Warm-Up Problem

Let $\mathcal{L}$ be the language consisting of a, b, c as constant symbols, $f^{(3)}$ as a function symbol and $P^{(2)}$ as a predicate symbol. Give an interpretation where

$$\big(P(f(a,a,a), f(b,b,c)) \rightarrow P(f(c,c,c), f(b,b,c)\big)$$

is false. Use a finite domain in your interpretation. (Two early clicker questions are coming!)

Predicate Logic: Semantics, Interpretations and Environments

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Lecture 13

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Last Time

- Did another substitution example (Please Review!)
- Discussed Interpretations with respect to Predicate Logic

Learning Goals

- Define an **interpretation** and an **environment**.
- Give examples of interpretations and environments in specific situations.
- Define validity, satisfiable and unsatisfiable.

Environments

How do we deal with variables and quantifiers?

Definition: An environment is a function that assigns a value in the domain to every variable symbol in the language.

- An environment needs to be defined on **all** variables!
- We will see in practice, environments will only be used to interpret free variables but must nonetheless be defined on all variables including bound variables.
- Bound variables will get their meaning primarily through the corresponding quantifier.

Meaning of Terms

The combination of an interpretation and an environment supplies a value for every term.

Definition: Fix an interpretation $\mathcal{I}$ and environment E. For each term t, the value of t under $\mathcal{I}$ and E, denoted $t^{(\mathcal{I},E)}$, is as follows.

- If t is a constant c, the value $t^{(\mathcal{I},E)}$ is $c^{\mathcal{I}}$.
- If t is a variable x, the value $t^{(\mathcal{I},E)}$ is x^{E}.
- If t is $f(t_1, \ldots, t_n)$, the value $t^{(\mathcal{I},E)}$ is $f^{\mathcal{I}}(t_1^{(\mathcal{I},E)}, \ldots, t_n^{(\mathcal{I},E)})$.

To extend this definition to formulas, we must consider quantifiers.

But first, a few examples.

Constants Vs. Variables

Example: Let α_1 be $P(c)$ (where c is a constant), and let α_2 be $P(x)$ (where x a variable).

Let $\mathcal{I}$ be the interpretation with domain $\mathbb{N}$, $c^{\mathcal{I}} = 2$ and $P^{\mathcal{I}} =$ "is even". Then $\alpha_1^{\mathcal{I}} = \mathtt{T}$, but $\alpha_2^{\mathcal{I}}$ is undefined.

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To give α_2 a value, we must also specify an environment. For example, if $E(x) = 2$, then $\alpha_2^{(\mathcal{I},E)} = \mathtt{T}$.

Example

Let $f^{(1)}$ and $h^{(2)}$ be function symbols. Let $P^{(1)}$ and $Q^{(2)}$ be predicate symbols, let a, b, c be constant symbols and let x, y, z be variable symbols. Define an interpretation $\mathcal{I}$ by:

- Domain: $\mathcal{D} = \{1, 2, 3\}$
- Constants: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}} : f^{\mathcal{I}}(1) = 2,\ f^{\mathcal{I}}(2) = 3,\ f^{\mathcal{I}}(3) = 1$
- $h^{\mathcal{I}} : (x, y) \mapsto \min\{x, y\}$ (min is the minimum function)
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$
- $Q^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$

and define an environment E by

$$E(x) = 3, E(y) = 3, E(z) = 1.$$

(see next slide...)

What is the meaning of each of these formulas in the interpretation and environment on the previous slide ?

- $f(h(f(a), z))^{(\mathcal{I}, E)}$
- $f(h(y, c))^{(\mathcal{I}, E)}$
- $Q(x, h(a, b))^{(\mathcal{I}, E)}$
- $P(h(f(a), x))^{(\mathcal{I}, E)}$

Meaning of Terms—Example

Example. Suppose a language has constant symbol 0, a unary function s, and a binary function $+$. We shall write $+$ in infix position: $x + y$ instead of $+(x, y)$.

The expressions $s(s(0) + s(x))$ and $s\big(x + s(x + s(0))\big)$ are both terms.

