Warm-Up Problem

Prove that $\{(p \wedge (\neg q))\} \vdash (\neg(p \to q))$. (We'll use this later!)

Propositional Logic: Soundness and Completeness for Natural Deduction

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Lecture 9

Based on work by J. Buss, A. Gao, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Learning goals

Soundness and Completeness of Natural Deduction in Propositional logic

- Explain the difference between soundness and completeness
- Show that Natural Deduction is both sound and complete.

What is a proof?

A proof in our system:

- Starts with a set of premises Σ,
- Transforms the premises using a set of rules (Natural Deduction for us)
- Ends with the conclusion.

A proof is purely syntactic; in fact, one could write a computer program that would verify the proof for us.

Reminder: Difference Between Entailment and Proof

Recall that

Definition

$\Sigma \models \alpha$ if and only if every valuation satisfying $\Sigma^t = \mathtt{T}$ implies that $\alpha^t = \mathtt{T}$.

Definition

$\Sigma \vdash_{\mathsf{ND}} \alpha$ if and only if there is a proof in our Natural Deduction system beginning with the premises of Σ and ending with α.

These two concepts **are not the same in general!** However...

Soundness and Completeness of Natural Deduction

We will prove that Natural Deduction is both sound and complete.

Soundness of Natural Deduction means that the conclusion of a proof is always a logical consequence of the premises. That is,

$$\text{If } \Sigma \vdash_{ND} \alpha, \text{ then } \Sigma \vDash \alpha \ .$$

Completeness of Natural Deduction means that all logical consequences in propositional logic are provable in Natural Deduction. That is,

$$\text{If } \Sigma \vDash \alpha, \text{ then } \Sigma \vdash_{ND} \alpha \ .$$

High Brow Comments

- The proof of soundness will work even for infinite sets. The key idea is that a proof is finite and so for an infinite set Σ, we would show that there is a finite set $\Sigma_0 \subseteq \Sigma$ satisfying $\Sigma_0 \vdash \alpha$.
- Completeness also works for infinite sets but for our proof, we will be assuming that Σ is finite for simplicity.
- Notice that we have $\leftrightarrow$ for entailment but not for Natural Deduction. It's best to remember that if and only if is semantically equivalent to implication and its converse; rewrite the formula using this fact.

Notes

Not all proof systems have these properties.

- Intuitionist's Natural Deduction (Natural Deduction without the double elimination) is sound but not complete (we cannot prove $(p \vee (\neg p))$).
- For a system that is not sound but is complete, consider Natural Deduction but add $(p \wedge (\neg p))$ as an axiom. This is not sound since it contains a contradiction and hence we can prove anything. However, this is complete (Since anything that we can semantically entail will have a proof after we have our contradiction).

Semantic Entailment Revisited

Once proven, soundness implies that to prove a semantic entailment $\Sigma \models \alpha$ holds, we can use:

- Truth Table
- Direct Proof (Assume we have a valuation that evaluates true for all premises and show that this implies that the conclusion also evaluates to true under the same valuation)
- Proof by Contradiction
- Natural Deduction proof of $\Sigma \vdash \alpha$.

Example

Show that $\{(p \vee q)\} \nvdash p$.

Example

Show that $\{(p \lor q)\} \nvdash p$. **Proof:**
Assume we've proved the soundness of Natural Deduction. Then, the contrapositive is

$$(\{(p \lor q)\} \nvDash p) \to (\{(p \lor q)\} \nvdash p)$$

It suffices then to show that the entailment does not hold. This is true since if we take the valuation t satisfying $p^t = \texttt{F}$ and $q^t = \texttt{T}$, then the left hand side of the entailment is true and the right hand side is false. Hence $\{(p \lor q)\} \vdash p$.

Proof of Soundness

We prove soundness, that is

$$\text{If } \Sigma \vdash_{ND} \alpha, \text{ then } \Sigma \models \alpha \ .$$

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We will use structural induction... on the length of the proof!

Let $P(n)$ be the statement that "For any well-formed formula φ and any set of well-formed formulas Σ, if Σ proves φ in n lines, then $\Sigma \vDash \varphi$." We prove $P(n)$ is true for all positive integers n.

Base Case and IH

If $\Sigma \vdash \alpha$ with a proof of size 1 for any well formed formula α, then $\alpha \in \Sigma$. Thus, we see that $\Sigma \vDash \alpha$ by definition.(Whenever a valuation makes Σ true, since α is inside Σ, it must by definition make α true).

Now, assume that $P(i)$ holds for all integers $1 \leq i \leq k$ for some integer k.

Inductive Conclusion

Consider the last line of $\Sigma \vdash \alpha$, namely

1.		
$\vdots$	$\vdots$	
k.		
$k+1$.	α	Some Rule

What rule could give us α?

