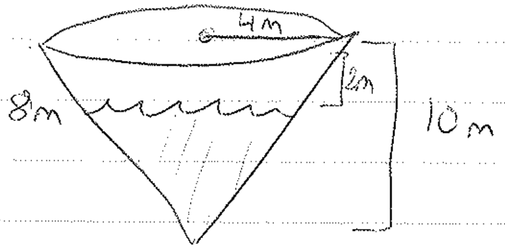
Chain is now lifted $2x$

So $W = \int_0^5 (2x)(78.4)\,dx$

$= 156.8\left(\frac{x^2}{2}\right)\Big|_0^5 = 156.8(12.5-0) = 1960J.$

$\boxed{\text{GOTO 3.5}}$

Ex: A tank has the shape

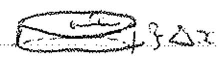


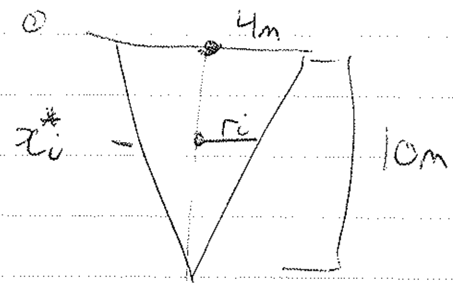
It is filled with water up to 8m. Find the work required to empty the tank by pumping all water from the top. Note: water density is $1000 kg/m^3$

Ans: Let x be the distance from the ~~bottom~~ top

Divide $[2,10]$ into equal subintervals $x_0, x_1, \ldots x_n$ with $x_i^* \in [x_{i-1}, x_i]$.

i^{th} layer is





$$\frac{r_i}{10-x_i^*} = \frac{4}{10}$$

So $r_i = \frac{2}{5}(10-x_i^*)$.

V_i = Volume at i^{th} layer $= \pi r_i^2 \Delta x = \frac{4\pi}{25}(10-x_i^*)^2 \Delta x$

m_i = mass of i^{th} layer = density $\times V_i = 1000 \cdot \frac{4\pi}{25}(10-x_i^*)^2 \Delta x = 160\pi(10-x_i^*)^2 \Delta x$

F_i = Force on i^{th} layer $= m_i g = (9.8)\,160\pi(10-x_i^*)^2 \Delta x = 1568\pi(10-x_i^*)^2 \Delta x$.

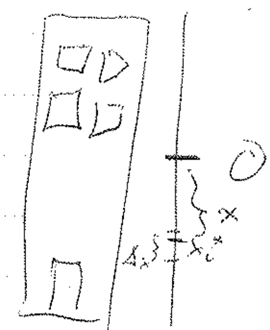
W_i = work on ith layer $= F_i \underbrace{x_i^*}_{\text{distance}} = 1568\pi\, x_i^*(10-x_i^*)^2 \Delta x.$

Now, take limits:

3.5

Ex: A 200-lb cable is 100 ft long and hangs vertically from a building. How much work is required to lift the bottom end of the cable to the top of the building?

Note: 200 lb is <u>NOT</u> a <u>mass</u>. It is a <u>weight</u> and weight means force.

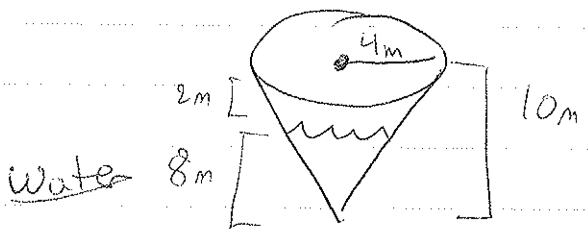


Force per foot $= \dfrac{200\text{ lb}}{100\text{ ft}} = 2\text{ lb/ft}.$

$W = F \cdot d \approx (2\Delta x)(2x)$

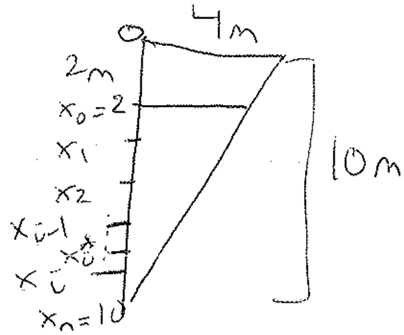
$W = \int_0^{50} 4x\,dx = (2x^2)\Big|_0^{50} = 5000$ ft-lb

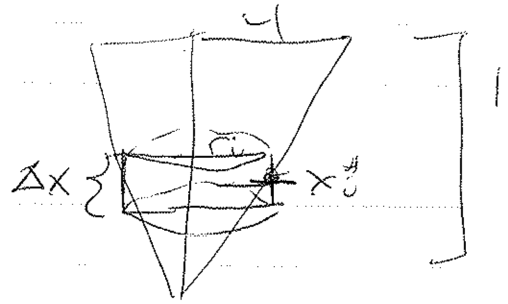
Ex: A tank has the shape

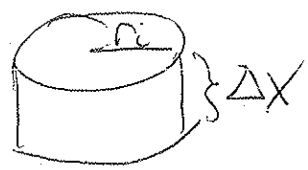


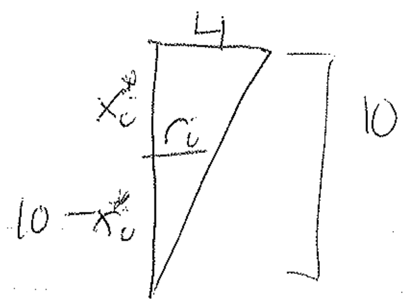
?: Work to empty tank from top.
density of water = 100 Kg/m^3

Ans: Let x be the distance from the top.
Divide [2, 10] into subintervals ~~$x_0, \ldots x_n$~~ with $x_i^* \in [x_{i-1}, x_i]$









$$\frac{r_i}{10-x_i^*} = \frac{4}{10} \Rightarrow r_i = \frac{2}{5}(10-x_i^*)$$

Volume of ith ~~slice~~ piece $= \pi r_i^2 \Delta x = \frac{4\pi}{25}(10-x_i^*)^2 \Delta x$

Mass of ith piece $=$ density $\times V_i$
$= (1000)\left(\frac{4\pi}{25}(10-x_i^*)^2\right)$

Force on ith layer $= m_i a = 160\pi(10-x_i^*)^2 \Delta x (9.8)$

Work on ith layer $= F_i$ displacement $= 1568\pi(10-x_i^*)^2 \Delta x (x_i^*)$

THUS,

(2)

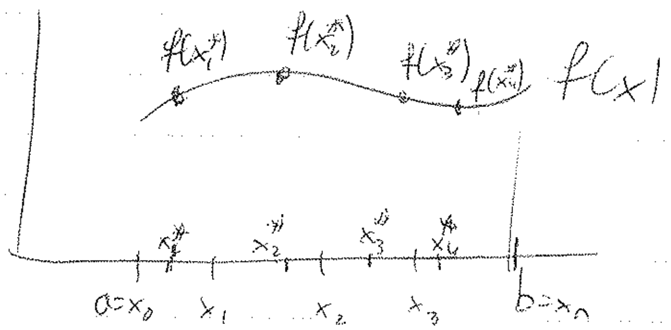
$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 1568\pi x_i^* (10 - x_i^*)^2 \Delta x$$

$$= \int_2^{10} 1568\pi x(10-x)^2\,dx = 1568\pi \int_2^{10} (100x - 20x^2 + x^3)\,dx$$

$$= 1568\pi \left(50x^2 - \frac{20x^3}{3} + \frac{x^4}{4}\right)\Big|_2^{10} = 1568\pi\left(\frac{2048}{3}\right) \approx 3.4 \times 10^6 J$$

§6.5 Average Value:

- Usual setup



$n=4$ above. What is the average value of $f(x_i^*)$?

$$\frac{f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)}{4}$$

In General:

$$\text{Average value of } f \approx \frac{1}{n} \sum_{i=1}^{n} f(x_i^*)$$

Want Riemann sum. Missing Δx.

<u>BUT</u> $\Delta x = \frac{b-a}{n} \Rightarrow \frac{1}{n} = \frac{\Delta x}{b-a}$

<u>So</u>

Def'n The average value of f on $[a,b]$ is

$$f_{av} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i^*) = \lim_{n\to\infty} \frac{\Delta x}{b-a} \sum_{i=1}^{n} f(x_i^*) = \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Ex: Find f_{av} for $f(x) = x^n$ on $[0,1]$ $(n \geq 0)$

Ans $f_{av} = \frac{1}{1-0}\int_0^1 x^n dx = \frac{1}{n+1} x^{n+1}\Big|_0^1 = \frac{1}{n+1}$.

Ex: Find f_{av} for $y = \sin(x)$ on $[0, \frac{\pi}{2}]$.

OMIT $f_{av} = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \sin(x)dx = \frac{2}{\pi}(-\cos(x))\Big|_0^{\pi/2} = \frac{-2}{\pi}\left(\cos(\tfrac{\pi}{2}) - \cos(0)\right)$

$= \frac{-2}{\pi}(0-(1)) = \frac{2}{\pi}$.

Does a function have to obtain its average value?

Thm. ~~MVTI~~ (Mean Value Theorem for Integrals MVTI).
If f is continuous on $[a,b]$, then there exists a $c \in [a,b]$ s.t.

$$f(c) = f_{av} = \frac{1}{b-a}\int_a^b f(x)dx$$

That is

$$\int_a^b f(x)dx = f(c)(b-a)$$

Ex: If f is continuous and $\int_1^3 f(x)dx = 8$ show f takes the value 4 at least once on $[1,3]$.

Pf. By ~~the MVT for Integrals~~ MVTI, there is a $c \in [1,3]$ such that

$$f(c)(3-1) = \int_1^3 f(x)dx = 8$$
$$2f(c) = 8$$
$$f(c) = 4$$

§7.1 Integration By Parts

Let's undo the product rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + g(x)f'(x)$$

Integrate both sides:

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx.$$

Rearranging gives:

Theorem: (Integration By Parts)
Let f and g be differentiable functions. Then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Ex: $\int xe^x\,dx$. Which function do we make f & g?

Two choices:

Choice 1: $f(x) = e^x$, $g'(x) = x$.
So $f'(x) = e^x$ $g(x) = \frac{x^2}{2}$

Hence $\int xe^x\,dx = \int g'(x)f(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$

$$= \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\,dx$$

Harder than before!

5

Let's try the other substitution
Choice 2: $f(x)=x$ $g'(x)=e^x$
$f'(x)=1$ $g(x)=e^x$

Hence $\int xe^x dx = \int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$

$= xe^x - \int e^x(1)dx$ ⇒ Easy!

$= xe^x - e^x + C$

Ex $\int_0^{\pi} x\cos(x)\,dx$

Use "table method

$u = x$ $v = \sin(x)$
$du = dx$ $dv = \cos(x)\,dx$

$$\int_0^{\pi} x\cos(x)\,dx = x\sin(x)\Big|_0^{\pi} - \int \sin(x)\,dx$$

$$= x\sin(x)\Big|_0^{\pi} - (-\cos(x))\Big|_0^{\pi}$$

$$= \pi\sin(\pi) - 0\sin(0) + \cos(\pi) - \cos(0) = 0 - 0 + (-1) - (1) = -2$$

Ex: $\int x\ln(x)\,dx$

$u = \ln x$ $v = \frac{x^2}{2}$
$du = \frac{1}{x}dx$ $dv = x\,dx$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

Ex: $\int \ln(x)\,dx = \int (1)\ln(x)\,dx$

$u = \ln x$ $v = x$
$du = \frac{1}{x}dx$ $dv = (1)dx$

$$\int \ln(x)\,dx = x\ln x - \int (1)dx = x\ln x - x + C$$

Rule of thumb:

Make the u function...

- L: Logarithms $\ln x$
- I: Inverse trig functions $\arctan x$
- A: Algebraic functions x^2, $1+x^3$,
- T: Trig functions $\sin(x)$, $\tan(x)$
- E: Exponential function e^x, 2^x.

Ex: $\int x^6 \ln(x)\,dx$

$u = \ln(x)$ $\quad v = \frac{x^7}{7}$

$du = \frac{dx}{x}$ $\quad dv = x^6\,dx$

$$= \frac{x^7}{7}\ln(x) - \int \frac{x^6}{7}\,dx$$

$$= \frac{x^7}{7}\ln(x) - \frac{x^7}{49} + C$$

Ex: $\int_0^1 \arctan(x)\,dx$

$u = \arctan(x)$ $\quad v = x$

$du = \frac{dx}{1+x^2}$ $\quad dv = dx$

$$= x\arctan(x)\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

Let $u = 1+x^2$ so $du = 2x\,dx$

$u(0) = 1+(0)^2 = 1$ $\quad u(1) = 1+(1)^2 = 2$

$$= (1)\arctan(1) - (0)\arctan(0) - \int_1^2 \frac{1}{2u}\,du$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(u)\Big|_1^2$$

$$= \frac{\pi}{4} - \frac{1}{2}\left(\ln(2) - \ln(1)\right) = \frac{\pi}{4} - \frac{\ln(2)}{2}.$$

Ex: $\int \frac{\log(x)}{x}\,dx$ Gotcha! Let $u = \log(x)$ so $du = \frac{1}{x}dx$

$$= \int u\,du = \frac{u^2}{2} + C = \frac{\log(x)^2}{2} + C$$

Ex: $\int t^2 e^t\,dt$

$u = t^2$ $\quad v = e^t$
$du = 2t\,dt$ $\quad dv = e^t\,dt$

$$= t^2e^t - \int 2te^t\,dt$$

$$= t^2e^t - 2\int te^t\,dt$$

Use parts again!
$u = t$ $\quad v = e^t$
$du = dt$ $\quad dv = e^t\,dt$

$$= t^2e^t - 2\left(te^t - \int e^t\,dt\right)$$

$$= t^2e^t - 2te^t + 2e^t + C$$

Ex: $\int \sin(t)e^t\,dt$

$u = \sin(t)$ $\quad v = e^t$
$du = \cos(t)dt$ $\quad dv = e^t\,dt$

$$= \sin(t)e^t - \int \cos(t)e^t\,dt$$

$u = \cos(t)$ $\quad v = e^t$
$du = -\sin(t)$ $\quad dv = e^t\,dt$

$$= \sin(t)e^t - \left(\cos(t)e^t - \int (-\sin(t))e^t\,dt\right)$$

$$= \sin(t)e^t - \cos(t)e^t - \int \sin(t)e^t\,dt$$

Rewriting

$$\boxed{\int \sin(t)e^t\,dt} = \sin(t)e^t - \cos(t)e^t \boxed{- \int \sin(t)e^t\,dt}$$

$$2\int \sin(t)e^t\,dt = \sin(t)e^t - \cos(t)e^t + C$$

$$\int \sin(t)e^t\,dt = \frac{\sin(t)e^t - \cos(t)e^t}{2} + C'$$

§7.2 Trigonometric Integrals

We want to integrate

$$\int \sin^a(x)\cos^b(x)\,dx \quad \text{and} \quad \int \tan^a(x)\sec^b(x)\,dx.$$

Recall:

$1 = \sin^2(x) + \cos^2(x)$ $\quad \downarrow \div$ by $\cos^2(x)$.

$\sec^2(x) = \tan^2(x) + 1$

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1 = \cos^2(x) - \sin^2(x)$.

