Warm-Up Problem

Let α be a Predicate logic formula and t a term. Using the fact that

$$\alpha[t/x]^{(\mathcal{I},E)} = \alpha^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}$$

(which can be proven by structural induction) show that

$$\emptyset \models ((\forall x\ \alpha) \to \alpha[t/x]) \ .$$

Predicate Logic: Natural Deduction

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Lecture 15

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

- State and prove the Relevance Lemma.
- Define what it means for a set of [well-formed] Predicate Logic formulas to semantically entail a [well-formed] formula.
- Solve problems using this definition.

Learning Goals

- Use new rules for Natural Deduction over Predicate Logic.
- Solve problems using these new rules.

Natural Deduction for Predicate Logic

Natural Deduction for Predicate Logic extends Natural Deduction for propositional logic by including rules for introduction and elimination of quantifiers.

Other proof techniques and tricks remain the same as Natural Deduction for propositional logic. In fact, all the rules from Natural Deduction *extend* to our Predicate Logic setting, however we need new rules to handle quantifiers.

Elimination of ∀ and introduction of ∃ are fairly straightforward.

Name	**$\vdash$-notation**	**inference notation**
∀-elimination (∀e)	If $\Sigma \vdash (\forall x\ \alpha)$ then $\Sigma \vdash \alpha[t/x]$	$\dfrac{(\forall x\ \alpha)}{\alpha[t/x]}$
∃-introduction (∃i)	If $\Sigma \vdash \alpha[t/x]$, then $\Sigma \vdash (\exists x\ \alpha)$	$\dfrac{\alpha[t/x]}{(\exists x\ \alpha)}$

Given that a formula is true for every value of x,
conclude it is true for any particular value, such as that of t.

Given that a formula is true for a particular value (of t),
conclude it is true for some value.

Notice the similarity between $\forall$e and $\wedge$e, and between $\exists$i and $\vee$i:

Name	$\vdash$**-notation**	**inference notation**
$\forall$-elimination ($\forall$e)	If $\Sigma \vdash (\forall x\ \alpha)$ then $\Sigma \vdash \alpha[t/x]$	$\dfrac{(\forall x\ \alpha)}{\alpha[t/x]}$
$\wedge$-elimination ($\wedge$e)	If $\Sigma \vdash (\alpha \wedge \beta)$, then $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$	$\dfrac{(\alpha \wedge \beta)}{\alpha}$ $\dfrac{(\alpha \wedge \beta)}{\beta}$
$\exists$-introduction ($\exists$i)	If $\Sigma \vdash \alpha[t/x]$, then $\Sigma \vdash (\exists x\ \alpha)$	$\dfrac{\alpha[t/x]}{(\exists x\ \alpha)}$
$\vee$-introduction ($\vee$i)	If $\Sigma \vdash \alpha$, then $\Sigma \vdash (\alpha \vee \beta)$ and $\Sigma \vdash (\beta \vee \alpha)$	$\dfrac{\alpha}{(\alpha \vee \beta)}$ $\dfrac{\alpha}{(\beta \vee \alpha)}$

Example

Using ∀e and ∀e, prove that

$$\left\{(\forall x\ P(x))\right\} \vdash_{\mathit{ND}} (\exists x\ P(x))$$

Example

Using $\forall$e and $\forall$e, prove that

$$\{(\forall x\ P(x))\} \vdash_{ND} (\exists x\ P(x))$$

Proof:

1.	$(\forall x\ P(x))$	Premise
2.	$P(u)$	$\forall$e: 1
3.	$(\exists x\ P(x))$	$\exists$i: 2

Notice that the u we use must be a symbol that hasn't appeared yet.

Example: $\forall$e

Example. Show $\{P(t), \left(\forall x \left(P(x) \to (\neg Q(x))\right)\right)\} \vdash (\neg Q(t))$.

Example. Show $\{P(t), \left(\forall x \left(P(x) \rightarrow (\neg Q(x))\right)\right)\} \vdash (\neg Q(t))$.

Proof:

1.	$P(t)$	Premise
2.	$\left(\forall x \left(P(x) \rightarrow (\neg Q(x))\right)\right)$	Premise
3.	$\left(P(t) \rightarrow (\neg Q(t))\right)$	∀e: 2
4.	$(\neg Q(t))$	→e: 1,3

Example. Show $\{(\neg P(y))\} \vdash \left(\exists x\, (P(x) \to Q(y))\right)$.

Example. Show $\{(\neg P(y))\} \vdash \left(\exists x\,(P(x) \rightarrow Q(y))\right)$.

Proof:

1.	$\neg P(y)$	Premise
2.	$P(y)$	Assumption
3.	$\bot$	$\neg$e: 2, 1
4.	$Q(y)$	$\bot$e: 3
5.	$P(y) \rightarrow Q(y)$	$\rightarrow$i: 2–4
6.	$\exists x\left(P(x) \rightarrow Q(y)\right)$	$\exists$i: 5

Note to the example

The general form of rule ∃i:

$$\frac{\alpha[t/x]}{(\exists x\ \alpha)}$$

Use in the previous example:

$$\frac{(P(y) \to Q(y))}{\big(\exists x\, (P(x) \to Q(y))\big)}$$

We took $(P(x) \to Q(y))$ for α.

