Warm-Up Problem

Let $\mathcal{L}$ be a language with constant symbol a, function symbols $h^{(3)}$, $f^{(1)}$, Predicate symbol $P^{(1)}$ and variables x, z.

- Define what it means for a Predicate logic formula to be valid. Do the same for satisfiable and unsatisfiable.
- Determine if there is an interpretation and environment such that $\mathcal{I} \not\models_E \left(\exists x\ P(h(f(a), x, z))\right)$.

Predicate Logic: Semantic Entailment

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Lecture 14

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Last Time

- Discussed Interpretations and Environments and how these model (give meaning to) predicate logic formulas
- Discussed validity, satisfiability, unsatisfiable.
- Note: Apparently, we shouldn't be writing $P^{\mathcal{I}}(a, b)$ (where a and b are domain elements). We should always be writing $\langle a, b \rangle \in P^{\mathcal{I}}$. See today's lecture for some formal examples (we'll be lenient on A5 about this however).

Learning Goals

- State and prove the Relevance Lemma.
- Define what it means for a set of [well-formed] Predicate Logic formulas to semantically entail a [well-formed] formula.
- Solve problems using this definition.

Lemma:

Let α be a well-formed Predicate formula, $\mathcal{I}$ be an interpretation, and E_1 and E_2 be two environments such that

$$E_1(x) = E_2(x) \text{ for every } x \text{ that occurs free in } \alpha.$$

Then

$$\mathcal{I} \models_{E_1} \alpha \text{ if and only if } \mathcal{I} \models_{E_2} \alpha \ .$$

Proof by induction on the structure of α.

Semantic Entailment

Let Σ be a set of well-formed Predicate logic formulas and α is a well-formed predicate logic formula.

For interpretation $\mathcal{I}$ and environment E, we write $\mathcal{I} \vDash_E \Sigma$ if and only if for every $\varphi \in \Sigma$, we have that $\mathcal{I} \vDash_E \varphi$.

We say that Σ is a *semantically entails* α, written as $\Sigma \vDash \alpha$, if and only if for any interpretation $\mathcal{I}$ and environment E, we have $\mathcal{I} \vDash_E \Sigma$ implies $\mathcal{I} \vDash_E \alpha$. This can also be written as $\alpha^{(\mathcal{I}, E)} = \mathtt{T}$.

$\emptyset \vDash \alpha$ means that α is valid.

Notes

Suppose $\Sigma = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ Then $\Sigma \models \beta$ means...

- ...that every pair of interpretation and environment that makes Σ true must also make β true.
- ...that $\emptyset \models ((\alpha_1 \wedge (\alpha_2 \wedge (... \wedge \alpha_n))) \rightarrow \beta)$
- ...that $((\alpha_1 \wedge (\alpha_2 \wedge (... \wedge \alpha_n))) \rightarrow \beta)$ is valid

To prove these, take an arbitrary interpretation $\mathcal{I}$ and environment E and show that if this satisfied Σ then it must also satisfy β. You may also assume towards a contradiction that $\mathcal{I} \not\models_E \beta$ and proceed from there if this helps.

To prove that $\Sigma \not\models \beta$ find an interpretation $\mathcal{I}$ and environment E that satisfies Σ but that doesn't satisfy β, that is, show that $\mathcal{I} \not\models_E \beta$.

Example: Semantic Entailment

Example: Show that for any well-formed Predicate formulas α and β:

$$\emptyset \models \big((\forall x\,(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))\big)\ .$$

Example: Semantic Entailment

Example: Show that for any well-formed Predicate formulas α and β:

$$\emptyset \models \big((\forall x\,(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))\big)\ .$$

Proof by contradiction. Suppose there are $\mathcal{I}$ and E such that

$$\mathcal{I} \not\models_E \big((\forall x\,(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))\big)\ .$$

Then we must have $\mathcal{I} \models_E (\forall x\,(\alpha \to \beta))$ and $\mathcal{I} \not\models_E ((\forall x\ \alpha) \to (\forall x\ \beta))$;

the second gives $\mathcal{I} \models_E (\forall x\ \alpha)$ and $\mathcal{I} \not\models_E (\forall x\ \beta)$.

Using the definition of $\models$ for formulas with $\forall$, we have
for every a $\in dom(\mathcal{I})$, $\mathcal{I} \models_{E[x \mapsto \mathsf{a}]} (\alpha \to \beta)$ and $\mathcal{I} \models_{E[x \mapsto \mathsf{a}]} \alpha$.
Thus also $\mathcal{I} \models_{E[x \mapsto \mathsf{a}]} \beta$ for every a $\in dom(\mathcal{I})$.