The following are examples of interpretations and environments.

- $\boldsymbol{\mathcal{D}} = dom\{\boldsymbol{\mathcal{I}}\} = \mathbb{N}$, $0^{\boldsymbol{\mathcal{I}}} = 0$, $s^{\boldsymbol{\mathcal{I}}}$ is the successor function and $+^{\boldsymbol{\mathcal{I}}}$ is the addition operation. Then, if $E(x) = 3$, the terms get values $\big(s(s(0) + s(x))\big)^{(\boldsymbol{\mathcal{I}}, \boldsymbol{E})} = 6$ and $\Big(s\big(x + s(x + s(0))\big)\Big)^{(\boldsymbol{\mathcal{I}}, \boldsymbol{E})} = 9$.

Meaning of Terms—Example 2

- $\mathcal{D} = dom\{\mathcal{J}\}$ is the collection of all words over the alphabet $\{a, b\}$,
  $0^{\mathcal{J}} = a$,
  $s^{\mathcal{J}}$ appends a to the end of a string, and
  $+^{\mathcal{J}}$ is concatenation.

  Let $G(x) = aba$. Then

$$\big(s(s(0) + s(x))\big)^{(\mathcal{J},G)} = aaabaaa$$

  and

$$\Big(s\big(x + s(x + s(0))\big)\Big)^{(\mathcal{J},G)} = abaabaaaaa \ .$$

Quantifiers

- Finally, we can evaluate formulas with free and bound variables.
- How can we evaluate a formula of the form $(\forall x\ \alpha)$ or $(\exists x\ \alpha)$?
  - For $(\forall x\ \alpha)$, we need to verify that α is true for every possible value of x in the domain
  - For $(\exists x\ \alpha)$ we need to verify that α is true for at least one possible value of x in the domain

We formalize this on the next few slides.

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... but first a small aside...

Quantifiers Over Finite Domains

The universal and existential quantifiers may be understood respectively as generalizations of conjunction and disjunction. If the domain $D = \{a_1, \ldots, a_k\}$ is finite then:

For all x, $P(x)$ iff $P(a_1)$ and ... and $P(a_k)$

There exists x, $P(x)$ iff $P(a_1)$ or ... or $P(a_k)$

where P is a predicate (a property).

Quantified Formulas

Definition: For any environment E and domain element d, the environment "E with x re-assigned to d", denoted $E[x \mapsto \mathsf{d}]$, is given by

$$E[x \mapsto \mathsf{d}](y) = \begin{cases} \mathsf{d} & \text{if } y \text{ is } x \\ E(y) & \text{if } y \text{ is not } x. \end{cases}$$

In other words, $E[x \mapsto \mathsf{d}](x) = d$ and for any other variable y, $E[x \mapsto \mathsf{d}](y) = E(y)$.

Key point: $E[x \mapsto \mathsf{d}]$ is just a new environment!

Example

Let $\mathcal{D} = \{1, 2, 3\}$ for some interpretation $\mathcal{I}$ and consider E as defined by

$$E(x) = 3 \qquad E(y) = 3 \qquad E(z) = 1$$

Then

$$E[x \mapsto 2](x) = 2 \qquad E[x \mapsto 2](y) = 3 \qquad E[x \mapsto 2](z) = 1$$

What about the following?

$$E[x \mapsto 2][y \mapsto 2](x) \qquad E[x \mapsto 2][y \mapsto 2](y) \qquad E[x \mapsto 2][y \mapsto 2](z)$$

Values of Quantified Formulas

Definition: The values of $(\forall x\ \alpha)$ and $(\exists x\ \alpha)$ are given by