Case 1

What if α came from $\rightarrow$ e?

Then, we must have proven that $\Sigma \vdash (\beta \rightarrow \alpha)$ and we also have proved that $\Sigma \vdash \beta$ in k or fewer lines.

By the induction hypothesis, we have that $\Sigma \models (\beta \rightarrow \alpha)$ and $\Sigma \models \beta$. Let t be a truth valuation satisfying $\Sigma^t = \texttt{T}$. Then, by entailment, $(\beta \rightarrow \alpha)^t = \texttt{T}$ and $\beta^t = \texttt{T}$. Hence, by the definition of implication, we see that $\alpha^t = \texttt{T}$. Thus $\Sigma \models \alpha$.

Case 2

What if α came from $\to$ i?

This means that $\alpha = (\beta \to \gamma)$ for well formed formulas β and γ.

1.		
$\vdots$	$\vdots$	
j.	β	Assumption
$\vdots$	$\vdots$	
k.	γ	Some Rule
$k+1$.	α	$\to$ i: j-k

Notice that by removing line $k+1$ we no longer have a complete proof! (We ended in a subproof!)

However, if we add β to our list of premises, we can then prove γ in k or fewer lines! This means that

$$\Sigma \cup \{\beta\} \vdash \gamma.$$

Now, by the Induction Hypothesis, we have that $\Sigma \cup \{\beta\} \models \gamma$. (Notice here that since our induction hypothesis holds for ANY Σ, we could take a new sigma, say $\Sigma_1 = \Sigma \cup \{\beta\}$ and use the IH on this.)

We claim that this implies that $\Sigma \models (\beta \to \gamma)$.

Proof: Assume towards a contradiction that there is a truth valuation t satisfying $\Sigma^t = \mathtt{T}$ but $(\beta \to \gamma)^t = \mathtt{F}$. Then by definition of implication, we must have that $\beta^t = \mathtt{T}$ and $\gamma^t = \mathtt{F}$. However, we have that $\Sigma \cup \{\beta\} \models \gamma$ and so this means that $\gamma^t = \mathtt{T}$. This is a contradiction. Hence $\Sigma \models (\beta \to \gamma)$.

Completeness of Natural Deduction

We now turn to completeness.

Recall that *completeness* means the following.

Let Σ be a set of formulas and φ be a formula.

$$\text{If } \Sigma \vDash \varphi, \text{ then } \Sigma \vdash \varphi \ .$$

That is, every consequence has a proof.

How can we prove this?

Three Lemmata

Set up: Let $\Sigma = \{\alpha_0, \alpha_1, ..., \alpha_n\}$ for well-formed formulas α_i (for all $1 \leq i \leq n$). We want to show for any well-formed formula β,

$$\text{If } \Sigma \vDash \beta \text{ holds, then } \sigma \vdash \beta \text{ is valid.}$$

- **Lemma 1:** If $\Sigma \vDash \beta$, then

$$\emptyset \vDash (\alpha_0 \to (\alpha_1 \to (... \to (\alpha_n \to \beta)..)))$$

- **Lemma 2:** Tautologies are provable, that is, for any well-formed formula γ, if $\emptyset \vDash \gamma$, then $\emptyset \vdash \gamma$. (This is the key step!)
- **Lemma 3:** If $\emptyset \vdash (\alpha_0 \to (\alpha_1 \to (... \to (\alpha_n \to \beta)..)))$, then

$$\{\alpha_0, \alpha_1, ..., \alpha_n\} \vdash \beta$$

that is, $\Sigma \vdash \beta$.

Lemma 1: If $\Sigma \vDash \beta$, then

$$\emptyset \vDash (\alpha_0 \to (\alpha_1 \to (... \to (\alpha_n \to \beta)..)))$$

Proof: Assume that $\Sigma \vDash \beta$. Assume towards a contradiction that

$$\emptyset \nvDash (\alpha_0 \to (\alpha_1 \to (... \to (\alpha_n \to \beta)..)))$$

Then, there exists a truth valuation t such the long implication on the right is false. Unwinding implication by implication, this means that $(\alpha_i)^t = \mathtt{T}$ for all $1 \leq i \leq n$ and that $\beta^t = \mathtt{F}$. This contradicts the fact that $\Sigma \vDash \beta$.

Hence $\Sigma \vDash \beta$. This completes lemma 1. (Technically we should be formal and use induction but this obscures the main idea).

Proof of Lemma 2 - Tautologies are Provable

Lemma 2: For any well-formed formula γ, if $\emptyset \vDash \gamma$, then $\emptyset \vdash \gamma$.

Throughout assume that γ contains atoms $p_1, .., p_n$.

