Warm-Up Problem

What is $\alpha[y/x]$ when

$$\alpha \stackrel{\text{def}}{=} \big(\forall y \,(\forall y \;\, P(y,x))\big)$$

where P is a binary predicate?

Substitution—Formal Definition

Let x be a variable and t be a term. For a formula α,

1. If α is $P(t_1, \ldots, t_k)$, then $\alpha[t/x]$ is $P\Big(t_1[t/x], \ldots, t_k[t/x]\Big)$.
2. If α is $(\neg\beta)$, then $\alpha[t/x]$ is $(\neg\beta[t/x])$.
3. If α is $(\beta \star \gamma)$, then $\alpha[t/x]$ is $(\beta[t/x] \star \gamma[t/x])$.
4. If α is $(\forall x\ \beta)$ or $(\exists x\ \beta)$, then $\alpha[t/x]$ is α.
5. If α is $(\mathcal{Q}y\ \beta)$ for some other variable y and some quantifier $\mathcal{Q} \in \{\exists, \forall\}$, then
   (a) If y does not occur in t, then $\alpha[t/x]$ is $(\mathcal{Q}y\ \beta[t/x])$.
   (b) Otherwise, select a variable z that occurs in neither α nor t; then $\alpha[t/x]$ is $(\mathcal{Q}z\ (\beta[z/y])[t/x])$.

The last case prevents capture by renaming the quantified variable to something harmless.

Predicate Logic: Semantics, Interpretations and Environments

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Lecture 12

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Last Time

- Finished off translations
- Discussed Parse Trees
- Discussed Substitutions... (clicker question coming!)

Learning Goals

- Define an **interpretation**.
- Give examples of interpretations in specific situations.

Leading Question

Given a well-formed Predicate logic formula, is it `T` or `F` in some context?

- In Propositional logic, a truth valuation was enough to assign a meaning to a formula
- In Predicate logic, we need **a lot** more.

Motivating Example

For example, if we consider the formula (for $P^{(2)}$ a predicate symbol and variable x)

$$\alpha \stackrel{\text{def}}{=} (\forall x \; P(x, x))$$

and we use an interpretation $\mathcal{I}$ of $P(x, x)$ to mean x is equal to x and consider a domain $\mathcal{D} = \{1\}$, then indeed, α is true under this interpretation.

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We will formalize the notation of interpretations (and later environments) and explain what it means for a Predicate logic formula to be **valid**, **satisfiable**, and **unsatisfiable**.

Brief Definition

An interpretation consists of a domain as well as meanings for all of the constant, function and predicate symbols.

Huth and Ryan use the term “model” instead of “interpretation”.

More formally...

Semantics: Interpretations

Definition: Fix a set $\mathcal{L}$ of constant symbols, function symbols, variable symbols and predicate symbols. (The "language" of our formulas.)

An *interpretation* $\mathcal{I}$ (for the set $\mathcal{L}$) consists of

- A non-empty set $dom(\mathcal{I})$ or $\mathcal{D}^{\mathcal{I}}$ or more simply $\mathcal{D}$, called the domain (or universe) of $\mathcal{I}$.
- For each constant symbol c, a member $c^{\mathcal{I}}$ of $dom(\mathcal{I})$.
- For each function symbol $f^{(i)}$, an i-ary function $f^{\mathcal{I}}$.
- For each predicate symbol $P^{(i)}$, an i-ary predicate $P^{\mathcal{I}}$.

When there are no variables and no quantifiers, this is more than enough to specify meaning to a formula.

High Brow Comment

Technically, our language should have all of the variable symbols we will ever need. In practice this is a bit cumbersome so we will usually forgo including variables in our language explicitly and simply use them as they appear in our formulas.

Values of Variable-Free Terms

Definition: Fix an interpretation $\mathcal{I}$. For each term t containing no variables, the value of t under interpretation $\mathcal{I}$, denoted $t^{\mathcal{I}}$, is as follows.

- If t is a constant c, the value $t^{\mathcal{I}}$ is $c^{\mathcal{I}}$.
- If t is $f(t_1, \dots, t_n)$, the value $t^{\mathcal{I}}$ is $f^{\mathcal{I}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}})$.

The value of a term is always a member of the domain of $\mathcal{I}$.

For example, consider f a unary function and 0 a constant. If we have an interpretation $\mathcal{I}$ with domain $\mathbb{N}$, $0^{\mathcal{I}}$ the usual zero and $f^{\mathcal{I}}$ the usual successor function (increment by 1), then

$$f(0)^{\mathcal{I}} = f^{\mathcal{I}}(0^{\mathcal{I}}) = 1$$

Watch out!

Notice that in the previous example, even though we used the constant symbol 0, we still needed to specify that the interpretation of 0 is indeed the usual zero.

For example, consider f a unary function and 0 a constant. If we have an interpretation $\mathcal{I}$ with domain $\mathbb{N}$, $0^{\mathcal{I}}$ to be the usual number 1 and $f^{\mathcal{I}}$ the usual successor function (increment by 1), then

$$f(0)^{\mathcal{I}} = f^{\mathcal{I}}(0^{\mathcal{I}}) = 2$$

While doable, this is not advised...