- $(\forall x\ \alpha)^{(\boldsymbol{\mathcal{I}}, E)} = \begin{cases} \mathtt{T} & \text{if } \alpha^{(\boldsymbol{\mathcal{I}}, E[x \mapsto \mathsf{d}])} = \mathtt{T} \text{ for every } \mathsf{d} \text{ in } dom(\boldsymbol{\mathcal{I}}) \\ \mathtt{F} & \text{otherwise} \end{cases}$
- $(\exists x\ \alpha)^{(\boldsymbol{\mathcal{I}}, E)} = \begin{cases} \mathtt{T} & \text{if } \alpha^{(\boldsymbol{\mathcal{I}}, E[x \mapsto \mathsf{d}])} = \mathtt{T} \text{ for some } \mathsf{d} \text{ in } dom(\boldsymbol{\mathcal{I}}) \\ \mathtt{F} & \text{otherwise} \end{cases}$

Note: The values of $(\forall x\ \alpha)^{(\boldsymbol{\mathcal{I}}, E)}$ and $(\exists x\ \alpha)^{(\boldsymbol{\mathcal{I}}, E)}$ do not depend on the value of $E(x)$.

The value $E(x)$ only matters for free occurrences of x *but nonetheless environments must be specified for all variables!*

Examples: Value of a Quantified Formula

Example. Let $dom(\mathcal{I}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{a} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{b}, \mathsf{b} \rangle\}$.

Let $E(x) = a$ and $E(y) = b$. We have

- $P(x, x)^{(\mathcal{I}, E)} = \mathtt{T}$, since $\langle E(x), E(x) \rangle = \langle \mathsf{a}, \mathsf{a} \rangle \in P^{\mathcal{I}}$.
- $P(y, x)^{(\mathcal{I}, E)} = \mathtt{F}$, since $\langle E(y), E(x) \rangle = \langle \mathsf{b}, \mathsf{a} \rangle \notin P^{\mathcal{I}}$.
- $\left(\exists y\ P(y, x)\right)^{(\mathcal{I}, E)} = \mathtt{T}$, since $P(y, x)^{(\mathcal{I}, E[y \mapsto \mathsf{a}])} = \mathtt{T}$.

  (That is, $\langle E[y \mapsto \mathsf{a}](y), E[y \mapsto \mathsf{a}](x) \rangle = \langle \mathsf{a}, \mathsf{a} \rangle \in P^{\mathcal{I}}$).
- What is $\left(\forall x\ (\forall y\ P(x, y))\right)^{(\mathcal{I}, E)}$?

Examples: Continued

Example. Let $dom(\boldsymbol{\mathcal{I}}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\boldsymbol{\mathcal{I}}} = \{\langle \mathsf{a}, \mathsf{a} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{b}, \mathsf{b} \rangle\}$.

Let $E(x) = \mathsf{a}$ and $E(y) = \mathsf{b}$.

- What is $\big(\forall x\ (\forall y\ P(x, y))\big)^{(\boldsymbol{\mathcal{I}}, E)}$?

  Since $\langle \mathsf{b}, \mathsf{a} \rangle \notin P^{\boldsymbol{\mathcal{I}}}$, we have

$$P(x, y)^{(\boldsymbol{\mathcal{I}}, E[x \mapsto \mathsf{b}][y \mapsto \mathsf{a}])} = \mathtt{F} \ ,$$

  and thus

$$\big(\forall x\ (\forall y\ P(x, y))\big)^{(\boldsymbol{\mathcal{I}}, E)} = \mathtt{F} \ .$$

Examples: Continued

Example. Let $dom(\mathcal{I}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{a} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{b}, \mathsf{b} \rangle\}$.

Let $E(x) = \mathsf{a}$ and $E(y) = \mathsf{b}$.

- What is $\left(\forall x \ (\forall y \ P(x, y))\right)^{(\mathcal{I}, E)}$?

  Since $\langle \mathsf{b}, \mathsf{a} \rangle \notin P^{\mathcal{I}}$, we have

  $$P(x, y)^{(\mathcal{I}, E[x \mapsto \mathsf{b}][y \mapsto \mathsf{a}])} = \mathtt{F} \ ,$$

  and thus

  $$\left(\forall x \ (\forall y \ P(x, y))\right)^{(\mathcal{I}, E)} = \mathtt{F} \ .$$

- What about $\left(\forall x \ (\exists y \ P(x, y))\right)^{(\mathcal{I}, E)}$?