Ex: Odd powers

$$\int \sin^5(x)\,dx = \int \sin(x)\left(1-\cos^2(x)\right)^2 dx$$

$$= \int \sin(x)\left(1 - 2\cos^2(x) + \cos^4(x)\right)dx.$$

Let $u = \cos(x)$ so $du = -\sin(x)\,dx$.

$$= -\int \left(1 - 2u^2 + u^4\right)du$$

$$= -u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + C.$$

$$= -\cos x + \frac{2}{3}\cos^3(x) + \frac{1}{5}\cos^5(x) + C.$$

Ex: Even powers: Double angle formulas

$$\int \cos^4(x)\,dx = \int \left(\cos^2(x)\right)^2 dx = \int \left(\frac{\cos 2x + 1}{2}\right)^2 dx \qquad 2\cos^2(x) = \cos 2x + 1$$

$$= \int \left(\frac{\cos^2 2x}{4} + \frac{\cos 2x}{2} + \frac{1}{4}\right)dx \qquad 2\cos^2 2x = \cos 4x + 1$$

$$= \int \left(\frac{\cos(4x)+1}{8} + \frac{\cos 2x}{2} + \frac{1}{4}\right)dx$$

②

$$= \frac{1}{8}\int \left(\cos(4x) + 4\cos(2x) + 3\right)dx$$

$$= \frac{1}{8}\left(\frac{1}{4}\sin(4x) + 2\sin(2x) + 3x\right) + C$$

Strategy for $\int \sin^m(x)\cos^n(x)\,dx$

(a) If the power of cosine (or sine) is odd, save a power of cosine (or sine) and change the other powers using $\cos^2(x) = 1-\sin^2(x)$ (or $\sin^2 = 1-\cos^2(x)$). Then substitute $u=\sin(x)$ (OR $u=\cos(x)$)

$$\int \sin^m(x)\cos^n(x) = \int \sin^m(x)\cos^{2k}(x)\cos(x)\,dx = \int \sin^m(x)\left(1-\sin^2(x)\right)^k \cos(x)\,dx$$

$u = \sin(x)$

OR $$\int \sin^m(x)\cos^n(x) = \int \sin^{2k}(x)\sin(x)\cos^n(x)\,dx = \int \left(1-\cos^2(x)\right)^k \sin(x)\cos^n(x)\,dx$$

$u = \cos(x)$.

(b) If both powers of sine & cosine are even, use double angle formulas.

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ $\quad\cos^2(x) = \frac{1}{2}(1+\cos(2x))$ $\quad\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$.

Key idea: $\sin(x)$ & $\cos(x)$ are derivatives of each other!

Try with $\tan(x)$ & $\sec(x)$

$\frac{d}{dx}\tan(x) = \sec^2(x)$ $\qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$.

Ex: $$\int \tan^2(x)\sec^4(x)\,dx = \int \tan^2(x)\sec^2(x)\sec^2(x)\,dx$$

$$= \int \tan^2(x)\left(1+\tan^2(x)\right)\sec^2(x)\,dx \qquad u = \tan(x),\ du = \sec^2(x)$$

$$= \int u^2(1+u^2)\,du = \cdots = \frac{\tan^3(x)}{3} + \frac{\tan^5(x)}{5} + C.$$

$= \cdots$ ~~$\frac{1}{5}\tan^3(x)\sec^2(x) + \frac{2}{15}\tan^3(x)$~~

$$\int \tan^3(x)\sec^7(x)\,dx = \int \tan^2(x)\sec^6(x)\,(\sec(x)\tan(x)\,dx) \qquad \cancel{u=\sec}$$

$$= \int (\sec^2(x)-1)\sec^6(x)\,(\sec(x)\tan(x)\,dx) \qquad u=\sec(x),\ du=\sec(x)\tan(x)\,dx$$

$$= \int (u^2-1)\,u^6\,du$$

$$= \dots = \frac{\sec^9(x)}{9} - \frac{\sec^7(x)}{7} + C$$

Strategy for $\int \tan^m(x)\sec^n(x)\,dx$

(a) If the power of $\sec(x)$ is even (≥ 2), save a $\sec^2(x)$ and use $\sec^2(x) = 1+\tan^2(x)$

$$\int \tan^m(x)\sec^{2k}(x)\,dx = \int \tan^m(x)\,(\sec^2(x))^{k-1}\sec^2(x)\,dx$$

$$= \int \tan^m(x)\,(1+\tan^2(x))^{k-1}\sec^2(x)\,dx \qquad \text{Then sub } u=\tan(x).$$

b) If $\tan(x)$ power is odd, save $\sec(x)\tan(x)$ and use $\tan^2(x) = \sec^2(x)-1$.

$$\int \tan^{2k+1}(x)\sec^n(x)\,dx = \int (\tan^2(x))^k \sec^{n-1}(x)\,\sec(x)\tan(x)\,dx$$

$$= \int (\sec^2(x)-1)^k \sec^{n-1}(x)\,\sec(x)\tan(x)\,dx$$

Then sub $u=\sec(x)$.

For other cases: No general algorithm. Recall:

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C.$$

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(B).

(1) (9)

Ex: $\int \tan^4(x)\,dx = \int \tan^2(x)(1+\sec^2(x))\,dx$

$= \int \tan^2(x)\sec^2(x)\,dx + \int \tan^2(x)\,dx$

$u = \tan(x)$
$du = \sec(x)\,dx$

$= \int u^2\,du + \int (1+\sec^2(x))\,dx$

$= \frac{u^3}{3} + x + \tan(x) + C$

$= \frac{\tan^3(x)}{3} + x + \tan(x) + C.$

Ex: $\int \sec^3(x)\,dx = \int \sec^2(x)\sec(x)\,dx$

$u = \sec(x)$ $\quad v = \tan(x)$
$du = \sec(x)\tan(x)\,dx$ $\quad dv = \sec^2(x)\,dx$

$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$

$= \sec(x)\tan(x) - \int \sec(x)(-1+\sec^2(x))\,dx$

$= \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| - \int \sec^3(x)\,dx$

$\Rightarrow \int \sec^3(x)\,dx = \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|) + C.$

Similar tricks work for $\int \cot^m(x)\csc^n(x)\,dx$.
Lastly, to evaluate
(a) $\int \sin(mx)\cos(nx)\,dx$ (b) $\int \sin(mx)\sin(nx)\,dx$ (c) $\int \cos(mx)\cos(nx)\,dx$

Use
(a) $\sin A\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
(b) $\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
(c) $\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$.

IGNORE

2

Ex: $\cancel{\int \sin(6x)\cos(7x)\,dx = \int \frac{1}{2}\left(\sin(-x) + \sin(13x)\right) dx}$

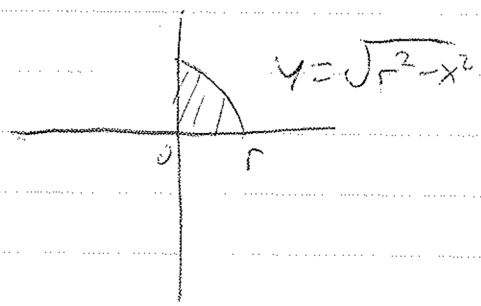
$\cancel{\sin(-x) = \sin(x)}$

$= \frac{1}{2}\int \sin(x) + \frac{1}{2}\int \sin(9x)\,dx$

$= \frac{1}{2}\cos(x) - \frac{1}{9}\cos(9x) + C.$

IGNORE

§ 7.3 Trigonometric Substitutions:



Area of a circle $= 4\int_0^r \sqrt{r^2-x^2}\,dx = \pi r^2$ ← why???

Trick: $\sqrt{r^2 - r^2\sin^2\theta} = \sqrt{r^2\cos^2\theta} = |r\cos\theta|.$

So we substitute $x = r\sin\theta$ so $dx = r\cos\theta\,d\theta$

End points $x=0 \Rightarrow 0 = r\sin\theta \Rightarrow \theta = 0$

$x = r \Rightarrow r = r\sin\theta \Rightarrow 1 = \sin\theta \Rightarrow \theta = \frac{\pi}{2}$ (★)

NOTE we want this substitution to make sense and inverse trig substitutions make sense when $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. (more later)

So $4\int_0^r \sqrt{r^2-x^2}\,dx = 4\int_0^{\frac{\pi}{2}} |r|\sqrt{1-\sin^2\theta}\; r\cos\theta\,d\theta = 4\int_0^{\frac{\pi}{2}} r^2\,|\cos\theta|\cos\theta\,d\theta$ $(r>0)$

$\cos\theta > 0$ on region $\Rightarrow = 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 4r^2\int_0^{\frac{\pi}{2}} \frac{(1+\cos(2\theta))}{2}\,d\theta = 2r^2\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{\frac{\pi}{2}}$

$= 2r^2\left(\frac{\pi}{2} + 0 - 0 - 0\right) = \pi r^2.$

This is called an inverse substitution.

Table of Trig Substitutions

one to one here. ie defined inverses.

Expression	Substitution		Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \le \theta < \frac{\pi}{2}$ OR $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Ex: $\int \frac{dx}{x^2+4x+7} = \int \frac{dx}{(x+2)^2-4+7} = \int \frac{1}{(x+2)^2+3}$

$x^2+4x+4-4+7 = x^2+4x+7$

Use ~~$\tan\theta$~~ $x+2 = \sqrt{3}\tan\theta$. so $dx = \sqrt{3}\sec^2\theta\, d\theta$

$$= \int \frac{\sqrt{3}\sec^2\theta\, d\theta}{(\sqrt{3}\tan\theta)^2+3} = \int \frac{\sec^2\theta\, d\theta}{3(\tan^2\theta+1)} = \frac{1}{3}\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{3}\theta + C = \frac{1}{3}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + C$$

Ex: $\int \frac{dx}{x^2\sqrt{x^2-16}}$ Let $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta\, d\theta$

$$= \int \frac{4\sec\theta\tan\theta\, d\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}} = \int \frac{\tan\theta}{4\sec\theta\sqrt{16\tan^2\theta}} = \int \frac{\tan\theta}{4\sec\theta\, 4|\tan\theta|}\, d\theta$$

Now on $0 \le \theta < \frac{\pi}{2}$ OR $\pi \le \theta < \frac{3\pi}{2}$, $|\tan\theta| = \tan\theta$ so

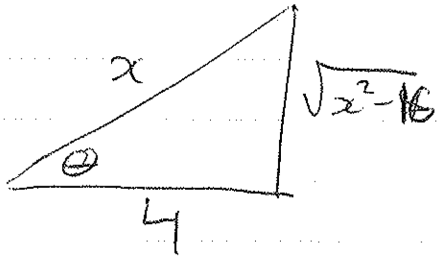
$$= \int \frac{d\theta}{16\sec\theta} = \frac{1}{16}\int \cos\theta\, d\theta = \frac{1}{16}\sin\theta + C$$

2

Now, need to change $\sin\theta$ to x.

$\sec\theta = \frac{x}{4}$

$\Rightarrow \cos\theta = \frac{4}{x}$



So $\sin\theta = \frac{\sqrt{x^2-16}}{x}$ since.

$$\int \frac{dx}{x^2\sqrt{x^2-16}} = \frac{\sqrt{x^2-16}}{16x} + C.$$

Ex: $\displaystyle\int \frac{2x}{\sqrt{x^2+1}}\,dx$ $\qquad u = x^2+1$, $du = 2x\,dx$

$$= \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{x^2+1} + C$$

Moral: Sometimes other methods are easier.

Did not do

Class try:

$\frac{1}{3}\left(3\arcsin\left(\frac{x}{3}\right)\right) + C$

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{3\sqrt{1-\left(\frac{x}{3}\right)^2}} = \arcsin\left(\frac{x}{3}\right) + C$$

OR $x = 3\sin\theta$ $\qquad dx = 3\cos\theta\,d\theta$

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta\,d\theta}{\sqrt{9-(3\sin\theta)^2}} = \int \frac{3\cos\theta\,d\theta}{3\sqrt{1-\sin^2\theta}} = \int d\theta = \theta + C = \arcsin\left(\frac{x}{3}\right) + C$$

3 8

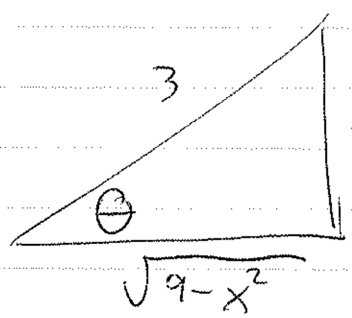
$$\int \sqrt{9-x^2}\,dx \qquad \text{let } x = 3\sin\theta, \quad dx = 3\cos\theta\,d\theta$$

$$= \int \sqrt{9-(3\sin\theta)^2}\,(3\cos\theta\,d\theta)$$

$$= \int 9\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$= \int 9\cos^2\theta\,d\theta = 9\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + C = \frac{9}{2}\left(\theta + \sin\theta\cos\theta\right) + C.$$



$\sin\theta = \frac{x}{3}$ $\quad \cos\theta = \frac{\sqrt{9-x^2}}{3}$

$\theta = \arcsin\left(\frac{x}{3}\right)$

$$= \frac{9}{2}\arcsin\frac{x}{3} + \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\arcsin\frac{x}{3} + \frac{x\sqrt{9-x^2}}{2} + C$$

§7.4 Integration By Partial Fractions

Recall: $\int \frac{dx}{x+a} = \ln|x+a| + C$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{(\frac{x}{a})^2+1} = \frac{1}{a^2}\left(a\arctan\left(\frac{x}{a}\right)\right) + C = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

Example: $\int \frac{(3x+1)}{x^2+2x-3}\,dx$ ← seems hard.

Ex: $\int\left(\frac{1}{x-1} + \frac{2}{x+3}\right)dx = \ln|x-1| + 2\ln|x+3| + C$ ← easy.

<u>Note</u> $\frac{1}{x-1} + \frac{2}{x+3} = \frac{(x+3)+2(x-1)}{(x-1)(x+3)} = \frac{x+3+2x-2}{x^2+2x-3} = \frac{3x+1}{x^2+2x-3}$

<u>So</u> $\int \frac{(3x+1)}{x^2+2x-3}dx = \int\left(\frac{1}{x-1} + \frac{2}{x+3}\right)dx = \ln|x-1| + 2\ln|x+3| + C.$

$f(x) = \frac{P(x)}{Q(x)}$ where P, Q are polynomials

Q: How do we go from "ugly rational function" to "nice rational functions"?

(1) If $\deg(P) \geq \deg(Q)$, then do long division.

Ex $\int \frac{(x^4+2x)}{x-1}dx$ $\quad \left.\begin{array}{l}\deg(x^4+2x) = 4 \\ \deg(x-1) = 1\end{array}\right\} 4 \geq 1$. so long division.

$$= \int\left(x^3+x^2+x+3 + \frac{3}{x-1}\right)dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + 3x + 3\ln|x-1| + C.$$

$$\begin{array}{r|l}
 & x^3 + x^2 + x + 3 \\ \hline
x-1 & x^4 \qquad\qquad + 2x \\
 & -(x^4 - x^3) \\ \hline
 & x^3 + 0x^2 \\
 & -(x^3 - x^2) \\ \hline
 & x^2 + 2x \\
 & -(x^2 - x) \\ \hline
 & 3x + 0 \\
 & -(3x - 3) \\ \hline
 & 3
\end{array}$$

(2) Simplify into simpler pieces depending on denominator factors.

$$\frac{A}{(x-a)^n} \qquad \frac{Bx+C}{(x^2+bx+c)^M}$$

Ex: If $Q(x) = (x-3)^3(x+2)(x^2+x+4)^2$, then we want

$$\frac{P(x)}{Q(x)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{x+2} + \frac{Ex+F}{x^2+x+4} + \frac{Gx+H}{(x^2+x+4)^2}$$

(3) Find the constants. $A, B, \dots, H$

(4) Integrate term by term.

