Approximate Integration

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then

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has no nice formula.However, we should be able to compute or at least approximate definite integrals of $f(x)$, for example

$$\int_1^5 f(x)dx.$$

After all, definite integrals just represent area.

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1. Trapezoid Rule
2. Simpson's Rule

Left Endpoint Rule for $f(x) = \sqrt{1+x^3}\sin(6x) + 14$

Let's try each of these methods with our function $f(x)$ on $[1, 5]$ and using a partition of four subintervals. In these cases, $\Delta x = \frac{5-1}{4} = 1$.

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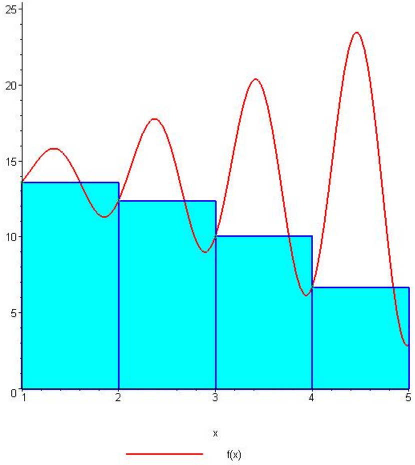

Left Endpoint Rule for $f(x) = \sqrt{1+x^3}\sin(6x) + 14$

Using the formula, we see that

$$\int_1^5 f(x)dx \approx L_4 \quad = \quad f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x$$

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&= 42.72027090
\end{aligned}$$

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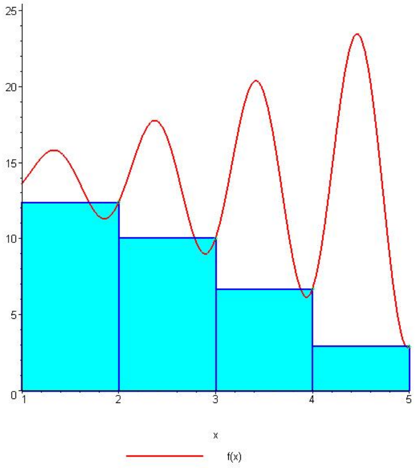

Right Endpoint Rule for $f(x) = \sqrt{1+x^3}\sin(6x) + 14$

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Midpoint Rule for $f(x) = \sqrt{1+x^3}\sin(6x) + 14$

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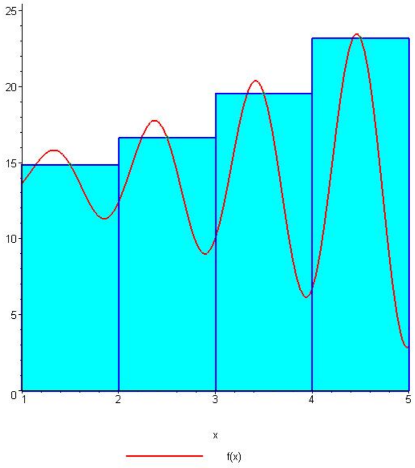

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&= 74.23479834
\end{aligned}$$

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How do we get new rules? One way is to take combinations of the above rules. For example, lets try taking half the left endpoint rule and half the right endpoint rule. This gives us the *trapezoid rule*.

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&\qquad\qquad + (f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x + f(x_n)\Delta x)\Big) \\
&= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right)
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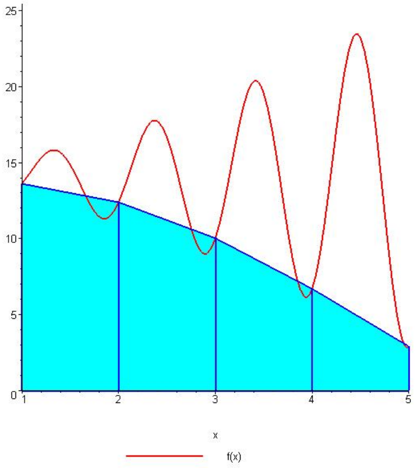

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Note that we could only compute the error term exactly if we knew the value of the integral already (and if we knew that then what's the point of doing this!) However, it turns out that we can bound the error.

A Bound on the Error

Theorem: Suppose that $|f''(x)| \leq K$ for all $x \in [a, b]$ and for some constant $K \in \mathbb{R}$. If E_T and E_M are the errors in the trapezoid and midpoint rules respectively, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

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Notice that the bound for E_M is twice as good as the bound for E_T.