A Question of Syntax

In the previous example, we wrote

$$P(x, y)^{(\mathcal{I}, E[x \mapsto \mathsf{b}][y \mapsto \mathsf{a}])} = \mathtt{F} \ .$$

Why did we not write simply

$$P(\mathsf{b}, \mathsf{a}) = \mathtt{F}$$

or perhaps

$$P(\mathsf{b}, \mathsf{a})^{(\mathcal{I}, E)} = \mathtt{F} \ ?$$

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Why did we not write simply

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or perhaps

$$P(\mathsf{b}, \mathsf{a})^{(\mathcal{I}, E)} = \texttt{F} \ ?$$

Because "$P(\mathsf{b}, \mathsf{a})$" is not a formula. The elements a and b of $dom(\mathcal{I})$ are not symbols in the language; they cannot appear in a formula.

Satisfaction of Formulas

An interpretation $\mathcal{I}$ and environment E *satisfy* a formula α, denoted $\mathcal{I} \vDash_E \alpha$, if $\alpha^{(\mathcal{I},E)} = \texttt{T}$;
they do not satisfy α, denoted $\mathcal{I} \nvDash_E \alpha$, if $\alpha^{(\mathcal{I},E)} = \texttt{F}$.

Form of α	*Condition for* $\mathcal{I} \vDash_E \alpha$
$P(t_1, \ldots, t_k)$	$\left\langle t_1^{(\mathcal{I},E)}, \ldots, t_k^{(\mathcal{I},E)} \right\rangle \in P^{\mathcal{I}}$
$(\neg\beta)$	$\mathcal{I} \nvDash_E \beta$
$(\beta \wedge \gamma)$	both $\mathcal{I} \vDash_E \beta$ and $\mathcal{I} \vDash_E \gamma$
$(\beta \vee \gamma)$	either $\mathcal{I} \vDash_E \beta$ or $\mathcal{I} \vDash_E \gamma$ (or both)
$(\beta \rightarrow \gamma)$	either $\mathcal{I} \nvDash_E \beta$ or $\mathcal{I} \vDash_E \gamma$ (or both)
$(\forall x\ \beta)$	for every $\mathsf{a} \in dom(\mathcal{I})$, $\mathcal{I} \vDash_{E[x \mapsto \mathsf{a}]} \beta$
$(\exists x\ \beta)$	there is some $\mathsf{a} \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x \mapsto \mathsf{a}]} \beta$

If $\mathcal{I} \vDash_E \alpha$ for every E, then $\mathcal{I}$ *satisfies* α, denoted $\mathcal{I} \vDash \alpha$.

Example: Satisfaction

Example. Consider the formula $(\exists y\ P(x, y \oplus y))$.

(For P a binary predicate and $\oplus$ a binary function.)

Suppose $dom(\mathcal{I}) = \{1, 2, 3, ...\}$,
$\oplus^{\mathcal{I}}$ is the addition operation, and
$P^{\mathcal{I}}$ is the equality predicate.

Give a simple condition that determines when
$\mathcal{I} \vDash_E (\exists y\ P(x, (y \oplus y)))$ holds.

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$\mathcal{I} \vDash_E (\exists y\ P(x, (y \oplus y)))$ iff $E(x)$ is an even number.

Validity and Satisfiability

Validity and satisfiability of formulas have definitions analogous to the ones for propositional logic.

Definition: A formula α is

- *valid* if every interpretation and environment satisfy α; that is, if $\mathcal{I} \models_E \alpha$ for every $\mathcal{I}$ and E,
- *satisfiable* if some interpretation and environment satisfy α; that is, if $\mathcal{I} \models_E \alpha$ for some $\mathcal{I}$ and E, and
- *unsatisfiable* if no interpretation and environment satisfy α; that is, if $\mathcal{I} \not\models_E \alpha$ for every $\mathcal{I}$ and E.

(The term "tautology" is not used in predicate logic.)

Note

If there is no need to specify an environment, then simply defining an interpretation $\mathcal{I}$ and writing $\mathcal{I} \vDash \alpha$ will suffice.