Let's try this with our example.

$$\frac{3x+1}{x^2+2x-3} \qquad (1)\ \deg(3x+1) = 1 \leq 2 = \deg(x^2+2x-3) \text{ so (1) is okay.}$$

(2) $$\frac{3x+1}{x^2+2x-3} = \frac{3x+1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

(3) $$\frac{3x+1}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)} = \frac{(A+B)x+3A-B}{(x-1)(x+3)}$$

Method (1): Compare coefficients. Since denominators are equal, numerators must be.

$$3x+1 = (A+B)x + (3A-B)$$

So $A+B=3$ $\qquad 3A-B=1$

$A = 3-B$ $\qquad 3(3-B)-B=1$

$\qquad 9-3B-B=1$

$\qquad -4B=-8$

$\qquad B=2$

$A = 3-2$

$A=1$

Method (2)

$$3x+1 = A(x+3) + B(x-1)$$

Since this holds for all values of x, we can plug in values of x and solve for A and B. Choose easy values for x (ie zeroes, 0, small integers etc.)

At $x=-3$,

$$3(-3)+1 = A((-3)+3) + B((-3)-1)$$
$$-9+1 = -4B$$
$$-8 = -4B$$
$$\underline{2} = B.$$

At $x=1$

$$3(1)+1 = A((1)+3) + B((1)-1)$$
$$3+1 = 4A$$
$$4 = 4A$$
$$1 = A.$$

<u>So</u> $\displaystyle\int \frac{(3x+1)}{x^2+2x-3}dx = \int\left(\frac{A}{x-1}+\frac{B}{x+3}\right)dx = \int\left(\frac{1}{x-1}+\frac{2}{x+3}\right)dx = \ln|x-1| + 2\ln|x+3| + C$

$= \ln|(x-1)(x+3)^2| + C.$

Ex: $\displaystyle\int \frac{x^2-9x+17}{x^3-3x^2+4}\,dx$

We need to factor the denominator. Notice that $x=-1$ is a root!

$$\begin{array}{r|l} & x^2-4x+4 \\ \hline x+1 & x^3-3x^2+4 \\ & \underline{x^3+x^2} \\ & \quad -4x^2 \\ & \quad \underline{-4x^2-4x} \\ & \qquad 4x+4 \\ & \qquad \underline{4x+4} \\ & \qquad R\ 0 \end{array}$$

So $x^3-3x^2+4 = (x+1)(x^2-4x+4)$

$= (x+1)(x-2)^2$

So $\displaystyle\frac{x^2-9x+17}{(x-2)^2(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+1)} = \frac{A(x-2)(x+1)+B(x+1)+C(x-2)^2}{(x-2)^2(x+1)}.$

So $\quad x^2-9x+17 = A(x-2)(x+1) + B(x+1) + C(x-2)^2$

at $x=-1$, $\quad 27 = C(-1-2)^2 = 9C \quad \Rightarrow C=3$

$x=2$ $\quad 3 = 3B \quad \Rightarrow B=1$

at $x=0$ $\quad 17 = A(-2)(1) + B(1) + C(-2)^2$

$17 = -2A + B + 4C$

$17 = -2A + 1 + 4(3)$

$17 = -2A + 13$

$4 = -2A$

$-2 = A$

So $$\int \frac{x^2-9x+17}{x^3-3x^2+4}\,dx = \int \frac{-2}{(x-2)}\,dx + \int \frac{dx}{(x-2)^2} + \int \frac{3}{(x+1)}\,dx$$

$$= -2\ln|x-2| - \frac{1}{(x-2)} + 3\ln|x+1| + C.$$

Maybe omit for time reasons.

Ex: $\displaystyle\int \frac{x+1}{x^2-2x+5}\,dx$ $\qquad$ (Tempting to try $u = x^2-2x+5$).

Key: complete the square $\quad x^2-2x+5 = x^2-2x+1-1+5 = (x-1)^2+4$.
Now, let $u = x-1$ so $du = dx$ and $x+1 = u+2$.

$$\int \frac{x+1}{x^2-2x+5}\,dx = \int \frac{x+1}{(x-1)^2+4}\,dx = \int \frac{u+2}{u^2+4}\,du = \int \frac{u}{u^2+4}\,du + 2\int \frac{du}{u^2+4}$$

$$= \frac{1}{2}\ln|u^2+4| + 2\left(\frac{1}{2}\arctan\left(\frac{u}{2}\right)\right) + C$$ (see page 1)

$$= \frac{1}{2}\ln|(x-1)^2+4| + \arctan\left(\frac{x-1}{2}\right) + C$$

Ex: $\displaystyle\int \frac{2x^2+8x+115}{(x-1)(x^2+8x+116)}\,dx$

$$\frac{2x^2+8x+115}{(x-1)(x^2+8x+116)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+8x+116} = \frac{A(x^2+8x+116)+(Bx+C)(x-1)}{(x-1)(x^2+8x+116)}$$

So $\quad 2x^2+8x+115 = A(x^2+8x+116) + (Bx+C)(x-1)$

At $x=1$ $\quad 125 = 125A + (B(1)+C)\underbrace{(1-1)}_{=0}$

$125 = 125A \qquad$ so $A=1$

At $x=0$ $\quad 115 = 116A + (B(0)+C)(0-1)$

$115 = 116(1) - C$

$-1 = -C \qquad$ so $C=1$

At $x=-1$ $\quad 2(-1)^2+8(-1)+115 = A((-1)^2+8(-1)+116) + (B(-1)+C)((-1)-1)$

$2-8+115 = A(1-8+116) + (-B+C)(-2)$

$109 = 109A + 2B - 2C$

$109 = 109(1) + 2B - 2(1)$

$2 = 2B \qquad$ so $B=1$

$$\int \frac{2x^2+8x+115}{(x-1)(x^2+8x+116)}\,dx = \int \frac{dx}{x-1} + \int \frac{x+1}{x^2+8x+116}\,dx$$

$$\cancel{= \ln|x-1| + \int \frac{x\,dx}{x^2+8x+116} + \int \frac{dx}{x^2+8x+116}}$$

For the last integral, complete the square.

$$x^2+8x+116 = x^2+8x+16-16+116 = (x+4)^2+100$$

$$\int \frac{2x^2+8x+115}{(x-1)(x^2+8x+116)}\,dx = \int \frac{dx}{x-1} + \int \frac{(x+1)}{(x+4)^2+100}\,dx$$

let $u = x+4$ so $x = u-4$.
$du = dx$

6

$$= \ln|x-1| + \int \frac{(u-4)+1}{u^2+100}\,du$$

$$= \ln|x-1| + \int \frac{u}{u^2+100} + \int \frac{-3\,du}{u^2+100}$$

$$= \ln|x-1| + \frac{1}{2}\ln|u^2+100| - 3\left(\frac{1}{100}\arctan\left(\frac{x}{100}\right)\right) + C$$

Jxxx.

0

Announcements
(0) Alternate midterm.
(1) Quiz Monday 7.2 & 7.3
(2) Webwork due Sunday
(3) Strike possibility.
(4) Note learning objectives on commonsite.
(5) Survey results.
* Write neater, draw better.
* Slow down on advanced steps.
* What to review from other courses. (depends on person) derivatives & algebra
* ALWAYS ASK QUESTIONS! There are no stupid questions.
* Quiz time is short; should match midterm.

(1.5) CHANGE OFFICE HOURS.
No Tu 11:00-12:00 or W 3:00-4:00
Instead Tu 2:30-4:30 ONE WEEK ONLY.

Just right 2/3rds
Lectures mix of clear, ok, confusing
Attempt to be organized.

to review
Int by parts (online?)
Work, Volume, Force (no...)
Trig integrals (when to use which sub) Piazza?
identities ("I should tatoo them to my arm").
→ I will help.

Piazza
* Most like ability to ask questions whenever even Fri nights.
* It's another site to check.
* Spams email. Make a filter.
* I'm teaching myself outside the classroom ⇒ GOOD THING!

Favourite quotes
"Very personable human professor.
Goals of the course
* Learn to integrate like a boss.
~~Undo math 100~~
* Integrate ~~everything~~
* to eliminate the weak.

$$\int_{-\pi}^{\pi} x^{101} \cos(x)\, dx$$

Theorem: (Euler's Formula) Let $i = \sqrt{-1}$. Then

$$e^{i\theta} = \cos(\theta) + i\sin\theta$$

Ex:

$$e^{i(\theta+\gamma)} = \cos(\theta+\gamma) + i\sin(\theta+\gamma)$$

$$\begin{aligned}
e^{i\theta}\cdot e^{i\gamma} &= (\cos(\theta) + i\sin(\theta))(\cos(\gamma) + i\sin(\gamma)) \\
&= \cos(\theta)\cos(\gamma) + \cos\theta\sin(\gamma)i + \cos(\gamma)\sin(\theta)i + \sin\theta\sin(\gamma)\underbrace{i^2}_{=-1} \\
&= \cos\theta\cos\gamma - \sin\theta\sin\gamma + (\cos\theta\sin\gamma + \cos\gamma\sin\theta)i
\end{aligned}$$

So

$$\cos(\theta+\gamma) = \cos\theta\cos\gamma - \sin\theta\sin\gamma$$

$$\sin(\theta+\gamma) = \cos\theta\sin\gamma + \cos\gamma\sin\theta$$

At $\theta = \gamma$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \overset{\cos^2\theta - 1 = -\sin^2\theta}{=} 2\cos^2\theta - 1$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Reminder

$$\int \frac{dx}{x+a} = \ln|x+a| + C$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Ex: $\displaystyle\int \frac{6x^2+9x+11}{(x-1)(x^3+x^2+8x-10)}\,dx$

Step (1) is clear.

Step (2) First factor cubic. Roots (if integers) are factors of 10 so try $\pm1, \pm2, \pm5, \pm10$

Notice that

$$(1)^3+(1)^2+8(1)-10=0$$

So $x-1$ is a factor of $x^3+x^2+8x-10$ (+1 is the root $x-1$ is the factor)

Using long division, note

$$x^3+x^2+8x-10=(x-1)(x^2+2x+10)$$

So $$\frac{6x^2+9x+11}{(x-1)(x^3+x^2+8x-10)} = \frac{6x^2+9x+11}{(x-1)^2(x^2+2x+10)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+2x+10}$$

$$= \frac{A(x-1)(x^2+2x+10)+B(x^2+2x+10)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+2x+10)}$$

Find A, B, C, D $\quad 6x^2+9x+11 = A(x-1)(x^2+2x+10)+B(x^2+2x+10)+(Cx+D)(x-1)^2,$

At $x=1$ $\quad 26 = B(13)$

$2 = B$

At $x=0$ $\quad 11 = -10A+10B+D$

$11 = -10A+20+D$

$-9 = -10A+D$

At $x=-1$ $\quad 6-9+11 = A(-2)(1-2+10)+B(1-2+10)+(-C+D)(-2)^2$

$8 = -18A+9B-4C+4D$

$-10 = 8-9(2) = 8-9B = -18A-4C+4D.$

At $x=2$ $\quad 24+18+11 = A(2-1)((2)^2+2(2)+10)+B(2^2+2(2)+10)+(2C+D)(2-1)^2$

$53 = 18A+18B+2C+D.$

After a bit of algebra... we see

$A=1,\ B=2,\ C=-1,\ D=1.$

Hence

$$\int \frac{6x^2+9x+11}{(x-1)(x^3+x^2+8x-10)}\,dx = \int \frac{dx}{x-1} + \int \frac{2\,dx}{(x-1)^2} + \int \frac{(-x+1)\,dx}{x^2+2x+10}$$

$$= \ln|x-1| - \frac{2}{x-1} + \int \frac{-x+1}{x^2+2x+1-1+10}\,dx$$

LEAVE SPACE $\longrightarrow$ $= \ln|x-1| - \dfrac{2}{x-1} - \dfrac{1}{2}\ln|(x+1)^2+9| + \dfrac{2}{3}\arctan\left(\dfrac{x+1}{3}\right) + C.$

Need to evaluate $\displaystyle\int \frac{-x+1}{(x+1)^2+9}\,dx.$ Let $u=x+1$, $du=dx$. $-x = -u+1$

$$= \int \frac{-u+2}{u^2+9}\,du$$

$$= -\frac{1}{2}\int \frac{2u}{u^2+9}\,du + 2\int \frac{1}{u^2+9}$$

$$= -\frac{1}{2}\int \frac{dw}{w} + \frac{2}{3}\arctan\left(\frac{u}{3}\right)$$ Let $w=u^2+9$, $dw=2u\,du$

$$= -\frac{1}{2}\ln|w| + \frac{2}{3}\arctan\left(\frac{u}{3}\right)$$

$$= -\frac{1}{2}\ln|u^2+9| + \frac{2}{3}\arctan\left(\frac{u}{3}\right)$$

$$= -\frac{1}{2}\ln|(x+1)^2+9| + \frac{2}{3}\arctan\left(\frac{x+1}{3}\right) + C$$ ✓

§7.5 Strategy For Integration

0. Just do it eg $\int \frac{dx}{x}$
1. Simplify the integrand eg $\int \sqrt{x}(1+\sqrt{x})\,dx$.
2. Try ~~an ob~~ a substitution eg. $\int \frac{x}{x^2+1}\,dx$
3. Classify the integrand
   (a) Trig functions eg. $\int \tan^3(x)\sec^7(x)\,dx$
   (b) Rational functions eg. $\int \frac{x^2+x+1}{(x-1)^3}\,dx$ use partial fractions
   (c) Integration by parts eg. $\int x\ln x\,dx$
   (d) Radicals eg. $\int \frac{dx}{x^2\sqrt{x^2-9}}$ use trig sub.
4. Repeat the above (try non-obvious methods).
   eg. $\int \frac{dx}{1-\cos(x)}$ (multiply top & bottom by $\frac{1+\cos(x)}{1+\cos(x)}$)

No 7.6

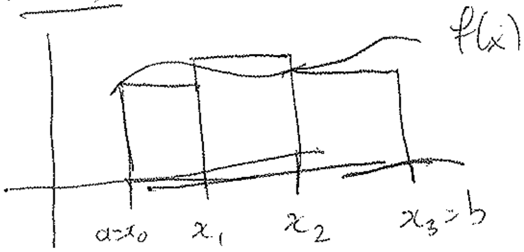
§7.7 Approximate Integration

What if we can't ~~a~~ antidifferentiate a definite integral?, eg

$$\int_0^1 e^{x^2} \quad \text{or} \quad \int_{-1}^1 \sqrt{x^3+3}\,dx$$

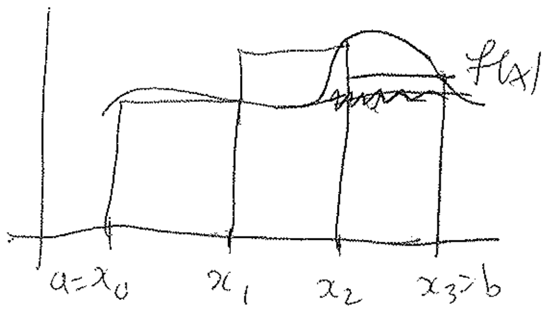
We should be able to at least approximate it (recall end point rules).