Thus $\mathcal{I} \models_E (\forall x\ \beta)$, a contradiction.

Example

Example. Show that $\{(\forall x\,(\neg\gamma))\} \vDash (\neg(\exists x\ \gamma))$.

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Proof: Suppose that $\mathcal{I} \vDash_E (\forall x\,(\neg\gamma))$. By definition, this means

for every $\mathsf{a} \in dom(\mathcal{I})$, $\mathcal{I} \vDash_{E[x\mapsto \mathsf{a}]} (\neg\gamma)$. that is, $(\neg\gamma)^{(\mathcal{I},E[x\mapsto \mathsf{a}])} = \mathtt{T}$.

Again by definition (for a formula with $\neg$), this is equivalent to

for every $\mathsf{a} \in dom(\mathcal{I})$, $\mathcal{I} \nvDash_{E[x\mapsto \mathsf{a}]} \gamma$ that is, $\gamma^{(\mathcal{I},E[x\mapsto \mathsf{a}])} = \mathtt{F}$.

and also

there is no $\mathsf{a} \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x\mapsto \mathsf{a}]} \gamma$.

Assuming towards a contradiction that $(\exists x\ \gamma)^{(\mathcal{I},E)} = \mathtt{T}$, this would mean that there is an $\mathsf{b} \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x\mapsto \mathsf{b}]} \gamma$ which contradicts the previous line. Hence $\mathcal{I} \vDash_E (\neg(\exists x\ \gamma))$ holds as required.

Example

Example: Find well-formed Predicate formulas α and β such that

$$\{((\forall x\ \alpha) \to (\forall x\ \beta))\} \not\models (\forall x\,(\alpha \to \beta))\ .$$

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Key idea: $\varphi_1 \to \varphi_2$ yields true whenever φ_1 is false.

Let α be $P(x)$. Let $\mathcal{I}$ have domain $\{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\mathsf{a}\}$. Then $\mathcal{I} \models (\forall x\ \alpha) \to (\forall x\ \beta)$ for any β. (Why?)

Example

Example: Find well-formed Predicate formulas α and β such that

$$\{((\forall x\ \alpha) \to (\forall x\ \beta))\} \not\models (\forall x\, (\alpha \to \beta))\ .$$

Key idea: $\varphi_1 \to \varphi_2$ yields true whenever φ_1 is false.

Let α be $P(x)$. Let $\mathcal{I}$ have domain $\{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\mathsf{a}\}$. Then $\mathcal{I} \models (\forall x\ \alpha) \to (\forall x\ \beta)$ for any β. (Why?)

To obtain $\mathcal{I} \not\models \forall x\, (\alpha \to \beta)$, we can use $\neg P(x)$ for β. (Why?)

Thus $((\forall x\ \alpha) \to (\forall x\ \beta)) \not\models (\forall x\, (\alpha \to \beta))$, as required. (Why?)

Example

Example: For any formula α and term t, show that

$$\emptyset \vDash ((\forall x\ \alpha) \rightarrow \alpha[t/x]) \ .$$

Recall that functions must be total!

Another Example

Let α be any well-formed Predicate formula **without** a free variable x. Let $\mathcal{I}$ be any interpretation and let E be any environment. Then

$$\alpha^{(\mathcal{I},E)} = (\forall x\ \alpha)^{(\mathcal{I},E)}.$$

Another Example

Let α be any well-formed Predicate formula **without** a free variable x. Let $\mathcal{I}$ be any interpretation and let E be any environment. Then

$$\alpha^{(\mathcal{I},E)} = (\forall x\ \alpha)^{(\mathcal{I},E)}.$$

Proof. Let $\mathcal{D}$ be the domain of $\mathcal{I}$. Since x is not free in α, therefore $E(y) = E[x \mapsto a](y)$, for every $a \in \mathcal{D}$ and for every y that occurs free in α.

Then by the Relevance Lemma, we have that

- $\mathcal{I} \vDash_E \alpha$
- if and only if $\mathcal{I} \vDash_{E[x \mapsto a]} \alpha$, for any $a \in \mathcal{D}$.
- if and only if $\mathcal{I} \vDash_E (\forall x\ \alpha)$,

which establishes the desired result.