Randomness

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Let’s Play a Game

Suppose you play a coin flipping game with a friend. It costs a dollar to play. Each time heads is flipped you lose your dollar and each time tails is flipped you win 10 dollars. Being a good mathematician you agree to play. You play the game 30 times and your friend leaves, flips the coin and tells you the results:
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```
HHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHH
```

You lost every time! You say to your friend that he cheated! Your friend claims he didn’t. Do you think he cheated? Why?
Which is more random?

Suppose we take a fair coin and flip it 30 times. Which of the following is more random? Why?

1. HHHHHHHHHHHHHHHHHHHHHHHHHHHHHHH
2. HTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHT
3. THHTHTHTTTHTHTTTHTHTTTHTTTTTHTTTTTTTHT
4. HTHTTHHHTHTHTHTTHHTTHHTHTHTHTHTHTHTH
Which is more random?

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3. THHTHTTTTHTHTTTTHTHTTTTTTTTTTTHT
4. HTHTTHHTHTTHHTHTTHHTHTHTHTHTHTTH

But which is more probable?
Which is more random?

Suppose we take a fair coin and flip it 30 times. Which of the following is more random? Why?

1. HHHHHHHHHHHH
2. HTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHT
3. THHTHTHTTTHTHTTTHTHTTTHTTTTTHT
4. HTHTTHHHTHTHTHTTHHTHTHTHTHTHTTH

But which is more probable? They all have the same probability! \((1/2^{30})\)
This phenomenon has been known for a while. Andrey Kolmogorov (1903-1987) was one of the first people to give a way to measure this. He defined a concept known today as Kolmogorov Complexity.
Kolmogorov Complexity

Define the *Kolmogorov Complexity* of a string as the length of the minimum length computer program needed to output the string.
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- The idea here is if there is a description of the string that is shorter than just writing the string itself, then the string is *not random*.

- This of course depends on the program you use but one can show that differences in programs only introduce a small difference in the lengths of the programs which as the sample (input) size gets large becomes negligible.
Let’s look at the original examples:

Example 1: HHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHH

This is not very random since we could write a program like

```python
for i in range(30): print "H"
```
Let’s look at the original examples:

Example 2: HTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHT

This is also not very random since we could write a program like

```python
for i in range(15): print "HT"
```
Let’s look at the original examples:

Example 3: THHTHTTTTHTHTTTHTHTTTTHTTTTTHT

This is also not very random even though it might seem very random.
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This is also not very random even though it might seem very random.

```python
for i in range(30):
    if is_prime(i): print "H" else print "T"
```
Let’s look at the original examples:

Example 4: HTHTTHHHTHTHHTTHHTTTTHHTHTHTHTHTHTHTTH

This example doesn’t contain any noticeably large patterns (hopefully!)

print "HTHTTHHHTHTHHTTTTHHTHTHTHTHTHTHTTH"
Kolmogorov Complexity has applications in many areas of mathematics:

- Computer Science (Shell Sort Average Case Runtime, Formal Language Theory, etc.)
- Combinatorics and Optimization (Tournaments, lower bounds on Ramsey Numbers, Constructive proof of the Lovász Local Lemma etc.)
- Number Theory (Infinitely many primes, Asymptotic Prime Number Theorem etc.)
- Philosophy, Logic (Unprovability of a number being random, etc.)
- Biology (Complex Systems, Sequence Similarity, etc.)
- Statistics and Probability Theory (Probabilities of properties in random graphs, Measure Theory etc.)
Examples

Theorem

The average case runtime of shell sort is order $n \log n$.

Theorem

Let $r(k)$ be the least integer such that either a graph or it’s complement has a clique (complete subgraph) of size $k$. Then

$$r(k) \geq k2^{k/2} \left( \frac{1}{e\sqrt{2}} - \text{error term} \right)$$
Examples

Theorem

Suppose a tournament has the transitive property, that is, if $A$ plays and defeats $B$ and $B$ plays and defeats $C$, then $A$ plays and defeats $C$. Let $v(n)$ be the maximum number of players $v$ such that every tournament of size $n$ has a transitive sub-tournament on $v$ players. Then $v(n) \leq 2 \log(n) + 1$.

Theorem

Let $\pi(n) = \sum_{p \leq n} 1$ be the function that counts the number of primes less than or equal to $n$. Then

$$\pi(n) \sim \frac{n}{\log n}.$$
Where I Learned About This

Learned all about Kolmogorov Complexity in CS860 here at the University of Waterloo from Ming Li.

An Introduction to Kolmogorov Complexity and Its Applications

Ming Li
Paul Vitányi

Springer
Experience Here at Waterloo (Why Waterloo?)

- A Mathematics Faculty

Large student populations means that more courses can be offered. Consequently, many more opportunities for friendships and collaborators. Can combine many interests and degrees (I for example combined CS and Pure Mathematics). UWaterloo is well known; many of my job offers I've received were from people who knew the quality of education you get here. City is built around students; everything is conveniently located. Lots of restaurants and pubs (when you're old enough!). Low cost of living. Many of my friends went on to very successful careers and many obtained doctorates in mathematics.
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