Recall $\Delta x = \frac{b-a}{n}$



Left endpoint rule

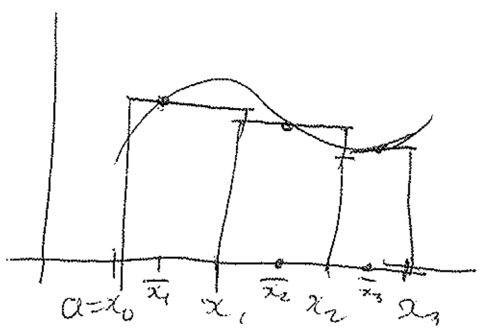
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x = L_n$$



Right endpoint rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\Delta x = R_n$$

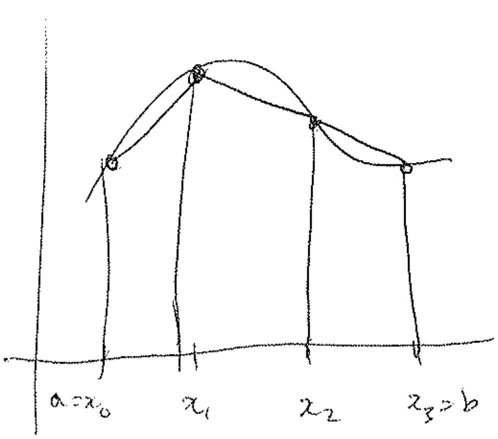
Let $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ be the midpoint of x_{i-1} and x_i



Midpoint rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = M_n$$

What if we combine the above rules?



Trapezoid Rule

$$\int_a^b f(x)\,dx = \frac{1}{2}L_n + \frac{1}{2}R_n$$

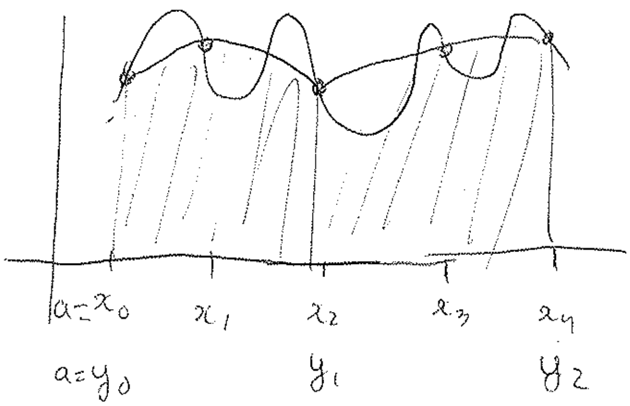
$$= \frac{1}{2}\left(f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x\right) + \frac{1}{2}\left(f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x\right)$$

$$= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

$$= T_n$$

Let's merge methods. Let

$\Delta x = \frac{b-a}{2n}$ $\quad$ $\Delta y = \frac{b-a}{n}$ $\quad$ so $\quad \Delta x = \frac{\Delta y}{2}$



Note $\bar{y}_i = x_{2i-1}$
$y_i = x_{2i}$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{6}L_n + \frac{1}{6}R_n + \frac{2}{3}M_n$$

$$= \frac{\Delta y}{6}\Big(\big(f(y_0)+f(y_1)+\dots+f(y_{n-1})\big) + f(y_1)+\dots+f(y_{n-1})+f(y_n) + 4\big(f(\bar{y}_1)+\dots+f(\bar{y}_{n-1})+f(\bar{y}_n)\big)\Big)$$

$$= \left(\frac{\Delta y}{2}\right)\left(\frac{1}{3}\right)\Big(\big(f(x_0)+f(x_2)+\dots+f(x_{2n-2})\big) + \big(f(x_2)+\dots+f(x_{2n-2})+f(x_{2n})\big) + 4\big(f(x_1)+f(x_3)+\dots+f(x_{2n-1})\big)\Big)$$

$$= \frac{\Delta x}{3}\Big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\dots+2f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})\Big)$$

$$\boxed{= S_{2n}}$$

It turns out that for the trapezoid, midpoint and Simpson's rule, we can get a bound on the error.

~~Error bounds~~ $\cancel{|f|\ f''(x)}$

Let $E_T = \int_a^b f(x)\,dx - T_n$
$E_M = \int_a^b f(x)\,dx - M_n$
$E_S = \int_a^b f(x)\,dx - S_n$ $\quad$ (n even)

Theorem (Error bounds)

Suppose $|f''(x)| \leq K$ for all $x \in [a,b]$ for some $K \in \mathbb{R}$. Then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$ smaller error

Theorem (Simpsons Rule Error Bound)

Suppose $|f^{(4)}(x)| \leq K$ for all $x \in [a,b]$ for some $K \in \mathbb{R}$, Then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$ smallest error

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See additional LaTeXed notes

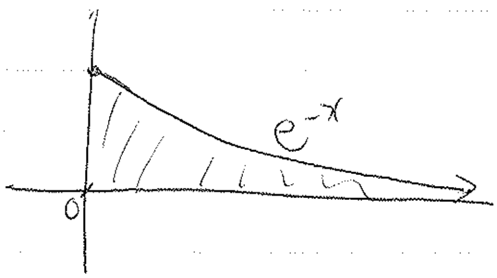
included in an extra pdf.

§7.8 Improper Integrals.

- So far, all functions have been continuous on nice regions.
- What happens when we include discontinuities or take integrals on infinite regions.

Ex: We want to make sense of

$$\int_0^\infty e^{-x}\,dx$$



ideally: $\displaystyle\int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty$

Rigorously use limits

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b$$

$$= \lim_{b\to\infty} -e^{-(b)} + e^{-(0)}$$

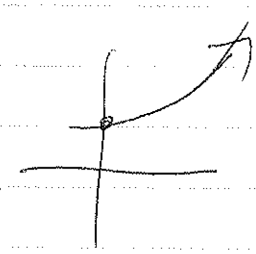
$$= 1 - \lim_{b\to\infty}\frac{1}{e^b}$$

$$= 1$$

Note: Infinite perimeter with finite area!!!

On the other hand,

$$\int_0^\infty e^{x}\,dx = \lim_{b\to\infty}\int_0^b e^{x}\,dx = \lim_{b\to\infty}(e^b - 1)$$



divergent. So integral doesn't exist (has infinite area)

Def'n: Improper integral of type I.

~~Def'n~~ If $\int_a^t f(x)dx$ exists for each $t \geq a$, then we define

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

If the limit exists and is finite.

- a similar def'n holds for $\int_{-\infty}^b f(x)dx$.
- If these limits exist, the integrals are said to be convergent. Otherwise they are divergent.
- If $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

for any $a \in \mathbb{R}$.

~~Ex~~ Q: For which $q \in \mathbb{R}$ is $\int_1^\infty x^q$ convergent?

Sol'n: If $q \neq -1$, then

$$\int_1^\infty x^q dx = \lim_{b\to\infty} \int_1^b x^q dx = \lim_{b\to\infty} \left(\frac{x^{q+1}}{q+1}\right)\Bigg|_1^b = \lim_{b\to\infty}\left(\frac{b^{q+1}}{q+1} - \frac{1}{q+1}\right)$$

the last limit exists if $q+1<0$ or $q<-1$. When $q=-1$, then

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \ln b - \ln(1) = \lim_{b\to\infty} \ln b$$

and the last limit does not exist.

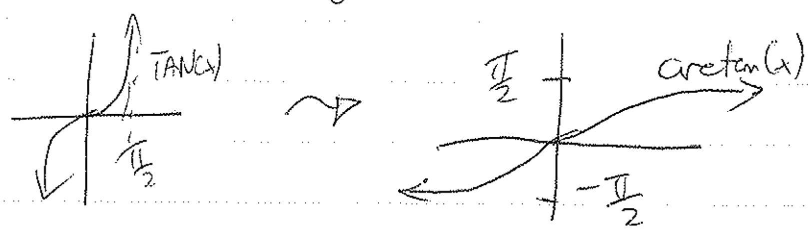
Hence the integral converges for all real $q<-1$ and diverges for all real $q \geq -1$.

Ex $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Break this up with 0.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b\to\infty} \arctan(x)\Big|_0^b = \lim_{b\to\infty} \arctan(b) - \arctan(0)$$
$$= \frac{\pi}{2}$$



Similarly, $$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \arctan(x)\Big|_a^0 = \arctan(0) - \lim_{a\to-\infty} \arctan(a)$$
$$= -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

Hence $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Recall: $\displaystyle\int_{-1}^{1} \frac{dx}{x^2} = \frac{-1}{x}\Big|_{-1}^{1} = -2$ <u>WRONG</u> Not cts at 0.

Def'n: Improper integral of Type 2.

- If f is continuous on $[a, b)$, and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$ if the limit exists and is finite.
- If f is continuous on $(a, b]$ and is discontinuous at a then
$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$ if the limit exists and is finite.

- If these limits exist, then the integral is convergent, else divergent.
- If f has a discontinuity at $c \in (a,b)$ and both

$$\int_a^c f(x)\,dx \quad \& \quad \int_c^b f(x)\,dx$$

are convergent, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Ex: $\displaystyle\int_2^6 \frac{dx}{\sqrt{x-2}}$ ← problem when $x=2$.

$$\int_2^6 \frac{dx}{\sqrt{x-2}} = \lim_{a\to 2^+}\int_a^6 \frac{dx}{\sqrt{x-2}} = \lim_{a\to 2^+} 2\sqrt{x-2}\Big|_a^6 = \lim_{a\to 2^-} 2\sqrt{6-2} - 2\sqrt{a-2}$$

$$= 4 - 0 = 4.$$

Ex: $$\int_{-1}^1 \frac{dx}{x^2} = \lim_{b\to 0^-}\int_{-1}^b \frac{dx}{x^2} + \lim_{a\to 0^+}\int_a^1 \frac{dx}{x^2}$$

$$= \lim_{b\to 0^-} -x^{-1}\Big|_{-1}^b + \lim_{a\to 0^+} -x^{-1}\Big|_a^1$$

$$= \underbrace{\lim_{b\to 0^-} -\frac{1}{b} - 1}_{\infty} + \underbrace{\lim_{a\to 0^+}\left(-1 + \frac{1}{a}\right)}_{\infty}$$

So the integral is divergent.

Lastly, Comparison Theorem

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent

Ex: Show $\int_1^\infty e^{-x^2}dx$ is convergent.

Ans: Note when $x \geq 1$, $x^2 \geq x$ so $-x^2 \leq x$ and $e^{-x^2} \leq e^{-x}$. Our first example showed $\int_1^\infty e^{-x}dx$ is convergent. Hence the comparison test tells us that $\int_1^\infty e^{-x^2}dx$ is convergent.

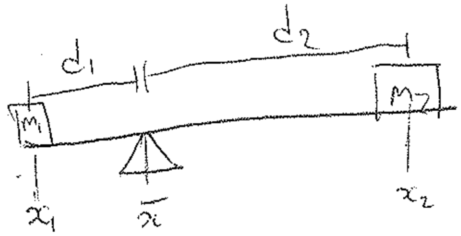
§8.3

Moments & Centre of Mass

TY for coming to my lecture!

(1)

To balance



need $m_1 d_1 = m_2 d_2$

$$m_1(\bar{x}-x_1) = m_2(x_2-\bar{x})$$

$$m_1\bar{x} - x_1 m_1 = m_2 x_2 - \bar{x} m_2$$

$$(m_1+m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1+m_2}.$$

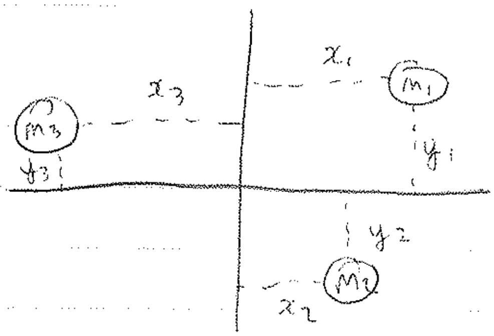
Each $m_i x_i$ is called a moment of mass m_i. In general

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \cancel{\frac{\sum m_i x_i}{n}} = \frac{M}{m}$$ = centre of mass ~~of~~ of.

Where $m = \sum_{i=1}^{n} m_i$ is the total mass and M is the moment of the system about the origin. (~~When the system is continuous with density $\rho(x)$, get~~

~~$\bar{x} = \frac{1}{m}\int_a^b x\,\rho(x)\,dx$. = centre of mass~~) omit?

In 2 dimensions, consider n particles of mass
$m_1, \dots m_n$
and locations
$(x_1, y_1), \dots (x_n, y_n)$



The moment of the system about the y-axis is
$$M_y = \sum_{i=1}^{n} m_i x_i$$
This is the tendency of the system to rotate around the y-axis.

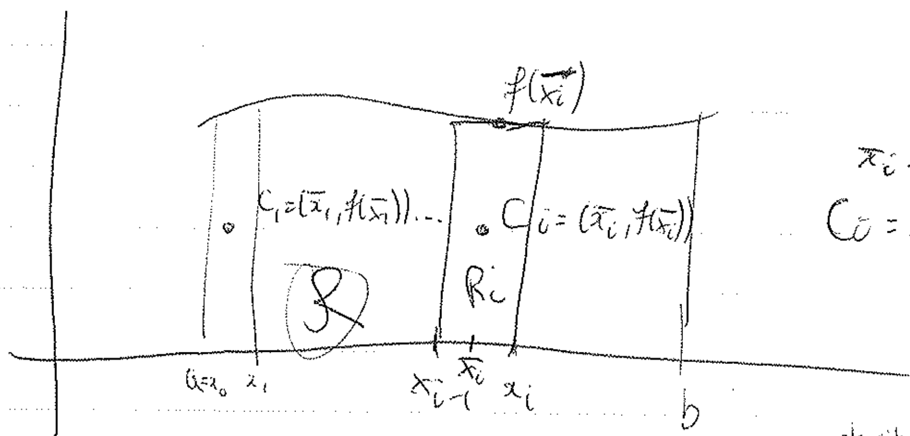
Similarly, the moment of the system about the x-axis is
$$M_x = \sum_{i=1}^{n} m_i y_i$$

Thus, $\underbrace{\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}}$

$(\bar{x}, \bar{y}) =$ Centre of mass

WANT centre of mass of a region.

Now, suppose we have constant density in a continuous system ~~with~~ between curves $y = f(x)$ and $y = 0$ ($= g(x)$ x-axis).



$\bar{x}_i$ = midpoint.
C_i = Centroid of rectangle = middle. (centre)

Vertical strip at position R_i has mass $\rho f(\bar{x}_i) \Delta x$. (density · area = mass)

Moment of R_i with y-axis is $\bar{x}_i \rho f(\bar{x}_i) \Delta x = $ (distance from y axis)(mass)

Moment of $\mathcal{R}$ about y-axis is

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i) \Delta x = \rho \int_a^b x f(x) dx$$

Moment at position R_i with respect to x axis is $\frac{f(\bar{x}_i)}{2} \rho f(\bar{x}_i) \Delta x$
= (distance from x-axis)(mass)

Moment of R about x-axis is

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\rho}{2} f(\bar{x}_i)^2 \Delta x = \frac{\rho}{2} \int_a^b f(\bar{x}_i)^2 dx.$$

Now, $m = \rho A = \rho \int_a^b f(x) dx$ A = area of curve.

