Randomness

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University of Waterloo

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Let's Play a Game

Suppose you play a coin flipping game with a friend. It costs a dollar to play. Each time heads is flipped you lose your dollar and each time tails is flipped you win 10 dollars. Being a good mathematician you agree to play. You play the game 30 times and your friend leaves, flips the coin and tells you the results:

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HHHHHHHHHHHHHHHHHHHHHHHHHHHHHH

You lost every time! You say to your friend that he cheated! Your friend claims he didn't. Do you think he cheated? Why?

Which is more random?

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Suppose we take a fair coin and flip it 30 times. Which of the following is more random? Why?

1. HHHHHHHHHHHHHHHHHHHHHHHHHHHHHH
2. HTHTHTHTHTHTHTHTHTHTHTHTHTHTHT
3. THHTHTHTTTHTHTTTHTHTTTHTTTTTHT
4. HTHTTHHHTHTHHTTHHTTHHTHTHTHTTH

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But which is more probable?

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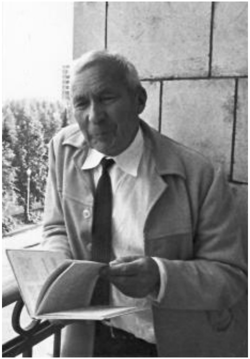
But which is more probable? They all have the same probability! $(1/2^{30})$

Kolmogorov

This phenomenon has been known for a while. Andrey Kolmogorov (1903-1987) was one of the first people to give a way to measure this. He defined a concept known today as Kolmogorov Complexity.



Kolmogorov Complexity

Define the *Kolmogorov Complexity* of a string as the length of the minimum length computer program needed to output the string.

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Define the *Kolmogorov Complexity* of a string as the length of the minimum length computer program needed to output the string.

- The idea here is if there is a description of the string that is shorter than just writing the string itself, then the string is not random.
- This of course depends on the program you use but one can show that differences in programs only introduce a small difference in the lengths of the programs which as the sample (input) size gets large becomes negligible.

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Let's look at the original examples:

Example 1: HHHHHHHHHHHHHHHHHHHHHHHHHHHHHH

This is not very random since we could write a program like

```
for i in range(30): print "H"
```

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Let's look at the original examples:

Example 2: HTHTHTHTHTHTHTHTHTHTHTHTHTHTHT

This is also not very random since we could write a program like

```
for i in range(15): print "HT"
```

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Let's look at the original examples:

Example 3: THHTHTHTTTHTHTTTHTHTTTHTTTTTHT

This is also not very random even though it might seem very random.

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Example 3: THHTHTHTTTHTHTTTHTHTTTHTTTTTHT

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```
for i in range(30):
    if is_prime(i): print "H" else print "T"
```

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Let's look at the original examples:

Example 4: HTHTTHHHTHTHHTTHHTTHHTHTHTHTTH

This example doesn't contain any noticeably large patterns (hopefully!)

```
print "HTHTTHHHTHTHHTTHHTTHHTHTHTHTTH"
```

Applications

Kolmogorov Complexity has applications in many areas of mathematics:

- Computer Science (Shell Sort Average Case Runtime, Formal Language Theory, etc.)
- Combinatorics and Optimization (Tournaments, lower bounds on Ramsey Numbers, Constructive proof of the Lovász Local Lemma etc.)
- Number Theory (Infinitely many primes, Asymptotic Prime Number Theorem etc.)
- Philosophy, Logic (Unprovability of a number being random, etc.)
- Biology (Complex Systems, Sequence Similarity, etc.)
- Statistics and Probability Theory (Probabilities of properties in random graphs, Measure Theory etc.)

Examples

Theorem

The average case runtime of shell sort is order $n \log n$.

Theorem

Let $r(k)$ be the least integer such that either a graph or it's complement has a clique (complete subgraph) of size k. Then

$$r(k) \geq k2^{k/2}\left(\frac{1}{e\sqrt{2}} - \textit{error term}\right)$$

Examples

Theorem

Suppose a tournament has the transitive property, that is, if A plays and defeats B and B plays and defeats C, then A plays and defeats C. Let $v(n)$ be the maximum number of players v such that every tournament of size n has a transitive sub-tournament on v players. Then $v(n) \leq 2\log(n) + 1$.

Theorem

Let $\pi(n) = \sum_{p \leq n} 1$ be the function that counts the number of primes less than or equal to n. Then

$$\pi(n) \sim \frac{n}{\log n}.$$

Where I Learned About This

Randomness
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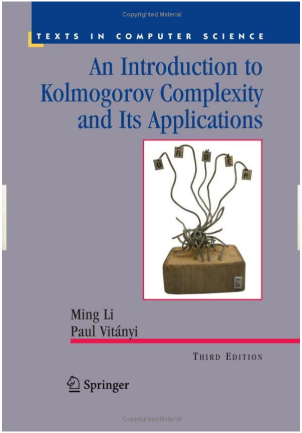
Randomness
Applications
Waterloo

Learned all about Kolmogorov Complexity in CS860 here at the University of Waterloo from Ming Li.



Experience Here at Waterloo (Why Waterloo?)

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- City is built around students; everything is conveniently located. Lots of restaurants and pubs (when you're old enough!). Low cost of living.
- Many of my friends went on to very successful careers and many obtained doctorates in mathematics.