Warm-Up Problem

Let α and β be two well formed formulas. Prove or disprove the following:

$$\text{If } \alpha \vDash (\beta \to \alpha) \text{ then } \emptyset \vdash (\alpha \to (\beta \to \alpha))$$

Predicate Logic: Informal Introduction, Syntax and Translation

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Lecture 10

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Learning Goals

- Describe the structure of Predicate Logic; this includes constants, variables, function symbols, terms and predicates.
- Translate sentences from English into Predicate Logic and vice versa

What can't we express using propositional logic?

Can we express the following ideas using propositional logic?

- Translate this sentence: Alice is married to Jay and Alice is not married to Leon.
- Translate this sentence: Every bear likes honey.
- Define what it means for a natural number to be prime.

What can't we express using propositional logic?

A few things that are difficult to express using propositional logic:

- Relationships among individuals: Alice is married to Jay and Alice is not married to Leon.
- Generalizing patterns: Every bear likes honey.
- Infinite domains: Define what it means for a natural number to be prime.

We can use predicate logic (first-order logic) to express all of these.

Elements of predicate logic

Predicate logic generalizes propositional logic.

New things in predicate logic:

- Domains
- Constants, Variables and Function symbols
- Terms
- Predicates
- Quantifiers

Domains

A *domain* is a non-empty set of objects. It is a world that our statement is situated within.

Examples of domains: natural numbers, people, animals, etc.

Why is it important to specify a domain? *The same statement can have different truth values in different domains.*

Consider this statement: There exists a number whose square is 2.

- If our domain is the set of natural numbers, is this statement true or false?
- If our domain is the set of real numbers, is this statement true or false?

Objects in a domain

Constants: concrete objects in the domain

- Natural numbers: 0, 6, 100, ...
- Alice, Bob, Eve, ...
- Animals: Winnie the Pooh, Mickey Mouse, Simba, ...

Variables: placeholders for concrete objects, e.g. x, y, z.

A variable lets us refer to an object without specifying which particular object it is.

Functions

Function Symbol: For now, just a symbol f followed by $($ then by some number n of comma separated symbols and then a final $)$. We say such a function has *arity* n and sometimes denote this by $f^{(n)}$. Later we will attach meaning so that a function symbol behaves like a function in the mathematical sense mapping from n copies of the domain into the domain:

$$f : \mathcal{D}^n \rightarrow \mathcal{D}$$

Terms

Terms: Defined inductively as:

1. Each constant symbol is a term and each variable is a term (atomic terms).
2. If t_1, ..., t_n are terms and f is a function symbol of arity n, then $f(t_1, ..., t_n)$ is a term.
3. Nothing else is a term.

Note: Binary functions, terms and later binary predicates are sometimes denoted like $(t_1 \ f \ t_2)$ instead of $f(t_1, t_2)$. For example, we usually write $(t_1 + t_2)$ instead of $+(t_1, t_2)$.

Examples

If 0 is a constant symbol, x and y are variables and $s^{(1)}$ and $+^{(2)}$ are function symbols, then 0, x, y, $s(0)$, $s(x)$, $s(y)$, $+(x, s(y))$ and $x + y$ are all examples of terms.

$s(x, y)$ is not a term (s is a unary function and $s + x$ is not a term either as s is a function and not a term on its own).

Predicates

A *predicate* represents

- a property of an individual, or
- a relationship among multiple individuals.

An *atomic formula* (or atom) is an expression of the form

$$P(t_1, \dots, t_n)$$

where P is an n-ary predicate symbol and each t_i is a term $(1 \leq i \leq n)$.

Note: Binary predicates are sometimes denoted like $(t_1 \ P \ t_2)$ instead of $P(t_1, t_2)$.

It helps to think of a predicate as a function mapping from n copies of the domain $\mathcal{D}^n$ into $\{\mathtt{T}, \mathtt{F}\}$, though we will actually attach this meaning to predicates later.

Examples:

- Define $L(x)$ to mean "x is a lecturer".
  (unary predicate)
  - Alice is a lecturer: $L(\text{Alice})$
  - Mickey Mouse is not a lecturer: $(\neg L(\text{Mickey Mouse}))$
  - y is a lecturer: $L(y)$
- Define $O(x, y)$ to mean "x is older than y".
  (binary predicate/relation)
  - Alex is older than Sam: $O(\text{Alex}, \text{Sam})$
  - a is older than b: $O(a, b)$

Representing Predicates

Mathematically, we represent a predicate by the set of all things that have the property. If S is the set of all students, then $x \in S$ means x is a student. The only restriction on a predicate is that it must be a subset of the domain.

A k-ary predicate (relation) is a set of k-tuples of domain elements. For example, the binary predicate less-than, over a domain $\mathcal{D}$, is represented by the set

$$\{ \langle x, y \rangle \in \mathcal{D}^2 \mid x < y \} \; .$$

Quantifiers

For how many objects in the domain is the statement true?

- The universal quantifier $\forall$: the statement is true for every object in the domain.
- The existential quantifier $\exists$: the statement is true for one or more objects in the domain.

General Formulas

We define the set of well-formed formulas of first-order logic inductively as follows.

1. A predicate (atomic formula) is a formula.
2. If α is a formula, then $(\neg\alpha)$ is a formula.
3. If α and β are formulas, and $\star$ is a binary connective symbol, then $(\alpha \star \beta)$ is a formula.
4. If α is a formula and x is a variable, then each of $(\forall x\ \alpha)$ and $(\exists x\ \alpha)$ is a formula.
5. Nothing else is a formula.

In case 4, the formula α is called the *scope* of the quantifier. The quantifier keeps the same scope if it is included in a larger formula.

Translating English into Predicate Logic

Translate the following sentences into predicate logic.

1. All animals like honey.
2. At least one animal likes honey.
3. Not every animal likes honey.
4. No animal likes honey.
5. No animal dislikes honey.
6. Not every animal dislikes honey.
7. Some animal dislikes honey.
8. Every animal dislikes honey.

Let the domain be the set of animals. *Honey*(x) means that x likes honey. *Bear*(x) means that x is a bear.

Multiple Quantifiers

Let the domain be the set of people. Let $L(x, y)$ mean that person x likes person y.

Translate the following formulas into English.

1. $(\forall x\, (\forall y\ L(x, y)))$
2. $(\exists x\, (\exists y\ L(x, y)))$
3. $(\forall x\, (\exists y\ L(x, y)))$
4. $(\exists y\, (\forall x\ L(x, y)))$

Food For Thought

- How to we express at least one animal likes honey?
- How to we express at most one animal likes honey?
- How to we express exactly one animal likes honey?
- How to we express at least two different animals like honey?
- How to we express exactly two different animals like honey?