Hence
$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x) dx}{\rho \int_a^b f(x) dx} = \frac{1}{A} \int_a^b x f(x) dx$$
$$\bar{y} = \frac{M_x}{m} = \frac{\frac{\rho}{2} \int_a^b f(x)^2 dx}{\rho \int_a^b f(x) dx} = \frac{1}{A} \int_a^b \frac{1}{2} f(x)^2 dx.$$
Coordinates of centre of mass.

$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2} f(x)^2$

1

Can do similar things when second curve is $y = g(x)$ (not necessarily x-axis).

Ex: Find the centre of mass of a parabolic plate $y = 1 - x^2$, $y = 0$, $-1 \le x \le 1$.

Sol'n

$$A = \int_{-1}^{1} (1 - x^2)\,dx = 2\int_0^1 (1 - x^2)\,dx = 2\left(x - \frac{x^3}{3}\right)\Big|_0^1 = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$$

Symmetric, Odd

$$\bar{x} = \frac{1}{A}\int_{-1}^{1} x f(x)\,dx = \frac{1}{A}\int_{-1}^{1} x(1-x^2)\,dx = \frac{1}{A}\int_{-1}^{1} x - x^3\,dx = 0$$

$$\bar{y} = \frac{1}{A}\int_{-1}^{1} \frac{1}{2}(1-x^2)^2\,dx = \frac{1}{2A}\int_{-1}^{1} (1 - 2x^2 + x^4)\,dx = \frac{1}{A}\int_0^1 (1 - 2x^2 + x^4)\,dx$$

$$= \frac{1}{A}\left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1 = \frac{1}{\left(\frac{4}{3}\right)}\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{3}{4}\left(\frac{8}{15}\right) = \frac{2}{5}$$

Hence, the centre of mass is at $(0, \frac{2}{5})$.

§9.3 Separable Equations

A very important topic that we will spend very little time on.

Ex: Solve for y when $\dfrac{dy}{dx} = \dfrac{-x}{y}$.

Pf: Treat as fractions

$$y\,dy = -x\,dx.$$

Integrate both sides

$$\int y\,dy = -\int x\,dx.$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C.$$

$$y^2 = x^2 + 2C$$

$$y = \pm\sqrt{-x^2+2C}.$$ ← a circle!

Def'n: A separable equation is one of the form

$$\frac{dy}{dx} = f(y)\,g(x)$$

ie the LHS is just a derivative and the RHS is a product of a function of y and a function of x.

The technique to solve these is to bring all terms depending on y to one side and similarly for x.

Ex: $y' = xy$.

Change to Leibniz notation $\frac{dy}{dx} = xy$.

Now, if $y \neq 0$,

$$\frac{dy}{y} = x\,dx.$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C.$$

$$|y| = e^{\frac{x^2}{2}+C}$$

$$y = \pm e^{C} e^{x^2/2}.$$

$$y = Ae^{x^2/2} \qquad (1)$$

Where A is any real number EXCEPT maybe 0 as $e^C > 0$.

Now, check $y = 0$. If $y=0$; then we have a sol'n. ~~$A=0$, If $A=0$ then $y=0$ and $y'=0=x(0)=xy$.~~ So (1) holds even for $A=0$.

Hence $y = Ae^{x^2/2}$ for any real A.

NOTE: There is a uniqueness theorem for differential equations saying if $\frac{dy}{dx} = 0$ for some $y=K$, a constant, then $y(x) = K$ for all x.

A standard mixing problem

Ex: A tank contains 20kg of salt dissolved in 5000L of water. Brine that contains 0.03 kg of salt per litre enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains out at the same rate. How much salt remains in the tank after half an hour?

Sol'n: Let $y(t)$ be the amount of salt in the tank in kg after time t in minutes. Given $y(0)=20$, we want to find $y(30)$.

Write a differential equation for y.

$$\frac{\text{Change in salt in tank}}{\text{Change in time}} = \frac{dy}{dt} = \text{rate in} - \text{rate out} \quad (\text{in kg/min}).$$

Rate in/out is concentration $\times$ flow.

$y = y(t)$.

$$\frac{dy}{dt} = 0.03\ \text{kg/L} \cdot 25\ \text{L/min} - y\ \text{kg/L} \cdot 25\ \text{L/min}.$$

$$= 0.75 - \frac{25y}{5000}$$

$$= \frac{3}{4} - \frac{y}{200}$$

$$= \frac{150 - y}{200}$$

$$\int \frac{dy}{150-y} = \int \frac{dt}{200}$$

$$-\ln|150-y| = \frac{t}{200} + C$$

when $t=0$, $y(0)=20$

$$-\ln|150-y(0)| = \frac{0}{200} + C$$

$$-\ln|150-20| = C$$

$$-\ln 130 = C.$$

Thus,
$$-\ln|150-y| = \frac{t}{200} - \ln 130$$

$$\ln|150-y| = \ln 130 - \frac{t}{200}$$

$$|150-y| = 130e^{-t/200}$$

$$150-y = \pm 130e^{-t/200}$$

$$y = 150 \mp 130e^{-t/200}$$

As $y(0)=20$, we want the negative above. Hence

$$y = 150 - 130e^{-t/200}$$

Thus
$$y(30) = 150 - 130e^{-30/200}$$
$$= 150 - 130e^{-3/20}.$$

Remark as $t \to \infty$, $y(t) \to 150$. This means if the tank is completely full of the incoming brine, then it would contain $5000 \times 0.03 = 150$ kg of salt.

§11.1 Sequences

Def'n: A sequence is a list of (real) numbers
$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

It can be thought of as a function from natural numbers to real numbers
$$f: \mathbb{N} \longrightarrow \mathbb{R} \quad a_n = f(n)$$

We write sequences as
$$\{a_1, a_2, a_3, \ldots\} \quad \text{or} \quad \{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}.$$

We call a_n the n^{th} term of the sequence

Ex: $\left\{\frac{1}{n^2}\right\}$ is $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

$\left\{\frac{1}{2n+1}\right\}$ is $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$

$1+\frac{1}{8}, 1+\frac{1}{16}, 1+\frac{1}{32}, \ldots$ is $\{1+2^{-n}\}_{n=3}^{\infty}$.

$\{2, 3, 5, 7, 11, 13, \ldots\}$ is the sequence of primes

$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ is the Fibonacci sequence. It is defined by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$.

Our main use for sequences is for limits.

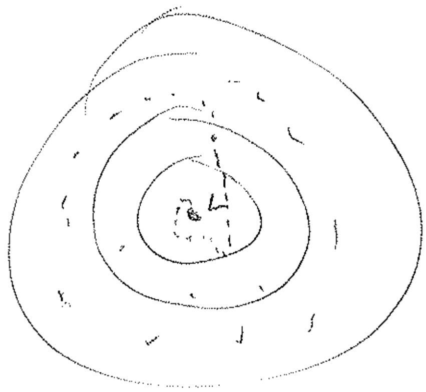
Ex: Let $a_n = 1 + \frac{1}{n}$.

As n gets large, $\frac{1}{n}$ gets small so a_n tends to 1. We write
$$\lim_{n\to\infty} a_n = 1 \quad \text{or} \quad a_n \xrightarrow{n\to\infty} 1.$$

Def'n: A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \xrightarrow{n\to\infty} L$$

If we can make the terms a_n as close to L as we like for all n sufficiently large. If this limit exists, then we say the sequence is convergent. Otherwise we say the sequence is divergent.



How do limits of sequences and limits of functions relate?

Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Ex: $\lim_{x\to\infty} e^{-x} = 0$ so $e^{-n} \xrightarrow{n\to\infty} 0$

$r>0$ $\lim_{x\to\infty} \frac{1}{x^r} = 0$ so $\frac{1}{n^r} \xrightarrow{n\to\infty} 0$.

Properties: Let $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ for two convergent sequences $\{a_n\}$ and $\{b_n\}$. Let $c \in \mathbb{R}$. Then

(1) $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

(2) $\lim_{n\to\infty} c\, a_n = cA$

(3) $\lim_{n\to\infty} a_n b_n = AB$

(4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ Provided $B \neq 0$.

(5) $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p>0$ and $a_n>0$ for all n.

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OR (1)

Ex: $\displaystyle\lim_{n\to\infty}\frac{3n}{2n+7} = \lim_{n\to\infty}\frac{3}{2+\frac{7}{n}} \overset{\text{by }(4)}{=} \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty}\left(2+\frac{7}{n}\right)} \overset{\text{by }(1)}{=} \frac{3}{\lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{7}{n}}$

$\displaystyle = \frac{3}{2+0} = \frac{3}{2}$

What about for $a_n = \frac{\sin(n)}{n}$? Squeeze Theorem.

Theorem: If $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences so that

$a_n \le b_n \le c_n$ for all n sufficiently large

and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ Then $\lim_{n\to\infty} b_n = L$.

Ex: $a_n = \frac{\sin(n)}{n}$. Notice

$$\frac{-1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$$

and * $\lim_{n\to\infty}\frac{1}{n} = \lim_{n\to\infty}\frac{-1}{n} = 0$ so $\lim_{n\to\infty}\frac{\sin(n)}{n} = 0$.

What about for $a_n = \frac{(-1)^n}{n}$?

Theorem: If $\lim_{n\to\infty}|a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Ex: $a_n = \frac{(-1)^n}{n}$ then $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n} = 0$ so $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$.

What about ~~$a_n = \sin(\frac{\pi}{n})$?~~ $a_n = \ln\left(1+\frac{1}{n}\right)$?

Theorem If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$

Ex: ~~$\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty}\frac{\pi}{n}\right) = \sin(0) = 0.$~~

$\displaystyle\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right) = \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right) = \ln(1) = 0.$

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Ex: $a_n = \frac{n!}{n^n}$

$$0 \le a_n = \frac{1\cdot 2\cdot 3 \cdots n}{n\cdot n\cdot n \cdots n} = \frac{1}{n}\left(\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}\right) \le \frac{1}{n}$$

So $\underbrace{0}_{\to 0} \le a_n \le \underbrace{\boxed{\tfrac{1}{n}}}_{\to 0}$ Hence, the squeeze theorem says $\lim_{n\to\infty} a_n = 0$

Theorem: $$\lim_{n\to\infty} r^n = \begin{cases} 1 & \text{if } r=1 \\ 0 & \text{if } -1<r<1 \\ \text{divergent} & \text{otherwise} \end{cases}$$

~~Pf: When r=1 this is clear~~
~~For r negative, use $|r^n|=|r|^n$ then a previous theorem to get to the positive case.~~
~~When r>1, then this is divergent (powers of numbers bigger than 1 grow.~~
~~When r<1, then powers of r decrease and tend to 0. ⊠.~~
~~At r=-1, notice~~

Pf: When $r=1$, then this is clear
When $r=-1$, then odd n give -1 and even n give 1 so there are two subsequences tending to different values hence divergent.
~~Similar when $r<-1$~~. When $r>1$, this is divergent (large powers blow up)
For $r<-1$, combine $r=-1$ and $r>1$. For $-1<r<1$, use the fact that
$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n$$
to reduce to positive case. For $0\le r<1$, then note powers of r get smaller and smaller ie since $\lim_{x\to\infty} a^x = 0$ when $0\le a<1$ then $\lim_{n\to\infty} r^n = 0$ when $0\le r<1$.
(draw exponential graph). ⊠.

Def'n: A sequence $\{a_n\}$ is
- Increasing if $a_n < a_{n+1}$ for all $n \geq 1$.
- decreasing ~~and~~ if $a_n > a_{n+1}$ for all $n \geq 1$
- Monotonic ~~if~~ if it is either increasing or decreasing.
- Bounded above if there is an M so that $a_n \leq M$ for all $n \geq 1$
- Bounded below if there is an m so that $m \leq a_n$ for all $n \geq 1$.
- Bounded if it is bounded above ~~or~~ and below.

Ex: $\left\{\frac{1}{n+2}\right\}_{n=1}^{\infty}$ is bounded above by $1/3$ bounded below by 0 and is decreasing since $\frac{1}{n+2} > \frac{1}{(n+1)+2}$ for all $n \geq 1$.

Does it converge? Yes! We can compute this or use

Theorem (Monotone Convergence Theorem)
Every bounded monotonic sequence is convergent.

Doesn't say what the limit is BUT tells us if we want to find it, we can.

1

OR(3)

§11.2 Series

Consider a sequence $\{a_n\}_{n=1}^{\infty}$. We wish to sum such sequences ie to compute

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

However what does this mean?

Ex: $1+2+3+4+5+\cdots$

Trick is to look at partial sums

$$S_1 = 1 = 1$$
$$S_2 = 1+2 = 3$$
$$S_3 = 1+2+3 = 6$$
$$\vdots$$
$$S_N = 1+2+\cdots+N = \frac{N(N+1)}{2}.$$

As $N \to \infty$, $S_N \to \infty$ so the partial sums diverge. So $1+2+3\cdots$ won't make sense.

~~What about~~ Ex: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$

Geometrically:

Perhaps this should be 1?

$$S_1 = \frac{1}{2} = \frac{1}{2}$$
$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$
$$\vdots$$
$$S_N = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^N} = \frac{2^N - 1}{2^N} = 1 - \frac{1}{2^N}.$$

As $N \to \infty$, $S_N \to 1$ so, we write $\sum_{n=1}^{\infty} 2^{-n} = 1$.

Sometimes it's not so clear if a sum diverges or converges

Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$.

Def'n: Let $\sum_{n=1}^{\infty} a_n$ be an infinite series and

$$S_N = a_1 + \dots + a_N = \sum_{n=1}^{N} a_n$$

denote the N^{th} partial sum. If $S_N \xrightarrow{N\to\infty} S$ for some $S \in \mathbb{R}$, then write

$$\sum_{n=1}^{\infty} a_n = S.$$

and say the series is convergent. If $\{S_n\}_{n=1}^{\infty}$ ~~diverges~~ is divergent, then say the series is divergent.

~~Theorem~~ Properties Let $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ be two convergent series. ~~Then~~ and $c \in \mathbb{R}$. Then

$$\sum_{n=1}^{\infty} c a_n = cA$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B.$$

NOTE: $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ are not easy to compute.

In general, series are very difficult to compute. However, sometimes we can compute them.

1 B

Theorem: Let $a, r \in \mathbb{R}$. The series

$$\sum_{n\geq 1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is called a geometric series. If $|r|<1$ or $a=0$, then it converges and

$$\sum_{n\geq 1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r|\geq 1$ and $a\neq 0$, then it diverges.

Pf: Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ ~~and~~

So $rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$ multiply both sides by r

$S_n - rS_n = a - ar^n$ subtract the above

$S_n = a\left(\dfrac{1-r^n}{1-r}\right)$ isolate. Provided $r\neq 1$

Now, if $a=0$, $S_n=0$ always so its limit is 0. ~~If $|r|<1$, then~~ Otherwise

$$\lim_{n\to\infty} S_n = a\left(\frac{1-\lim_{n\to\infty} r^n}{1-r}\right) = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r|>1. \end{cases}$$

When $r=1$, then $S_n = na \xrightarrow{n\to\infty} \infty$

$r=-1$ then $S_n = \begin{cases} a & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ ← has a subsequence with limit a and a subsequence with limit 0 so it diverges. ☐

④ ②

Ex: $9 - \frac{27}{5} + \frac{81}{25} - \frac{243}{125} + \cdots$

This is a geometric series (though it might not be obvious!). Note

$$a = 9$$
$$r = \frac{-27/5}{9} = \frac{-3}{5} \quad \text{(ratio of first two terms)}$$

Note $\frac{-27}{5} \cdot \left(\frac{-3}{5}\right) = \frac{81}{25}$, $\frac{81}{25} \cdot \left(\frac{-3}{5}\right) = \frac{-243}{125}$ etc.