However, knowing what $\alpha[t/x]$ is, does not determine what α is.

We could also take $(P(x) \to Q(x))$ for α; thus the derivation step would be

$$\frac{(P(y) \to Q(y))}{\big(\exists x\, (P(y) \to Q(y))\big)} \; .$$

But the formula $\big(\exists x\, (P(x) \to Q(x))\big)$ is not what we wanted to prove.

Proving a Universal

The $\forall$-introduction rule follows ordinary mathematical usage. To prove a property holds for all integers, one often starts with

> Let x be an integer....

This means the same as

> Assume that the variable "x" refers to an integer.

Then one proves that x has the property.

Since we know nothing about the value x, except that it is an integer, this justifies that every integer has the property.

One could also start the proof with

> Let x be anything. If x is an integer, then....

The conclusion is essentially the same.

Fresh Variables

Definition: A variable is *fresh* in a subproof if it occurs nowhere outside the box of the subproof.

- For $\forall$e we **do not need fresh variables!**
- Fresh variables are available inside a nested subproof but *not* outside its subproof.
- Do not reuse variable names for **fresh** variables in subproofs.
- For $\forall$i, you will need to start with a fresh variable u.
- For $\exists$e, you will need to start with $\alpha[w/x]$ for a fresh variable w (see next slides).

Not Free Variables

Definition: A variable y is *not free* in a set of well-formed Predicate logic formulas Σ if and only if it is not free in any $\gamma \in \Sigma$.

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As an example, let $\mathcal{L}$ be a language consisting of variables w, x, y, z and predicate symbols $P^{(2)}$, $Q^{(1)}$. Then if

$$\Sigma = \{(\forall w\ P(w, x)), (\exists y\ Q(y))\}$$

we see that each of w, y and z are not free in Σ (only x is free in a formula of Σ).

Rule $\forall$-Introduction

Name	$\vdash$-notation	inference notation
$\forall$-introduction ($\forall$i)	If $\Sigma \vdash \alpha[y/x]$ and y not free in Σ or α, then $\Sigma \vdash (\forall x\ \alpha)$	$\dfrac{\boxed{\begin{array}{c} y \text{ fresh} \\ \vdots \\ \alpha[y/x] \end{array}}}{(\forall x\ \alpha)}$

In other words, in order to prove $(\forall x\ \alpha)$, prove $\alpha[y/x]$ for arbitrary y.
Notes:

- It's safest to always use variables that aren't in any formula in Σ and not in α (these are always not free variables).
- Your fresh variable must be used only in the subproof. They cannot escape boxes.
- Use different fresh variables in different subproofs to avoid confusion.

The analogy to draw is with $\wedge$i:

Name	$\vdash$**-notation**	**inference notation**
$\forall$-introduction ($\forall$i)	If $\Sigma \vdash \alpha[y/x]$ and y not free in Σ or α, then $\Sigma \vdash (\forall x\ \alpha)$	[y fresh ⋮ $\alpha[y/x]$] / $(\forall x\ \alpha)$
$\wedge$-introduction ($\wedge$i)	If $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$, then $\Sigma \vdash (\alpha \wedge \beta)$	$\dfrac{\alpha \quad \beta}{(\alpha \wedge \beta)}$

Instead of two formulas, you're combining all the possible formulas that could occur.

Example: ∀i

Example. Show $\{(\forall x\ P(x))\} \vdash (\forall y\ P(y))$.

Example: $\forall$i

Example. Show $\{(\forall x\ P(x))\} \vdash (\forall y\ P(y))$.

Proof:

1.	$(\forall x\ P(x))$	Premise
2.	u fresh	
3.	$P(u)$	$\forall$e: 1
4.	$(\forall y\ P(y))$	$\forall$i: 2-3

Example: $\forall$i

Example. Show $\emptyset \vdash_{ND} \big(\forall x\,(P(x) \to P(x))\big)$

Example. Show $\emptyset \vdash_{ND} (\forall x\,(P(x) \to P(x)))$

Solution:

1.	u fresh	
2.	$P(u)$	Assumption
3.	$P(u)$	Reflexive: 2
4.	$(P(u) \to P(u))$	$\to$i: 2–3
5.	$(\forall x\,(P(x) \to P(x)))$	$\forall$i: 1–4

Note above that α is $(P(x) \to P(x))$ so that $\alpha[u/x]$ is $(P(u) \to P(u))$.

