Warm-Up Problem

Consider the statement

$$\left\{\Big(\forall x\,(\exists y\,(P(x) \vee Q(y)))\Big)\right\} \vdash \Big(\exists y\,(\forall x\,(P(x) \vee Q(y)))\Big)$$

and the following attempted proof:

1.	$\Big(\forall x\,(\exists y\,(P(x) \vee Q(y)))\Big)$	Premise
2.	u fresh	
3.	$(\exists y\,(P(u) \vee Q(y)))$	$\forall$e: 1
4.	$(P(u) \vee Q(z)),\ z$ fresh	Assumption
5.	$(P(u) \vee Q(z))$	Reflexive: 4
6.	$(P(u) \vee Q(z))$	$\exists$e: 3,4-5
7.	$(\forall x\,(P(x) \vee Q(z)))$	$\forall$i: 2-6
8.	$\Big(\exists y\,(\forall x\,(P(x) \vee Q(y)))\Big)$	$\exists$i: 7

Identify the main error.

Predicate Logic: Natural Deduction

Carmen Bruni

Lecture 16

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Last Time

- Use new rules for Natural Deduction over Predicate Logic.
- Solve problems using these new rules.

Learning Goals

- Use new rules for Natural Deduction over Predicate Logic.
- Solve problems using these new rules.
- Completeness of Natural Deduction
- Answer some upcoming clicker questions!

Try Some of the Handout Problems

Try the Handout Problems! Some extras include:

1. $\left\{\left(\forall x\left(\forall y\left(R(x,y)\rightarrow R(y,x)\right)\right)\right)\right\}\vdash$ $\left(\left(\forall x\left(\forall y\left(R(x,y)\rightarrow R(y,x)\right)\right)\right)\wedge\left(\forall x\left(\forall y\left(R(y,x)\rightarrow R(x,y)\right)\right)\right)\right)$
2. $\left\{\left(\forall x\left(\exists y\ R(x,y)\right)\right)\right\}\vdash\left(\forall x\left(\exists y\left(\exists z\left(R(x,y)\wedge R(y,z)\right)\right)\right)\right)$
3. $\left\{\left(\forall x\left(\forall y\left(\forall z\left(\left(R(x,y)\wedge R(x,z)\right)\rightarrow R(y,z)\right)\right)\right)\right),\left(\forall x\ R(x,x)\right)\right\}\vdash$ $\left(\forall x\left(\forall y\left(R(x,y)\rightarrow R(y,z)\right)\right)\right)$
4. $\emptyset\vdash\left(\left(\forall x\left(\exists y\ R(x,y)\right)\right)\vee\left(\neg\left(\forall x\ R(x,x)\right)\right)\right)$

Example: $\exists$e

Example. Show $\left\{(\exists x\ P(x)), \left(\forall x\left(P(x) \rightarrow Q(x)\right)\right)\right\} \vdash (\exists x\ Q(x))$.

Example. Show $\{(\exists x\ P(x)), (\forall x\,(P(x) \to Q(x)))\} \vdash (\exists x\ Q(x))$.

Proof:

1.	$(\exists x\ P(x))$	Premise
2.	$(\forall x\,(P(x) \to Q(x)))$	Premise
3.	$P(u),\ u$ fresh	Assumption
4.	$(P(u) \to Q(u))$	$\forall$e: 2
5.	$Q(u)$	$\to$e: 3,4
6.	$(\exists x\ Q(x))$	$\exists$i: 5
7.	$(\exists x\ Q(x))$	$\exists$e: 1,3–6

Example: ∃e

Example. Show $\left\{\left(\exists x\, (P(x) \vee Q(x))\right)\right\} \vdash \left((\exists x\ P(x)) \vee (\exists x\ Q(x))\right)$.

Example. Show $\{(\exists x\,(P(x) \vee Q(x)))\} \vdash ((\exists x\ P(x)) \vee (\exists x\ Q(x)))$.

Proof:

1.	$(\exists x\,(P(x) \vee Q(x)))$	Premise
2.	$(P(u) \vee Q(u))$, u fresh	Assumption
3.	$P(u)$	Assumption
4.	$(\exists x\ P(x))$	∃i:3
5.	$((\exists x\ P(x)) \vee (\exists x\ Q(x)))$	∨i:4
6.	$Q(u)$	Assumption
7.	$(\exists x\ Q(x))$	∃i:6
8.	$((\exists x\ P(x)) \vee (\exists x\ Q(x)))$	∨i:7
9.	$((\exists x\ P(x)) \vee (\exists x\ Q(x)))$	∨e:2,3–5,6–8
10.	$((\exists x\ P(x)) \vee (\exists x\ Q(x)))$	∃e:1,2–9

Soundness and Completeness of Natural Deduction

Theorem.

- Natural Deduction is sound for Predicate Logic: if $\Sigma \vdash \alpha$, then $\Sigma \models \alpha$.
- Natural Deduction is complete for Predicate Logic: if $\Sigma \models \alpha$, then $\Sigma \vdash \alpha$.

Proof outline:

Soundness: Each application of a rule is sound. By induction, any finite number of rule applications is sound. (See next class)

Completeness: We shall show the contrapositive:

$$\text{if } \Sigma \nvdash \alpha, \text{ then } \Sigma \not\models \alpha \ .$$

We shall not give the full proof, but we will sketch the main points.

Completeness of ND for Predicate Logic: Getting started

To show: if $\Sigma \nvdash \alpha$, then $\Sigma \not\models \alpha$.

Lemma I: If $\Sigma \nvdash \alpha$, then $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$.

By rule $\rightarrow$i, if $\Sigma \cup \{(\neg\alpha)\} \vdash \alpha$, then $\Sigma \vdash ((\neg\alpha) \rightarrow \alpha)$. Thus $\Sigma \vdash \alpha$.