Since $|r| < 1$, the sum converges and its sum is

$$\sum_{n=1}^{\infty} (9)\left(\frac{-3}{5}\right)^{n-1} = \frac{a}{1-r} = \frac{9}{1-\left(-\frac{3}{5}\right)} = \frac{45}{8}.$$

Ex: Telescoping sums. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$S_1 = \frac{1}{2}$$
$$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$
$$S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$$
$$\vdots$$
$$S_n = \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n+1)} \stackrel{?}{=} \frac{n}{n+1} \quad \leftarrow \text{why?}$$

Idea: Use partial fractions!

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} = \frac{(n+1) - (n)}{n(n+1)}$$

so $S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

③ 72

Hence $S_n = 1 - \frac{1}{n+1} \xrightarrow{n\to\infty} 1$ so the sum is convergent.

What can we say about the sequence of terms in a convergent series?

Theorem: If $\sum_{n\geq 1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Equivalently, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n\geq 1}^{\infty} a_n$ is divergent.

The second statement is known as the divergence test.

Ex: Note $\sum_{n=1}^{\infty} \frac{n+2}{2n+3}$ diverges since $\lim_{n\to\infty} \frac{n+2}{2n+3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{2+\frac{3}{n}} = \frac{1}{2} \neq 0$.

Intuitively, for sums to be convergent, the terms must be small.

NOTE The converse is false!

Ex: $\sum_{n\geq 1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ Note $\lim_{n\to\infty} \frac{1}{n} = 0$ BUT this sum diverges!

to see this, note

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2} = 1 + \frac{1}{2}$$
$$S_4 = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} > 1 + \frac{2}{2}$$
$$S_8 = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}} > 1 + \frac{3}{2}$$
$$\vdots$$
$$S_{2^n} > 1 + \frac{n}{2}.$$

Hence, the partial sums get bigger and bigger, but VERY slowly (this is logarithmically slow). Hence, the sum diverges.

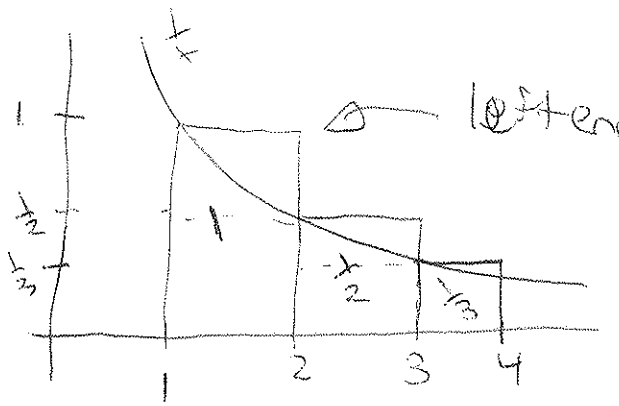
§11.3

(1)

Integral Test:

Remember $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges EVEN though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Question: How fast does it diverge?



So maybe $\sum_{n=1}^{\infty} \frac{1}{n} \approx \int_1^{\infty} \frac{1}{x} = \lim_{b\to\infty} \ln|b|$.

This turns out to be pretty close to the truth since

~~Theorem~~ $\gamma := \lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{1}{n} - \ln N\right) = 0.577216$. (Euler-Mascheroni Constant)

Open problem: Is γ rational?

Theorem: Suppose f is a continuous, positive, decreasing function on $[1,\infty)$. Let $a_n = f(n)$. Then

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex: $\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \ln|b|$ is divergent hence $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Ex: $\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \frac{-1}{b} + 1 = 1$ so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Note: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ so it differs from the answer in the integral test (and does not help in finding the answer).

Ex: For what p is $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Note ~~it~~ when $p < 0$, then

$$\lim_{n\to\infty} \frac{1}{n^p} = \infty$$

If $p=0$ $\lim_{n\to\infty} \frac{1}{n^p} = 1$ so in these two cases, the divergence test tells us that the sum diverges.

When $p > 0$, notice that $\frac{1}{x^p}$ is continuous, positive and decreasing on $[1,\infty)$. So use the integral test. Look at

we did this already!!!

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

Hence we have the p-series test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^{9973}}$ converges by the p-series test.

You try

$$\sum_{n=2}^{\infty} \frac{1}{n^2+1} \qquad \sum_{n=1}^{\infty} \frac{7n^6}{n^7+15} \qquad \sum_{n=1}^{\infty} \frac{\ln(n)}{n}.$$

Each corresponding function is continuous, positive, and decreasing so try the integral test.

$$\int_2^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \arctan(x)\Big|_1^b = \lim_{b\to\infty} \arctan(b) - \arctan(1)$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

So converges.

let $u = x^7+15$ $du = 7x^6$ $u(1) = 16$ $u(b) = b^7+15$

$$\int_1^{\infty} \frac{7x^6}{x^7+15}\,dx = \lim_{b\to\infty} \int_1^b \frac{7x^6}{x^7+15}\,dx = \lim_{b\to\infty} \int_{16}^{b^7+15} \frac{du}{u} = \lim_{b\to\infty} \ln|u|\Big|_{16}^{b^7+15}$$
$$= \lim_{b\to\infty} \ln|b^7+15| - \ln|16| \quad \text{diverges.}$$

So $\sum_{n=1}^{\infty} \frac{7n^6}{n^7+15}$ diverges.

$u = \ln(x)$ $du = \frac{dx}{x}$ $u(1) = 0$ $u(b) = \ln|b|$

$$\int_1^{\infty} \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \int_0^{\ln|b|} u\,du = \lim_{b\to\infty} \frac{u^2}{2}\Big|_0^{\ln|b|}$$
$$= \lim_{b\to\infty} \ln|b|^2 \quad \text{diverges.}$$

Take a look at this last integral, It looks like $\frac{1}{n}$. In fact,

$$\frac{1}{n} < \frac{\ln(n)}{n} \quad \text{for } n \geq 3.$$

So we should expect $\sum_{n=1}^{\infty} \frac{1}{n} < \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$.

and since the Harmonic series diverges, the sum we are looking at should diverge. This turns out to be true

Theorem The comparison test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with all positive terms,

(i) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all $n \geq 1$ then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent and $a_n \geq b_n$ for all $n \geq 1$ then $\sum_{n=1}^{\infty} b_n$ is also divergent.

Note: The $a_n \leq b_n$ or $a_n \geq b_n$ conditions only need to be true when n is large

Ex: Since $\frac{1}{n} < \frac{\ln(n)}{n}$ for $n \geq 3$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test, we have that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges.

Ex: $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

Intuition top is n bottom is n^3 so should behave like $\frac{1}{n^2}$ so should converge.

Actual proof: notice $2n^3 \leq 2n^3+1$ for all $n \geq 1$.

so $\frac{1}{2n^3+1} \leq \frac{1}{2n^3}$

and hence

$$\sum_{n=1}^{\infty} \frac{n}{2n^3+1} \le \sum_{n=1}^{\infty} \frac{n}{2n^3} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

by the p test.

converges by the comparison test.

Ex: $\sum_{n=1}^{\infty} \frac{n+5}{n^{4/3}}$

Intuition: like $\frac{n}{n^{4/3}} = \frac{1}{n^{1/3}}$ so diverges

Actual proof: Note $n < n+5$ so

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{n}{n^{4/3}} < \sum_{n=1}^{\infty} \frac{n+5}{n^{4/3}}$$

diverges by p test — diverges by comparison test.

Pf of comparison test

- Suppose $\sum_{n=1}^{\infty} b_n$ converges let
Let $s_N = \sum_{n=1}^{N} a_n$, $t_N = \sum_{n=1}^{N} b_n$, $t = \sum_{n=1}^{\infty} b_n$

Then notice s_N & t_N are increasing (all terms are positive).
~~As $t_n \xrightarrow{n\to\infty} t$ so $t_n \le t$.~~
Note $\{s_N\}$ & $\{t_N\}$ are increasing as $s_N \le s_N + a_{N+1} = s_{N+1}$.
Since $t_N \xrightarrow{N\to\infty} t$ and $a_i \le b_i$ for all i so $s_N \le t_N \le t$. So $\{s_N\}$ is bounded and increasing. Hence by the monotone convergence theorem, $\sum_{n=1}^{\infty} a_n$ converges.

- If $\sum_{n=1}^{\infty} b_n$ diverges then $t_n \xrightarrow{n\to\infty} \infty$. But $b_i \le a_i$ so $t_n \le s_n$
hence $s_n \xrightarrow{n\to\infty} \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges.

One more test.

The limit Comparison test
Suppose that $\sum_{n\geq 1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If

$$\lim \frac{a_n}{b_n} = c$$

where $0 < c < \infty$ (c is finite). Then either both series converge or diverge.

Pf: Let m, M be integers s.t.

$$0 < m < c < M.$$

as a_n/b_n is close to c for large n, there is an integer N so that

$$m < \frac{a_n}{b_n} < M \quad \text{when } n > N.$$

$$\Rightarrow \quad mb_n < a_n < Mb_n \quad \text{when } n > N.$$

If $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum_{n=1}^{\infty} Mb_n$ Converges so comparison test shows $\sum_{n=1}^{\infty} a_n < \infty$

If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} mb_n$ diverges so comparison test shows $\sum_{n=1}^{\infty} mb_n$ diverges.

Ex: Determine if $\sum_{n=1}^{\infty} \frac{3n^3+n^2}{\sqrt{12n+n^7}}$ converges or diverges.

Numerator $\sim 3n^3$ denominator $\sim n^{7/2}$.

Let $a_n = \frac{3n^3+n^2}{\sqrt{12n+n^7}}$ and $b_n = \frac{3n^3}{n^{7/2}} = \frac{3}{n^{1/2}}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{3n^3+n^2}{\sqrt{12n+n^7}} \cdot \frac{n^{1/2}}{3} = \lim_{n\to\infty} \frac{n^{7/2}\left(1+\frac{n^2}{3n^{7/2}}\right)}{n^{7/2}\sqrt{\frac{12}{n^6}+1}} = \frac{1+0}{\sqrt{0+1}} = 1$

So limit comparison test says that

$\sum_{n=1}^{\infty} a_n$ diverges since $\sum_{n=1}^{\infty} b_n$ diverges by the p-test.

Examples to try

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ $\sum_{n=1}^{\infty} \frac{1}{n!}$ $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+3}}{n^3+n^{2.5}}$

as $\sqrt{n^2+1} \le \sqrt{n^2+n^2} = \sqrt{2}\,n$

$n! \ge 2^{n-1}$

$\le \frac{1}{n}\,\frac{2}{n}\cdot 1 = \frac{2}{n^2}$

limit comparison with $\frac{1}{n}$

§11.4

Examples

(1) $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

Sol'n Note $\sqrt{n^2+1} \le \sqrt{n^2+n^2} \le \sqrt{2}\, n$ for $n \ge 1$. Hence

$$\frac{1}{\sqrt{2}\, n} \le \frac{1}{\sqrt{n^2+1}}$$

as $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2}\, n}$ diverges by the p-test, we have by the comparison test, ~~we have~~ that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ must also diverge.

(2) $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Sol'n: Note for $n \ge 3$

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n}{n} \le \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdots 1 = \frac{2}{n^2}.$$

Since $\frac{n!}{n^n} \le \frac{2}{n^2}$ and $\sum_{n=1}^{\infty} \frac{2}{n^2}$ Converges by the p-test; we have that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the Comparison test.

(3) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Note $n! = n\cdot(n-1)\cdot(n-2)\cdots(3)(2)(1) \geq \overbrace{2\cdot 2\cdots 2}^{n-1 \text{ times}}\cdot 1 = 2^{n-1}$

Hence $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$

Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a geometric series with $r<1$, it is convergent. Hence, by the Comparison test, we have that

$$\sum_{n=1}^{\infty} \frac{1}{n!} \text{ converges.}$$

(4) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+3}}{n^3+n^{2.5}}$

numerator $\sim \sqrt{n^4} = n^2$ denominator $= n^3$

Let $b_n = \frac{n^2}{n^3} = \frac{1}{n}$. Note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n\sqrt{n^4+3}}{n^3+n^{2.5}} = \lim_{n\to\infty} \frac{n^3\sqrt{1+\frac{3}{n^4}}}{n^3\left(1+\frac{1}{n^{0.5}}\right)} = 1$$

Since $0<1<\infty$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, the limit comparison test says that $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+3}}{n^3+n^{2.5}}$.

Examples: Find $\lim_{n\to\infty} a_n$ if

(1) $a_n = n\sin\left(\frac{1}{n}\right)$

Ans: let $m = \frac{1}{n}$ so $n = \frac{1}{m}$ and as $n \to \infty$, $m \to 0$

Hence

$$\lim_{n\to\infty} n\sin\left(\tfrac{1}{n}\right) = \lim_{m\to 0} \frac{\sin(m) - 0}{m - 0} = \text{derivative of } \sin \text{ at } 0$$
$$= \cos(0) = 1.$$

(2) $a_n = \left(1 + \frac{1}{n}\right)^n$

Sol'n exponents are hard. Take $e^{\log}$.

$$a_n = e^{\log\left(1+\frac{1}{n}\right)^n} = e^{n\log\left(1+\frac{1}{n}\right)}$$

want $\lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right)$ Let $m = \frac{1}{n}$ so $n = \frac{1}{m}$ and $m \to 0$ when $n \to \infty$

$$\text{So} = \lim_{m\to 0} \frac{\log(1+m) - \log(1)}{m - 0} = \text{derivative of } \log(1+x) \text{ at } 0.$$
$$= \left.\frac{1}{1+x}\right|_{x=0} = 1.$$

$$\underline{\text{So}} \lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\left(n\ln\left(1+\frac{1}{n}\right)\right)} = e^{\lim_{n\to\infty}\left(n\ln\left(1+\frac{1}{n}\right)\right)} = e^1 = e.$$

Note: sometimes e is defined as this limit!

§11.5 Alternating Series ~~Test~~

Def'n: An alternating series is a series ~~and~~ whose terms alternate in sign from positive to negative. They are of the form

$$a_n = (-1)^n b_n \quad \text{OR} \quad a_n = (-1)^{n+1} b_n$$

where b_n are positive.

Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

When do alternating series converge?

Alternating Series test

If $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - \cdots$ ~~satisfies~~ with b_n positive satisfies

(1) b_n are ~~decreasing~~ non-increasing ie $b_n \geq b_{n+1}$ for all n

(2) $\lim_{n \to \infty} b_n = 0$.

Then, the series is convergent.

Pf: Just MCT on the even ~~[illegible]~~ numbered terms. See p. 72 of text.

Examples

1) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

Note $\frac{1}{n+1} \leq \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.

Hence the alternating series test says that ~~the~~ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges.