Example: $\forall$i

Example. Show $\left\{(\forall x\,(P(x) \to Q(x))), (\forall x\ P(x))\right\} \vdash_{ND} (\forall x\ Q(x))$

Example: $\forall$i

Example. Show $\{(\forall x\,(P(x) \to Q(x))), (\forall x\; P(x))\} \vdash_{ND} (\forall x\; Q(x))$

Solution:

1.	$(\forall x\,(P(x) \to Q(x)))$	Premise
2.	$(\forall x\; P(x))$	Premise
3.	u fresh	
4.	$(P(u) \to Q(u))$	$\forall$e: 1
5.	$P(u)$	$\forall$e: 2
6.	$Q(u)$	$\to$e: 4,5
7.	$(\forall x\; Q(x))$	$\forall$i: 3–6

Elimination of an Existential Quantifier

Name	$\vdash$-notation	inference notation
$\exists$-elimination ($\exists$e)	If $\Sigma, \alpha[u/x] \vdash \beta$, with u fresh, then $\Sigma, (\exists x\ \alpha) \vdash \beta$	$\dfrac{(\exists x\ \alpha) \qquad \boxed{\begin{matrix}\alpha[u/x],\ u \text{ fresh} \\ \vdots \\ \beta\end{matrix}}}{\beta}$

In $\exists$e, the variable u should not occur free in Σ, α, or β. (Of course, u will normally be free in $\alpha[u/x]$.) Compare this to $\vee$e:

Name	$\vdash$-notation	inference notation
$\vee$-elimination ($\vee$e)	If Σ, $\alpha_1 \vdash \beta$ and Σ, $\alpha_2 \vdash \beta$, then Σ, $(\alpha_1 \vee \alpha_2) \vdash \beta$	$\dfrac{(\alpha_1 \vee \alpha_2) \qquad \boxed{\begin{matrix}\alpha_1 \\ \vdots \\ \beta\end{matrix}} \qquad \boxed{\begin{matrix}\alpha_2 \\ \vdots \\ \beta\end{matrix}}}{\beta}$

Example: ∃e

Example. Show $\{(\exists x\ P(x))\} \vdash (\exists y\ P(y))$.

Example. Show $\{(\exists x\ P(x))\} \vdash (\exists y\ P(y))$.

Proof:

1.	$(\exists x\ P(x))$	Premise
2.	$P(u)$ u fresh	Assumption
3.	$(\exists y\ P(y))$	∃i: 2
4.	$(\exists y\ P(y))$	∃e: 1,2-3

Example. Show $\{(\exists y\,(\forall x\ P(x,y)))\} \vdash (\forall x\,(\exists y\ P(x,y)))$.

Example: ∃e

Example. Show $\{(\exists y\,(\forall x\; P(x,y)))\} \vdash (\forall x\,(\exists y\; P(x,y)))$.

Proof:

1.	$(\exists y\,(\forall x\; P(x,y)))$	Premise
2.	$(\forall x\; P(x,w))$ w fresh	Assumption
3.	u fresh	
4.	$P(u,w)$	∀e: 2
5.	$(\exists y\; P(u,y))$	∃i: 4
6.	$(\forall x\,(\exists y\; P(x,y)))$	∀i: 3-5
7.	$(\forall x\,(\exists y\; P(x,y)))$	∃e: 1,2-6

Example: ∃e

Example. Show $\left\{(\exists x\ P(x)), \left(\forall x\left(P(x) \rightarrow Q(x)\right)\right)\right\} \vdash (\exists x\ Q(x))$.

Example. Show $\{(\exists x\ P(x)), (\forall x\,(P(x) \to Q(x)))\} \vdash (\exists x\ Q(x))$.

Proof:

1.	$(\exists x\ P(x))$	Premise
2.	$(\forall x\,(P(x) \to Q(x)))$	Premise
3.	$P(u)$, u fresh	Assumption
4.	$(P(u) \to Q(u))$	$\forall$e: 2
5.	$Q(u)$	$\to$e: 3,4
6.	$(\exists x\ Q(x))$	$\exists$i: 5
7.	$(\exists x\ Q(x))$	$\exists$e: 1,3–6

Example. Show $\left\{\left(\exists x\, (P(x) \vee Q(x))\right)\right\} \vdash \left((\exists x\ P(x)) \vee (\exists x\ Q(x))\right)$.

Example. Show $\{(\exists x\,(P(x) \vee Q(x)))\} \vdash ((\exists x\ P(x)) \vee (\exists x\ Q(x)))$.

Proof:

1.	$(\exists x\,(P(x) \vee Q(x)))$	Premise
2.	$(P(u) \vee Q(u))$, u fresh	Assumption
3.	$P(u)$	Assumption
4.	$(\exists x\ P(x))$	∃i:3
5.	$((\exists x\ P(x)) \vee (\exists x\ Q(x)))$	∨i:4
6.	$Q(u)$	Assumption
7.	$(\exists x\ Q(x))$	∃i:6
8.	$((\exists x\ P(x)) \vee (\exists x\ Q(x)))$	∨i:7
9.	$((\exists x\ P(x)) \vee (\exists x\ Q(x)))$	∨e:2,3–5,6–8
10.	$((\exists x\ P(x)) \vee (\exists x\ Q(x)))$	∃e:1,2–9