Example Question using the error bound

Question: How large does n have to be in order to estimate $\int_0^1 e^{-x^2}\,dx$ within 0.001 using the trapezoid rule? How about with the midpoint rule?

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I claim that this function is increasing form 0 to 1. To see this, notice that the third derivative is

$$f'''(x) = e^{-x^2}(12x - 8x^3) = 4xe^{-x^2}(3 - 2x^2)$$

and this function is greater than or equal to 0 between 0 and 1.

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and this function is greater than or equal to 0 between 0 and 1. Hence the derivative is positive on $[0, 1]$ and so the function is increasing there. Hence the maximum value of $f''(x)$ on $[0, 1]$ occurs at $x = 1$ which gives the value

$$K = f''(1) = e^{-(1)^2}(4(1)^2 - 2) = \frac{2}{e} = 0.7357588824$$

Example Question using the error bound

Question: How large does n have to be in order to estimate $\int_0^1 e^{-x^2}\,dx$ within 0.001 using the **trapezoid rule**? How about with the midpoint rule?

Solution: With $K = 0.7357588824$, we use the error bounding theorem and note that

$$E_T \leq 0.001 \quad \text{whenever} \quad \frac{K(b-a)^3}{12n^2} \leq 0.001.$$

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$$E_T \leq 0.001 \quad \text{whenever} \quad \frac{K(b-a)^3}{12n^2} \leq 0.001.$$

Plugging in our K, $a = 0$ and $b = 1$ values and solving for n, we see that we need n at least as large as

$$\begin{aligned} \frac{K(b-a)^3}{12n^2} \leq 0.001 \quad &\Rightarrow \quad (0.7357588824)(1-0)^3 \leq 12(0.001)n^2 \\ &\Rightarrow \quad \sqrt{\frac{0.7357588824}{12(0.001)}} \leq n \\ &\Rightarrow \quad 7.830277147 \leq n \end{aligned}$$

So we need n to be at least 8.

Example Question using the error bound

Question: How large does n have to be in order to estimate $\int_0^1 e^{-x^2}\,dx$ within 0.001 using the trapezoid rule? How about with the **midpoint rule**?

Solution: For the midpoint rule, we use the error bounding theorem and note that

$$E_M \leq 0.001 \quad \text{whenever} \quad \frac{K(b-a)^3}{24n^2} \leq 0.001.$$

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$$\frac{K(b-a)^3}{24n^2} \leq 0.001 \quad \Rightarrow \quad (0.7357588824)(1-0)^3 \leq 24(0.001)n^2$$
$$\Rightarrow \quad \sqrt{\frac{0.7357588824}{24(0.001)}} \leq n$$
$$\Rightarrow \quad 5.536842069 \leq n$$

So we need n to be at least 6.

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$$E_M \leq 0.001 \quad \text{whenever} \quad \frac{K(b-a)^3}{24n^2} \leq 0.001.$$

Plugging in our $K = 0.7357588824$, $a = 0$ and $b = 1$ values and solving for n, we see that we need n at least as large as

$$\begin{aligned} \frac{K(b-a)^3}{24n^2} \leq 0.001 \quad &\Rightarrow \quad (0.7357588824)(1-0)^3 \leq 24(0.001)n^2 \\ &\Rightarrow \quad \sqrt{\frac{0.7357588824}{24(0.001)}} \leq n \\ &\Rightarrow \quad 5.536842069 \leq n \end{aligned}$$

So we need n to be at least 6. Notice that this n is smaller than the one needed for the trapezoid rule because its a better estimation.

Other Rules? Simpson's Rule.

Named after English mathematician Thomas Simpson (1710-1761). For this rule, we can only apply it with an even number of sub intervals. We try to combine the left, right, and midpoint rules. We start by trying a third of each.

$$\frac{1}{3}L_n + \frac{1}{3}R_n + \frac{1}{3}M_n = \frac{1}{3}(L_n + R_n) + \frac{1}{3}M_n = \frac{2}{3}T_n + \frac{1}{3}M_n$$

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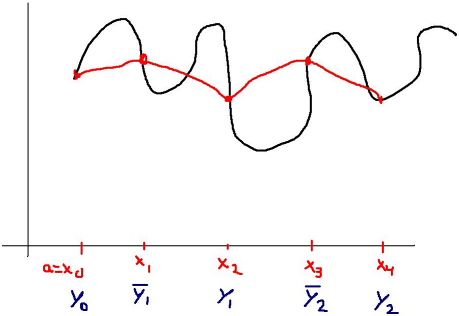
Remember the error bounds? We said that M_n was twice as good as T_n and so these fractions are the wrong way. To change this, we weigh the M_n piece twice as much. This gives

$$S_{2n} = \frac{1}{6}L_n + \frac{1}{6}R_n + \frac{2}{3}M_n = \frac{1}{3}T_n + \frac{2}{3}M_n.$$

Simpson's Rule expanded

Let's expand this rule. We need a diagram.