(2) $\sum_{n=1}^{\infty} \frac{(-1)^n 4n^2}{3n^2 - n}$

The series is alternating BUT

$$\lim_{n\to\infty} \frac{4n^2}{3n^2-n} = \lim_{n\to\infty} \frac{4}{3-\frac{1}{n}} = \frac{4}{3} \neq 0.$$

This suggests that the sum diverges. Try the divergence test.

$$\lim_{n\to\infty} \frac{(-1)^n 4n^2}{3n^2-n} = \lim_{n\to\infty} \frac{(-1)^n 4}{3-\frac{1}{n}}$$

? has a subsequence going to $\frac{4}{3}$ (even terms) and one going to $\frac{-4}{3}$ (odd terms).

Hence, this diverges.

(3) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+10^{12}}$

Clearly, this is an alternating series. Moreover,

$$\lim_{n\to\infty} \frac{n^3}{n^4+10^{12}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{10^{12}}{n}} = 0.$$

So is this decreasing eventually?

Let $f(x) = \frac{x^3}{x^4+10^{12}}$. So $f'(x) = \frac{3x^2(x^4+10^{12}) - 4x^6}{(x^4+10^{12})^2}$

When is $f'(x) < 0$?

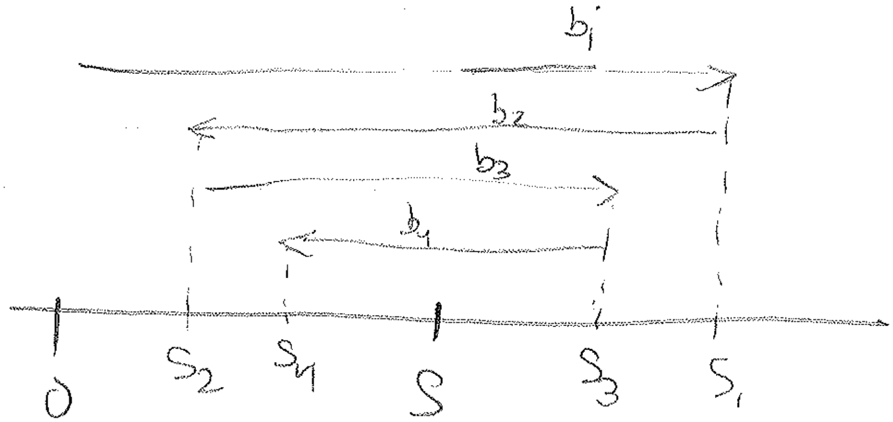
$$0 > 3x^2(x^4+10^{12}) - 4x^6 = -x^6 + 3\cdot 10^{12} x^2$$
$$= x^2(3\cdot 10^{12} - x^4)$$

So $0 > x^2$ NEVER HAPPENS

or $0 > 3\cdot 10^{12} - x^4 \Rightarrow x > \sqrt[4]{3}\cdot 10^{4}$

So eventually, $\frac{n^3}{n^4+10^{12}} > \frac{(n+1)^3}{(n+1)^4+10^{12}}$ ie is eventually decreasing.

Lastly, we estimate sums.



Notice that $|s - s_n| \leq b_{n+1}$

Theorem: Estimating alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ $\quad b_n > 0$

If $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$

Then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Ex: Compute $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ correct to 3 decimal places.

Sol'n: to be correct to 3 decimal places, use the alt series ~~Abst~~ estimate

Note $\frac{1}{n+1} < \frac{1}{n}$ & $\lim_{n\to\infty} \frac{1}{n} = 0$.

So $|s - s_n| \leq b_{n+1}$

Suffices to ask that $b_{n+1} < 0.0001$ 4 decimals since this won't affect the first 3 decimal places.

Actual answer $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$

So $\frac{1}{n+1} < \frac{1}{10^4} \Rightarrow 10^4 - 1 < n$. So Compute $\sum_{n=1}^{10{,}000} \frac{1}{n} \doteq 0.693$

Rearrangements

Look at

$$(1) \quad C := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \approx 0.693 > 0$$

Multiply by $\frac{1}{2}$

$$\frac{1}{2}C = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} \cdots$$

Add in 0's.

$$(2) \quad \frac{1}{2}C = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10}$$

Add (1) and (2)

$$(3) \quad \frac{3}{2}C = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} \cdots$$

Note that (1) and (3) are adding the same ~~num~~ terms!

So is (1) = (3) ? So $C = \frac{3}{2}C \rightsquigarrow \frac{1}{2}C = 0 \rightsquigarrow C = 0$

But we said $C > 0$??? What's going on!!!

Def'n: A series $\sum_{n\geq 1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n\geq 1}^{\infty} |a_n|$ converges.

Def'n: A series $\sum_{n\geq 1}^{\infty} a_n$ is called conditionally convergent if $\sum_{n\geq 1}^{\infty} a_n$ is convergent but not absolutely convergent.

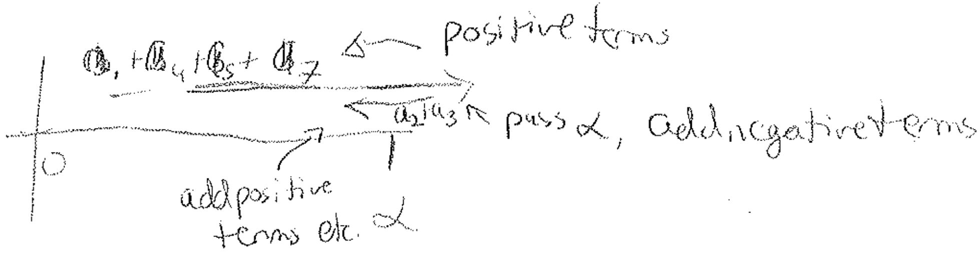
Ex: $\sum_{n\geq 1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent since it is convergent by the alternating series test but it is not absolutely convergent since $\sum_{n\geq 1}^{\infty} \frac{1}{n}$ diverges by the p-test.

If your series is only conditionally convergent, then the order you add terms matters! In fact, it is ~~real~~ much worse then you might think

Theorem (Riemann): Let $\alpha \in \mathbb{R}$. Then if $\sum_{n\geq 1}^{\infty} a_n$ is conditionally convergent, then there is a rearrangement of the a_n terms (say b_n) so that

$$\sum_{n\geq 1}^{\infty} b_n = \alpha.$$

Proof Sketch



Theorem: If $\sum_{n\geq 1}^{\infty} a_n$ is absolutely convergent, then the sum is invariant under rearrangements so

$$\sum_{n\geq 1}^{\infty} a_n = \sum_{n\geq 1}^{\infty} b_n$$

where the b_n are a permutation of the a_n terms.

This theorem explains the weird behaviour from before. Since $\sum_{n\geq1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n\geq1}^{\infty}\frac{1}{n}$ diverges, the series

$\sum_{n\geq1}^{\infty}\frac{(-1)^{n+1}}{n}$ is only conditionally convergent.

Hence

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots \neq 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\cdots$$

So conditionally convergent does not imply absolute convergence. Does ~~Convergent~~ absolute convergence imply convergence? Yes!

Theorem: If $\sum_{n\geq1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Pf: Notice that

$$\sum_{n\geq1}^{\infty} a_n = \underbrace{\sum_{n\geq1}^{\infty}(a_n+|a_n|)}_{\text{all positive terms}} - \sum_{n\geq1}^{\infty}|a_n|$$

$$\leq \sum_{n\geq1}^{\infty} 2|a_n| - \sum_{n\geq1}^{\infty}|a_n|$$

both converge since series is absolutely convergent.

~~Hence $\sum_{n\geq1}^{\infty} a_n$ is [illegible]~~

Hence $\sum_{n\geq1}^{\infty}(a_n+|a_n|)$ ~~and~~ $\sum_{n\geq1}^{\infty}|a_n|$ is convergent

by the comparison test. Since this is $\sum_{n\geq1}^{\infty} a_n$, we are done. ☐

Done have tests to see when series are absolutely convergent? Yes! The ratio test & the root test (not done this term).

Theorem (the ratio test)

Let $\sum_{n=1}^{\infty} a_n$ be a series.

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and hence is convergent.

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, OR $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the ratio test is inconclusive.

Ex: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$. Let $a_n = \frac{n!}{n^n}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)n!}{n!} \frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} e^{\ln\left(\frac{n}{n+1}\right)^n}$$

$$= \lim_{n\to\infty} e^{n \ln\left(\frac{n}{n+1}\right)} \qquad (*)$$

To evaluate

$$\lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} -n \ln\left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{-\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

Let $x = \frac{1}{n}$ So as $n \to \infty$, $x \to 0$. Hence

$$-\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = -\lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x} \qquad f'(a) = \lim_{h\to 0} \frac{f(h+a) - f(a)}{h}$$

$$= -\left(\ln(x)\right)'\Big|_{x=1}$$

$$= -\frac{1}{x}\Big|_{x=1}$$

$$= -1.$$

Hence from (*)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+1}\right)} = e^{\lim_{n\to\infty} -\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}} = e^{-1} = \frac{1}{e} < 1$$

hence the ratio test says that

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ converges. (absolutely)}$$

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{7^n}$. Does this converge absolutely?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{7^{n+1}} \cdot \frac{7^n}{n}$$

$$= \lim_{n\to\infty} \frac{1}{7} \cdot \frac{n+1}{n}$$

$$= \frac{1}{7} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= \frac{1}{7} < 1$$ Hence the sum converges absolutely.

Ex: $\sum_{n=1}^{\infty} (-1)^{n+1} n!$ Does this converge?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} \frac{(n+1)\,n!}{n!} = \lim_{n\to\infty}(n+1) = \infty$$

Hence this diverges.

§11.8 Power Series.

Def'n: A power series is of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

where $c_i \in \mathbb{R}$. are called the coefficients of the series.

Def'n: A power series centred at $a \in \mathbb{R}$ is

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

with $c_i \in \mathbb{R}$.

NB: Convention $(x-a)^0 = 1$ EVEN if $x=a$.

Ex: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

This is a geometric series. It converges whenever $|x|<1$ and diverges when $|x| \geq 1$.

Ex: For what values of x will $\sum_{n=0}^{\infty} n! x^n$ converge?

Ans: Use ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \begin{cases} \infty & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The ratio test says our series diverges if $x \neq 0$ and converges if $x=0$.
since $0<1$

NB: Power series centred at a always converge at $x=a$.

Ex: For what values does $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$ converge?

By the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{n+1} \div \frac{(x-2)^n}{n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(x-2)^n}\,\frac{n}{n+1}\right|$$
$$= \lim_{n\to\infty} |x-2|\left(\frac{1}{1+\frac{1}{n}}\right)$$
$$= |x-2|$$

The ratio test says our series converges provided $|x-2|<1$ so this converges when $-1<x-2<1$ ie when $1<x<3$. The ratio test says this diverges when $|x-2|>1$.

The ratio test says nothing when $|x-2|=1$ ie when $x=1,3$ so we check these points separately. When $x=3$, our series becomes

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{n} = \sum_{n=0}^{\infty}\frac{(3-2)^n}{n} = \sum_{n=0}^{\infty}\frac{1}{n}$$ divergent by p-series test

When $x=1$, we have

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}$$ Converges (we showed this using the alternating series test).

So, the interval of convergence of $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n}$ is $1\le x<3$.

Ex: For what values does $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converge?

Sol'n: By the ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!} \right|$$
$$= \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 < 1 \quad \text{for all } x \in \mathbb{R}.$$

Hence, the interval of convergence is $(-\infty, \infty) = \mathbb{R}$.

It turns out that this covers all cases.

Theorem: For $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only 3 possibilities

(i) The series converges ONLY when $x=a$

(ii) The series converges for all $x \in \mathbb{R}$

(iii) There is a positive real number R such that if $|x-a| < R$, then the series converges and if $|x-a| > R$, then the series diverges. When $|x-a| = R$, the series MIGHT converge and need to be checked separately.

In case (iii) R is the radius of convergence. In (i), $R=0$ and in (ii) $R=\infty$. The interval of converges is the interval I where $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges provided $x \in I$. It is always one of

$[a,a] = \{a\}$, $(-\infty,\infty) = \mathbb{R}$, $(a-R, a+R)$

$[a-R, a+R)$ $(a-R, a+R]$ OR $[a-R, a+R]$.

Ex: Find the radius of convergence and the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{n^2(x+4)^n}{5^n}$$

Sol: Use ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2(x+4)^{n+1}}{5^{n+1}} \div \frac{n^2(x+4)^n}{5^n}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^2 \frac{5^n}{5^{n+1}}\frac{(x+4)^{n+1}}{(x+4)^n}\right|$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(\frac{1}{5}\right)|x+4|$$

$$= \frac{|x+4|}{5}$$

The ratio test says this converges when $\frac{|x+4|}{5} < 1$ and diverges when $\frac{|x+4|}{5} > 1$. So we get convergence when $|x+4| < 5$ or $-5 < x+4 < 5$ so $-9 < x < 1$. The radius of convergence is 5. Lastly, we check when $x=-9$ or $x=1$ for our series.

When $x=-9$,

$$\sum_{n=0}^{\infty} \frac{n^2(x+4)^n}{5^n} = \sum_{n=0}^{\infty}\frac{n^2(-5)^n}{5^n} = \sum_{n=0}^{\infty}(-1)^n n^2$$

and this diverges by the divergence test since $\lim_{n\to\infty}(-1)^n n^2$ does not exist.

When $x=1$,

$$\sum_{n=0}^{\infty}\frac{n^2(1+4)^n}{5^n} = \sum_{n=0}^{\infty} n^2$$ and this diverges by the divergence test since $\lim_{n\to\infty} n^2 = \infty$.

Ex: Find the radius of convergence and interval of convergence of

$$\sum_{n=0}^{50} \frac{3^n \cdot n!\,(x-7)^n}{n^n}$$

Sol'n: This is a finite sum!!! It converges everywhere so $R = \infty$ and $I = \mathbb{R}$.

Ex: Find the radius of convergence and the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{n\sqrt{n}}$$

Sol'n: By the ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1} x^{n+1}}{(n+1)\sqrt{n+1}} \div \frac{(-2)^n x^n}{n\sqrt{n}} \right|$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{n\sqrt{n}}{(n+1)\sqrt{n+1}} \frac{|x|^{n+1}}{|x|^n}$$

$$= \lim_{n\to\infty} 2 \frac{n\sqrt{n}}{n(1+\frac{1}{n})\sqrt{n}\sqrt{1+\frac{1}{n}}} |x|$$

$$= 2|x|$$

The ratio test says this diverges when $2|x| > 1$ and converges when $2|x| < 1$ so when $-\frac{1}{2} < x < \frac{1}{2}$. Now, check the end points.

At $x = -\frac{1}{2}$,

$$\sum_{n=1}^{\infty} \frac{(-2)^n(-\frac{1}{2})^n}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} < \infty \text{ by p-series test}$$

At $x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{(-2)^n(\frac{1}{2})^n}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}} < \infty \text{ since it is absolutely convergent by the p-series test.}$$

Hence, the interval of convergence is $-\frac{1}{2} \le x \le \frac{1}{2}$.

§11.9 Representations of Functions as Power Series

Recall

$$\sum_{n=0}^{\infty} x^n$$

This is a geometric series which we can sum when $|x|<1$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \frac{a}{1-r}.$$

Similarly, we can write

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n$$

when $|-2x|<1$ so when $|x|<\frac{1}{2}$.