Clarity

Most of these issues are taken care of by not using a symbol that could be misinterpreted as being in the domain.

For example, consider f a unary function and a a constant. If we have an interpretation $\mathcal{I}$ with domain $\mathbb{N}$, $a^{\mathcal{I}}$ to be the usual number 1 and $f^{\mathcal{I}}$ the usual successor function (increment by 1), then

$$f(a)^{\mathcal{I}} = f^{\mathcal{I}}(a^{\mathcal{I}}) = 2$$

Total Function

Another issue arises that your function's interpretation must be defined on the entire domain! for a function with arity k, we need to define an interpretation such that the function $f^{\mathcal{I}}$ satisfies:

$$f^{\mathcal{I}} : \mathcal{D}^k \to \mathcal{D}$$

that is, *every* k-tuple from the domain must map into the domain. Such functions capable of doing this are called **total functions**.

- For example, the usual addition over the natural numbers is total since the sum of any two natural numbers gives another natural number.
- However, the usual subtraction over the natural numbers is not total. For example, we cannot perform $2 - 6$ and get a natural number.
- Similarly, square roots over the integers (or even the real numbers!) is not a total function since the square root of -1 is not an integer (or a real number).

Formulas with Variable-Free Terms

Formulas get values in much the same fashion as terms,
except that values of formulas lie in $\{\mathtt{T}, \mathtt{F}\}$.

Definition: Fix an interpretation $\mathcal{I}$. For each formula α containing no variables, the value of α under interpretation $\mathcal{I}$, denoted $\alpha^{\mathcal{I}}$, is as follows.

- If α is $P(t_1, \ldots, t_n)$, then

$$\alpha^{\mathcal{I}} = \begin{cases} \mathtt{T} & \text{if } \langle t_1^{\mathcal{I}}, \ldots, t_n^{\mathcal{I}} \rangle \in P^{\mathcal{I}} \\ \mathtt{F} & \text{otherwise.} \end{cases}$$

- If α is $(\neg\beta)$ or $(\beta \star \gamma)$, then $\alpha^{\mathcal{I}}$ is determined by $\beta^{\mathcal{I}}$ and $\gamma^{\mathcal{I}}$ in the same way as for propositional logic.

Example

Let $f^{(1)}$ and $h^{(2)}$ be function symbols. Let $P^{(1)}$ and $Q^{(2)}$ be predicate symbols and let a, b, c be constant symbols. Define an interpretation by:

- Domain: $\mathcal{D} = \{1, 2, 3\}$
- Constants: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}} : f^{\mathcal{I}}(1) = 2,\ f^{\mathcal{I}}(2) = 3,\ f^{\mathcal{I}}(3) = 1$
- $h^{\mathcal{I}} : (x, y) \mapsto \min\{x, y\}$ (min is the minimum function)
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$
- $Q^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$

What is the meaning of each of these formulas in this interpretation?

- $f(h(f(a), f(c)))^{\mathcal{I}}$
- $f(h(b, f(a)))^{\mathcal{I}}$
- $Q(f(c), a)^{\mathcal{I}}$
- $P(h(f(a), f(c)))^{\mathcal{I}}$

Follow Ups

We saw that under $\mathcal{I}$ we have that $Q(f(c), a)^{\mathcal{I}} = \mathtt{F}$. Is there another interpretation $\mathcal{J}$ such that $Q(f(c), a)^{\mathcal{J}} = \mathtt{T}$?

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Notice that $f(c)^{\mathcal{I}} = 1$ and $a^{\mathcal{I}} = 1$. So if we set $\mathcal{J} = \mathcal{I}$ except we define

$$Q^{\mathcal{J}} = \{\langle 1, 1 \rangle\}$$

then we see that $Q(f(c), a)^{\mathcal{J}} = \mathtt{T}$.

Follow Ups

We saw that under $\mathcal{I}$ we have that $P(h(f(a), f(c)))^{\mathcal{I}} = \texttt{T}$. Is there another interpretation $\mathcal{K}$ such that $P(h(f(a), f(c)))^{\mathcal{I}} = \texttt{F}$?

Follow Ups

We saw that under $\mathcal{I}$ we have that $P(h(f(a), f(c)))^{\mathcal{I}} = \texttt{T}$. Is there another interpretation $\mathcal{K}$ such that $P(h(f(a), f(c)))^{\mathcal{I}} = \texttt{F}$?

Notice that $f(c)^{\mathcal{I}} = 1$ and $f(a)^{\mathcal{I}} = 2$ and so $h(f(a), f(c))^{\mathcal{I}} = 1$. Thus, if we set $\mathcal{K} = \mathcal{I}$ except we define

$$P^{\mathcal{K}} = \emptyset$$

so that $1 \notin P^{\mathcal{K}}$, we see that $P(h(f(a), f(c)))^{\mathcal{K}} = \texttt{F}$.

Example

Let $f^{(1)}$ and $h^{(2)}$ be function symbols. Let $P^{(1)}$ and $Q^{(2)}$ be predicate symbols and let a, b, c be constant symbols. Define an interpretation with domain $\mathcal{D} = \{1, 2, 3\}$ such that with your interpretation, the following value is true

$$\big(P(a) \wedge Q(f(a), b)\big)$$