Revisit Example

Let $f^{(1)}$ and $h^{(2)}$ be function symbols. Let $P^{(1)}$ and $Q^{(2)}$ be predicate symbols, let a, b, c be constant symbols and let x, y, z be variable symbols. Define an interpretation by:

- Domain: $\mathcal{D} = \{1, 2, 3\}$
- Constants: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}} : f^{\mathcal{I}}(1) = 2,\ f^{\mathcal{I}}(2) = 3,\ f^{\mathcal{I}}(3) = 1$
- $h^{\mathcal{I}} : (x, y) \mapsto \min\{x, y\}$ (min is the minimum function)
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$
- $Q^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$

and define an environment E by

$$E(x) = 3, E(y) = 3, E(z) = 1.$$

- Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.
- Give a new interpretation $\mathcal{J}_2$ and environment G_2 satisfying $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

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Solution: Set $G_1 = E$ and $\mathcal{J}_1$ to be $\mathcal{I}$ except, let $P^{\mathcal{J}_1} = \emptyset$. Then $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution: Set $G_1 = E$ and $\mathcal{J}_1$ to be $\mathcal{I}$ except, let $P^{\mathcal{J}_1} = \emptyset$. Then $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Give a new interpretation $\mathcal{J}_2$ and environment G_2 satisfying $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution: Set $G_1 = E$ and $\mathcal{J}_1$ to be $\mathcal{I}$ except, let $P^{\mathcal{J}_1} = \emptyset$. Then $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Give a new interpretation $\mathcal{J}_2$ and environment G_2 satisfying $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$.

Solution: Set $G_2 = E$ and $\mathcal{J}_2$ to be $\mathcal{I}$ except, let $Q^{\mathcal{J}_2} = \emptyset$. Then $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$..

Relevance Lemma

Lemma:

Let α be a first-order formula, $\mathcal{I}$ be an interpretation, and E_1 and E_2 be two environments such that

$$E_1(x) = E_2(x) \text{ for every } x \text{ that occurs free in } \alpha.$$

Then

$$\mathcal{I} \vDash_{E_1} \alpha \text{ if and only if } \mathcal{I} \vDash_{E_2} \alpha \ .$$

Proof by induction on the structure of α.

Example: Satisfiability and Validity

Let $\mathcal{L}$ be a language consisting of variables x, y, z, function symbols $f^{(2)}$, $g^{(1)}$ and predicate symbol $P^{(2)}$. Let α be the formula $P\big(f(g(x), g(y)), g(z)\big)$. The formula is satisfiable:

- $dom(\boldsymbol{\mathcal{I}})$: $\mathbb{N}$
- $f^{\boldsymbol{\mathcal{I}}}$: summation
- $g^{\boldsymbol{\mathcal{I}}}$: squaring
- $P^{\boldsymbol{\mathcal{I}}}$: equality
- $E(x) = 3$, $E(y) = 4$ and $E(z) = 5$.

α is not valid. (Why?)

Another Example

Let $\mathcal{L}$ be a language consisting of variables x, y, predicate symbol $Q^{(2)}$.
Let $\mathcal{I}$ be the interpretation defined by

- Domain: $\mathcal{D} = \{1, 2\}$ $\qquad Q^{\mathcal{I}} = \emptyset$

and environment E given by:

- $E(x) = 1 \qquad E(y) = 2$

1. Show that $\mathcal{I} \not\models_E \alpha$ where $\alpha \stackrel{\text{def}}{=} \big(\exists x\,(\forall y\; Q(x, y))\big)$.
2. Give an interpretation $\mathcal{J}$ and an environment G such that $\mathcal{J} \models_G (\exists x\; Q(x, y))$
3. Give an interpretation $\mathcal{J}$ and an environment G such that $\mathcal{J} \models_G (\forall x\; Q(x, y))$
4. Give an interpretation $\mathcal{J}$ and an environment G such that $\mathcal{J} \models_G \big(\exists x\,(\forall y\; Q(x, y))\big)$
5. Is $\big(\forall x\,(\exists y\; Q(x, y))\big)$ valid? Why or why not?