A bit more work shows that

$$\frac{2x^5}{3-x} = x^5\left(\frac{2/3}{1-x/3}\right) = \frac{2}{3}x^5\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2x^{n+5}}{3^{n+1}}.$$

Provided $\left|\frac{x}{3}\right|<1$ so $|x|<3$.

What is perhaps more interesting is that on the regions of absolute convergence, we can also take integrals ~~fdiff~~ derivatives of power series termwise.

Theorem: If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R>0$ then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (hence continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

Both power series still have radius of convergence R.

Literally,

$$\left.\begin{aligned} \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n &= \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n \\ \text{and} \quad \int \sum_{n=0}^{\infty} c_n(x-a)^n &= C + \sum_{n=0}^{\infty} c_n \int (x-a)^n \end{aligned}\right\} \text{ in the interval } (a-R, a+R).$$

Examples: If $|x|<1$, then

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1-x)^{-1} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} \frac{d}{dx}x^n.$$

$$\int \frac{1}{1-x} = -\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = C + \sum_{n=1}^{\infty} \frac{x^n}{n} = C + \sum_{n=0}^{\infty} \int x^n$$

No absolute value since $|x|<1$ for this to hold so $1-x>0$ always.

NB: Since $\ln(1)=0$, we know $C=0$ so $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

Ex: Find the power series representation of $\ln(1+x)$ and its radius of convergence.

Pf: Since $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, we have $-\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

and so $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$. Since the radius of convergence of $-\ln(1-x)$ is $R=1$, the radius of convergence of $\ln(1+x)$ is also $R=1$.

Ex: Find a power series representation for arctan(x).

Sol'n: Clever: Recall that

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}.$$

So we work backwards. We know when $|x|<1$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

So

$$\arctan(x) = \int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx$$
$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Since $\arctan(0) = 0$, AND $|0|<1$ we have $C=0$ by plugging in $x=0$.

So when $|x|<1$, we have

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

and the radius of convergence is still $R=1$.

Ex: Evaluate $\int \frac{dx}{1+x^3}$ as a power series. Use it to approximate $\int_0^{0.5} \frac{dx}{1+x^3}$ correct to within 10^{-10}.

Sol'n: Note that

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}.$$

Integrating both sides yields

$$\int \frac{dx}{1+x^3} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1} \quad \text{Valid when } |x|<1$$

To find the ~~approximate~~ value of n within the bounds, we use FTC and to make our lives easier, set $C=0$. Then

$$\int_0^{0.5} \frac{dx}{1+x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n (0.5)^{3n+1}}{3n+1}$$

$$= \frac{1}{2} - \frac{1}{16\cdot 4} + \frac{1}{128\cdot 7} \mp \cdots \frac{\pm(-1)^n}{2^{3n+1}\cdot(3n+1)}$$

By the AST estimation theorem, we want to know when

$$\frac{1}{2^{3n+1}(3n+1)} < 10^{-10} \Rightarrow 10^{10} < 2^{3n+1}(3n+1)$$

A quick and easy bound is to find an n such that

$$10^{10} < 2^{3n+1}$$

So $\frac{(10\log_2 10)-1}{3} < n \Rightarrow 10.73976 < n$

So $n=11$ will do.

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Hence

$$\int_0^{0.5} \frac{dx}{1+x^3} = \frac{1}{2} - \frac{1}{16 \cdot 4} + \cdots - \frac{1}{2^{34}(34)} \approx 0.4854019421 5\ldots$$

Correction : When $|x|<1$

$$\int \frac{dx}{1-x} = \ln(1-x) + C$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{dx}{1-x} = D + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = D + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

So $\quad \ln(1-x) + C = D + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

$$\ln(1-x) = E + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \qquad \text{where } E = D - C.$$

to find E, set $x=0$ so

$$0 = \ln(1) = E + 0 = E$$

So $\quad \ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$

Ex: Find a power series representation for $f(x) = \arctan(x)$.

Sol'n: Notice that

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{Note } R=1.$$

Integrating both sides yields

$$\arctan(x) + C = D + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\arctan(x) = E + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad \text{where } E = D - C$$

Plugin $x=0$ to see that $E = \arctan(0) = 0$. Hence

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

The radius of convergence of $\arctan(x)$ is the same as the radius of convergence of $\frac{1}{1+x^2}$ which is $\underline{1}$.

Notes for lecture:

Radius of convergence stays the same in our calculations but the interval of convergence might change at the endpoints.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \quad \text{at } x=1 \quad \ln(2) = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \cdots$$

Alternating harmonic series!

If you forget the radius of convergence while doing computations, you can always use the ratio test to figure it out.

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Ex: Evaluate $\int_0^{0.2} \frac{dx}{1+x^5}$ using powerseries correct to 6 decimal places.

Sol'n:

$$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n\geq 1}^{\infty} (-x^5)^n = \sum_{n\geq 0}^{\infty} (-1)^n x^{5n}$$

Taking integrals

$$\int \frac{dx}{1+x^5} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{5n+1}$$

Hence

$$\int_0^{0.2} \frac{dx}{1+x^5} = x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \cdots \Big|_0^{0.2}$$

$$= 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \frac{(0.2)^{16}}{16} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (0.2)^{5n+1}}{5n+1}$$

This is an alternating series! It satisfies

$$\lim_{n\to\infty} \frac{(0.2)^{5n+1}}{5n+1} = \lim_{n\to\infty} \frac{1}{5^{5n+1}\cdot(5n+1)} = 0$$

and the terms are decreasing. Hence by the alternating series estimation test, if we use the first two terms, then the error is as large as the third term

ie $\frac{(0.2)^{11}}{11} \doteq 1.9\times 10^{-9} \to$ better than 6 decimal places.

So $I = 0.2 - \frac{(0.2)^6}{6} \doteq 0.199989\ldots$

Taylor & Maclaurin Series

Idea: Write $f(x)$ as a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \qquad |x-a|<R. \qquad (1)$$

We want to find each c_n. To find c_0, Plug in $x=a$ in (1) to get

$$f(a) = c_0.$$

To find c_1, differentiate (1) to see that

$$f^{(1)}(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \qquad (2)$$

plug in $x=a$ in (2) to see

$$f^{(2)}(a) = c_1.$$

To find c_2, differentiate (2) to see that

$$f^{(2)}(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots \qquad (3)$$

Plug in $x=a$ in (3) to see

$$\frac{f^{(2)}(x)}{2} = c_2.$$

One more. To find c_3, differentiate (3) to see

$$f^{(3)}(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots \qquad (4)$$

plug in $x=a$ in (4) to get $\dfrac{f^{(3)}(a)}{3!} = c_3$.

Theorem: If f has a power series expansion at a, it is of the form

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n. \qquad |x-a|<R.$$

$$(*) \qquad f(x)=\frac{f(a)}{0!}+\frac{f'(a)(x-a)}{1!}+\frac{f''(a)(x-a)^2}{2!}+\frac{f'''(a)(x-a)^3}{3!}+\cdots$$

When $a=0$ above, $(*)$ is called the Taylor (series) expansion at a.
In some textbooks, the equation $(*)$ when $a=0$ is called the Maclaurin series.

Ex: Find the Maclaurin series expansion of $f(x)=e^x$ and its radius of convergence.

Sol'n: If $f(x)=e^x$ then $f^{(n)}(x)=e^x$. So $f^{(n)}(0)=e^0=1$ for all $n\geq 0$.
Therefore, the Maclaurin series expansion for $f(x)=e^x$ is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots$$

To find the radius of convergence, use ratio test!

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\lim_{n\to\infty}\frac{|x|}{n+1}=0<1 \text{ for all } x. \text{ (ratio test)}$$

So $R=\infty$ by the ratio test.

Our theorem before says IF e^x has a ~~...~~ power series expansion, then
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
but does it have a power series representation. That is, if $f(x)$ is a smooth function (has derivatives of all orders) is it true that
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \ ?$$

To solve this question, use Taylor polynomials, that is
$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$
nth-degree Taylor polynomial of f at a.

Ex: for e^x, and $a=0$, we have
$$T_0(x) = 1 \qquad T_1(x) = 1+x = T_0(x)+x \qquad T_2(x) = 1+x+\frac{x^2}{2} = T_1(x)+\frac{x^2}{2}, \qquad T_3(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!} = T_2(x)+\frac{x^3}{3!}$$

In general, if $f(x)$ is the sum of it's Taylor series, then
$$f(x) = \lim_{n\to\infty} T_n(x).$$

Let $R_n(x) = f(x) - T_n(x)$ be the remainder of a taylor series. If $\lim_{n\to\infty} R_n(x) = 0$ then notice that
$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}(f(x) - R_n(x)) = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

(Taylor's Theorem)

Theorem: If $f(x) = T_n(x) - R_n(x)$ where T_n is the n^{th} degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0.$$

when $|x-a|<R$, then f is equal to the sum of its Taylor series on the interval $|x-a|<R$.

Theorem (Taylor's Inequality)
If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder term $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d.$$

We now show $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ but first, a lemma.

Lemma: For any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Proof: We showed by the ratio test that for any $x \in \mathbb{R}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges. Hence $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (converse of divergence test).

Prop'n: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Pf: By Taylor's Theorem, it suffices to show
$$\lim_{n\to\infty} R_n(x) = 0.$$
for all x. We pick any real number $d > 0$ then suppose $|x| \le d$. Then if $f(x) = e^x$, $f^{(n)}(x) = e^x$ for all n. So $|f^{(n+1)}(x)| = e^x \le e^d$ when $|x| \le d$.

Using Taylor's inequality with $a = 0$ and $M = e^d$, we have that for $|x| \le d$

$$0 \le |R_n(x)| \le \frac{e^d}{(n+1)!} |x|^{n+1} \xrightarrow{n\to\infty} 0 \text{ by the lemma.}$$

So by the squeeze theorem
$$\lim_{n\to\infty} |R_n(x)| = 0$$

So by another theorem
$$\lim_{n\to\infty} R_n(x) = 0$$

Holding for all $|x| \le d$. Since d was an arbitrary real number, this holds for all $x \in \mathbb{R}$. Taylor's theorem says that e^x equals its Maclaurin Series and hence

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R}.$$

$\blacksquare$

In fact, when $n = 1$,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

In fact, the same method shows for any $a \in \mathbb{R}$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$$

Prop'n: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x.

Proof: We find the Maclaurin series expansion for $\sin(x)$. Note that

$$\begin{aligned} f(x) &= \sin(x) & f(0) &= 0 \\ f'(x) &= \cos(x) & f'(0) &= 1 \\ f''(x) &= -\sin(x) & f''(0) &= 0 \\ f'''(x) &= -\cos(x) & f'''(0) &= -1 \\ f''''(x) &= \sin(x) & f''''(0) &= 0 \end{aligned}$$

These values are cyclic of order 4. Hence the Maclaurin series is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

To finish, we show this is actually $\sin(x)$ via Taylor's theorem. It suffices to show $\lim_{n\to\infty} R_n(x) = 0$.

Since $f^{(n+1)}(x) = \pm\sin(x)$ or $\pm\cos(x)$, we know $|f^{(n+1)}(x)| \le 1$ for all x. Taking $M=1$ in Taylor's inequality, we have

$$0 \le |R_n(x)| \le \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0 \text{ by lemma}$$

Hence, by the squeeze theorem and absolute value argument

$$\lim_{n\to\infty} R_n(x) = 0.$$

So $$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$ ✎.

Prop'n: $$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x$$

Proof: We could proceed as before OR...

$$\begin{aligned}
\cos(x) = \frac{d}{dx}\sin(x) &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{holds for all } x.
\end{aligned}$$

Ex: Find the Maclaurin series of $x^2\cos x$.

Sol'n $$x^2\cos(x^3) = x^2\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}.$$

FS

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

If you don't see the tricks, you can always compute Taylor series directly from the definition.

You should commit to memory the Table on pg. 762 except for $(1+x)^k$. This table has the power series expansion for
$\frac{1}{1-x}$, e^x, $\sin(x)$, $\cos(x)$, $\arctan(x)$, $\ln(1+x)$.

Ex: Find a closed form for the sum
$$\frac{1}{2}-\frac{1}{8}+\frac{1}{24}-\frac{1}{64}+\cdots=\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\cdots$$

Sol'n: This is
$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(\frac{1}{2}\right)^n}{n}$$

Recall:
$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$$

So
$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot 2^n}=\ln\left(1+\tfrac{1}{2}\right)=\ln\left(\tfrac{3}{2}\right).$$

How powerful is this technique?

Ex: Compute for $c\in\mathbb{R}$ $\lim_{n\to\infty} n^2(1-\cos(\frac{c}{n}))$.

Sol'n: Use Taylor series

$$n^2(1-\cos(\tfrac{c}{n}))=n^2\left(1-\left(1-\frac{(\frac{c}{n})^2}{2!}+\frac{(\frac{c}{n})^4}{4!}-\cdots\right)\right)$$
$$=n^2\left(\frac{c^2}{2!\,n^2}-\frac{c^4}{4!\,n^4}+\frac{c^6}{6!\,n^6}+\cdots\right)\ \ldots$$

$$n^2(1-\cos(\tfrac{c}{n})) = \cdots$$
$$= \frac{c^2}{2!} - \frac{c^4}{4!n^2} + \frac{c^6}{6!n^4} - \cdots$$
$$\xrightarrow{n\to\infty} \frac{c^2}{2}.$$

Multiplying and Dividing Taylor series.

Just like normal!

Ex: find the first three non-zero terms in the Maclaurin series for (a) $e^x\cos(x)$ and (b) $\tan(x)$

Sol'n: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

(a) $$e^x\cos(x) = 1 + x - \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^3}{2!} + \cdots$$
$$= 1 + x + \frac{x^3}{6} - \frac{x^3}{2} + \cdots$$
$$= 1 + x - \frac{2x^3}{6} + \cdots$$
$$= 1 + x - \frac{x^3}{3} + \cdots$$

(b) Note $$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

Use long division!

$$\begin{array}{r}
x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \\
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \;\big)\; x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots \\
-\left(x - \frac{1}{2}x^3 + \frac{1}{24}x^5 + \cdots\right) \\
\hline
\frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\
-\left(\frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots\right) \\
\hline
\frac{2}{15}x^5 + \cdots \\
\frac{2}{15}x^5 + \cdots \\
\hline
\text{other terms.}
\end{array}$$

$$-\frac{1}{6} + \frac{1}{2} = -\frac{1}{6} + \frac{3}{6} = \frac{1}{3}$$

$$\frac{1}{120} - \frac{1}{24} = \frac{-4}{120} = -\frac{1}{30}$$

$$-\frac{1}{30} + \frac{1}{6} = \frac{4}{30} = \frac{2}{15}$$

So ~~tan~~ $\mathrm{TAN}(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

Technically, we have not validated where these series converge. For multiplication, we still converge on $\mathbb{R}$, for division we should be careful and note it works only when $|x|$ is "small"

~~Let $i = \sqrt{-1}$. Th~~

Theorem (Euler) Let $i = \sqrt{-1}$ then

$$e^{ix} = \cos(x) + i\sin(x)$$

Pf: Assume our Taylor series techniques hold for complex numbers, then

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} \cdots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$i\sin(x) = i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} + \cdots$$

$$\cos(x) + i\sin(x) = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} \cdots$$

$$= e^{ix}$$

as claimed. ∎