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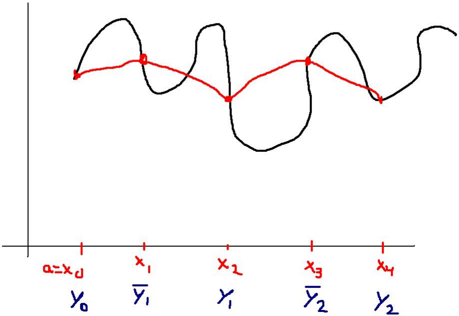

So notice that $y_i = x_{2i}$, $\bar{y}_i = x_{2i-1}$ and

$$\Delta y = \frac{b-a}{n} = 2\frac{b-a}{2n} = 2\Delta x$$

A Formula for Simpson's Rule

Hence,

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$$

A Formula for Simpson's Rule

Hence,

$$\begin{aligned} S_{2n} &= \frac{1}{3} T_n + \frac{2}{3} M_n \\ &= \frac{1}{3} \cdot \frac{\Delta y}{2} \left(f(y_0) + 2f(y_1) + \ldots + 2f(y_{n-1}) + f(y_n) \right) \\ &\quad + \frac{2}{3} \cdot \Delta y \left(f(\bar{y}_1) + f(\bar{y}_2) + \ldots + f(\bar{y}_n) \right) \end{aligned}$$

A Formula for Simpson's Rule

Hence,

$$\begin{aligned}
S_{2n} &= \frac{1}{3}T_n + \frac{2}{3}M_n \\
&= \frac{1}{3} \cdot \frac{\Delta y}{2}\left(f(y_0) + 2f(y_1) + \ldots + 2f(y_{n-1}) + f(y_n)\right) \\
&\qquad + \frac{2}{3} \cdot \Delta y \left(f(\bar{y}_1) + f(\bar{y}_2) + \ldots + f(\bar{y}_n)\right) \\
&= \frac{\Delta x}{3}\left(f(x_{2(0)}) + 2f(x_{2(1)}) + \ldots + 2f(x_{2(n-1)}) + f(x_{2(n)})\right) \\
&\qquad + \frac{4\Delta x}{3}\left(f(x_{2(1)-1}) + f(x_{2(2)-1}) + \ldots + f(x_{2(n)-1})\right)
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A Formula for Simpson's Rule

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&= \frac{\Delta x}{3}\Big((f(x_0) + 2f(x_2) + \ldots + 2f(x_{2n-2}) + f(x_{2n})) \\
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&= \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots \\
&\quad + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})\Big)
\end{aligned}$$

Simpson's Rule for $f(x) = \sqrt{1+x^3}\sin(6x) + 14$

Our last example,

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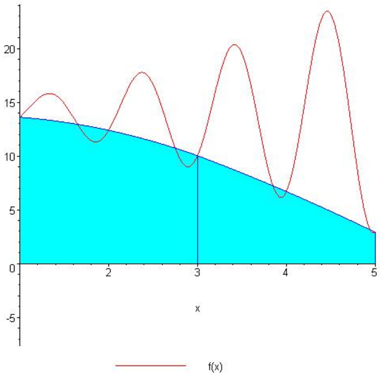

Simpson's Rule for $f(x) = \sqrt{1+x^3}\sin(6x) + 14$

Using the formula, we see that

$$\int_1^5 f(x)dx \approx S_4 \quad = \quad \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$$

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&= \tfrac{1}{3}\Big((\sqrt{1 + (1)^3}\sin(6(1)) + 14)(1) \\
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&= 37.04120590
\end{aligned}$$

A Bound on the Error

Let m be an even integer. As before, let

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Then

Theorem: Suppose that $|f''''(x)| \leq K$ for all $x \in [a, b]$ and for some constant $K \in \mathbb{R}$. Then

$$|E_S| \leq \frac{K(b-a)^5}{180m^4}.$$