Lemma II: If there are $\mathcal{I}$ and E s.t. $\mathcal{I} \models_E \Sigma \cup \{(\neg\alpha)\}$, then $\Sigma \not\models \alpha$.

$\mathcal{I}$ and E satisfy Σ but not α.

Lemma III (the big one):

If $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$, then there are $\mathcal{I}$ and E such that $\mathcal{I} \models_E \Sigma \cup \{(\neg\alpha)\}$.

Whence a Domain?

Given: $\Sigma \cup \{\neg\alpha\} \nvdash \alpha$.

Required: interpretation $\mathcal{H}$ and environment E that satisfy $\Sigma \cup \{\neg\alpha\}$.

To start, we need a domain. Where can we get one?

Whence a Domain?

Given: $\Sigma \cup \{\neg\alpha\} \nvdash \alpha$.

Required: interpretation $\mathcal{H}$ and environment E that satisfy $\Sigma \cup \{\neg\alpha\}$.

To start, we need a domain. Where can we get one?

We use the set of terms. That is, let the domain be

$$\{ \ulcorner t \urcorner \mid t \text{ is a term} \} \quad .$$

(The notation '$\ulcorner\ \urcorner$' indicates that we refer to the domain element, rather than to the expression.)

Interpretation of Terms

For a set Σ of premises, we want an interpretation $\mathcal{I}$ and an environment E, over the domain of terms.

Constants, variables, and functions are easy to handle.

- For a constant symbol c, we define $c^{\mathcal{I}} = \ulcorner c \urcorner$.
- For a variable x, we define $x^{E} = \ulcorner x \urcorner$.
- For a k-ary function symbol f, we define $f^{\mathcal{I}}(\ulcorner t_1 \urcorner, \dots, \ulcorner t_k \urcorner) = \ulcorner f(t_1, \dots, t_k) \urcorner$.

Interpretation of Terms (Continued)

Relations pose a problem, since they depend on Σ. For a relation symbol $R^{(k)}$, we must determine, for each tuple $\langle t_1, \ldots, t_k \rangle$, whether to put $\langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner \rangle$ into the set $R^{\mathcal{I}}$.

The basic idea is to consider each possible formula, one by one. For each, restrict the set of interpretations to account for it. Since we assume no proof of contradiction exists, we can guarantee that we always have some interpretations left as possible.

We suppress the details.

Checking a Formula

Suppose we have a list of all atomic formulas: φ_1, φ_2, φ_3,....

We will create a list of sets: Σ_0, Σ_1, Σ_2, etc., with $\Sigma_0 = \Sigma$ and Σ_{i+1} determined from Σ_i and α_{i+1}:

$\Sigma_0 \leftarrow \Sigma$
for $i \leftarrow 1, 2, 3, \ldots$
    if $\Sigma_{i-1} \cup \{\varphi_i\} \nvdash \neg\varphi_i$
        set $\Sigma_i \leftarrow \Sigma_{i-1} \cup \{\varphi_i\}$
    else
        set $\Sigma_i \leftarrow \Sigma_{i-1} \cup \{\neg\varphi_i\}$

Let β_k denote the formula added at step k; thus β_k is either φ_k or $\neg\varphi_k$.

Let Σ_∞ denote the union of all Σ_i: $\Sigma_\infty = \Sigma \cup \{\beta_1, \beta_2, \beta_3, \ldots\}$.

The Satisfying Interpretation—Almost

Once we have Σ_∞, it

- is consistent (there is no γ s.t. $\Sigma_\infty \cup \{\neg\gamma\} \vdash \gamma$), and
- contains either $R(t_1, \dots, t_k)$ or $\neg R(t_1, \dots, t_k)$ for every relation symbol R and terms $t_1, \dots, t_k$.

Thus Σ_∞ defines an interpretation and environment:

$$R^{\mathcal{I}} = \{ \langle \ulcorner t_1 \urcorner, \dots, \ulcorner t_k \urcorner \rangle \mid R(t_1, \dots, t_k) \in \Sigma_\infty \} \ .$$

This almost works, but we must handle two subtleties.

- We need a list of all formulas, in some order.
- Existential formulas (i.e., $\exists x \ \dots$) may never get a term to satisfy them. Thus they evaluate to $\mathtt{F}$ — even if $\neg \forall x \ \neg\varphi$ is true.

Ensuring Satisfaction of Existential Formulas

We use the following trick to ensure that each existential formula in Σ_∞ has a term that satisfies it.

Let c_1, c_2, ... be a list of fresh constant symbols, that do not occur in any formula of Σ. For each formula γ_i, add the formula

$$(\exists x\ \gamma_i) \to \gamma_i[c_i/x]$$

to the set Σ_0 at the start of the construction. (And include their terms in the domain!)

At the end of the construction, if $\Sigma_\infty \models \exists x\ \gamma_i$, then also $\Sigma_\infty \models \gamma_i[c_i/x]$.

One “Piece of the Puzzle”: Listing all formulas

As one part of the construction, we require a list of all possible formulas. Since we may have arbitrarily many constant, variable, functions and relation symbols, of any arity, we must take care that everything gets onto the list at some point. For example, if we take the ith formula to be $R(c_i)$, then many formulas ($R(x)$, $Q_4(f_7(y_{66}))$, etc.) never appear on the list.

We do the listing “in stages”, starting from stage 1. At stage j, consider the first j constants, variables, and function symbols. Form all terms that combine these, using at most j applications of a function. Apply each of the first j relation symbols to each of these terms, and then form all formulas from these that use at most j connectives or quantifiers.

The set of formulas formed this way is large, but finite. After all have been listed, continue to stage $j + 1$.