The idea will be to construct a subproof using the 2^n possible combinations of these atoms with their negations. Each subproof will contain one of p_i or $(\neg p_i)$ for each i and will prove γ and hence, by $\vee$ e, we will arrive at our result.

Lemma 2: For any well-formed formula γ, if $\emptyset \models \gamma$, then $\emptyset \vdash \gamma$.

Throughout, let t be a valuation and define for all $1 \leq i \leq n$,

$$\hat{p}_i = \begin{cases} p_i & \text{if } p_i^t = \mathtt{T} \\ \neg p_i & \text{if } p_i^t = \mathtt{F} \end{cases}$$

Note with this notation, $\hat{p}_i^t = \mathtt{T}$ for all $1 \leq i \leq n$.

Sublemma: For any formula γ containing atoms $p_1, ..., p_n$ and any valuation t with the above notation;

- If $\gamma^t = \mathtt{T}$ then $\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash \gamma$
- If $\gamma^t = \mathtt{F}$ then $\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash (\neg\gamma)$

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Concrete example: Let $\gamma = (p \to q)$. Consider this truth table:

p	q	$(p \to q)$	Claim in sublemma
T	T	T	$\{p, q\} \vdash (p \to q)$
T	F	F	$\{p, (\neg q)\} \vdash (\neg(p \to q))$
F	T	T	$\{(\neg p), q\} \vdash (p \to q)$
F	F	T	$\{(\neg p), (\neg q)\} \vdash (p \to q)$

The values in the last column would correspond to the $\hat{p}$ and $\hat{q}$ in the definition from before. So the second line above is the same as $\{\hat{p}, \hat{q}\} \vdash (\neg\gamma)$. Lines 1,3,4 are $\{\hat{p}, \hat{q}\} \vdash \gamma$.

Sublemma

Sublemma: For any formula γ containing atoms $p_1, ..., p_n$ and any valuation t with the above notation;

- If $\gamma^t = \texttt{T}$ then $\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash \gamma$
- If $\gamma^t = \texttt{F}$ then $\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash (\neg\gamma)$

Really, we're only interested in the first part above for our lemma as we will see, however in order to prove the first part, we need to both parts simultaneously (sometimes, this is called simultaneous induction).

We prove the given statement by structural induction. Let $P(\gamma)$ be the statement verbatim as above.

First Two Steps

Base Case: For an atom $\gamma = p_1$;

- If $(p_1)^t = \texttt{T}$ then $\{\hat{p}_1\} = \{p_1\} \vdash \{p_1\} = \gamma$.
- If $(p_1)^t = \texttt{F}$ then $\{\hat{p}_1\} = \{\neg p_1\} \vdash \{\neg p_1\} = (\neg\gamma)$.

(both of the above have one line proofs since the result is a premise).
Induction Hypothesis: Assume that $P(\gamma_1)$ and $P(\gamma_2)$ are true for some well-formed formulas γ_1 and γ_2.

Induction Conclusion

We have several cases.

Case 1: We have that $\gamma = (\neg\gamma_1)$.

Case 1a: If $\gamma^t = \texttt{T}$, then $(\neg\gamma_1)^t = \texttt{T}$ so $(\gamma_1)^t = \texttt{F}$ and since $P(\gamma_1)$ is true (and since γ_1 contains the same atoms as γ), we have that

$$\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash (\neg\gamma_1) = \gamma.$$

Case 1a: If $\gamma^t = \texttt{F}$, then $(\neg\gamma_1)^t = \texttt{F}$ so $(\gamma_1)^t = \texttt{T}$ and since $P(\gamma_1)$ is true (and since γ_1 contains the same atoms as γ), we have that

$$\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash \gamma_1.$$

By $\neg\neg$i, we see that

$$\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash (\neg(\neg\gamma_1)) = (\neg\gamma).$$

This completes the proof in this claim.

More Cases

Case 2: We have that $\gamma = (\gamma_1 \to \gamma_2)$.

Case 2a: Assume that $\gamma^t = \texttt{F}$ so that $(\gamma_1)^t = \texttt{T}$ and $(\gamma_2)^t = \texttt{F}$.

Nowm suppose that γ_1 contains atoms $q_1, .., q_k$ and that γ_2 contains atoms $r_1, ..., r_\ell$. Then by the induction hypothesis, we have that

$$\{\hat{q}_1, \hat{q}_2, ..., \hat{q}_k\} \vdash \gamma_1$$
$$\{\hat{r}_1, \hat{r}_2, ..., \hat{r}_\ell\} \vdash (\neg\gamma_2)$$

Since both $\{\hat{q}_1, \hat{q}_2, ..., \hat{q}_k\}$ and $\{\hat{r}_1, \hat{r}_2, ..., \hat{r}_\ell\}$ are subsets of $\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\}$, we also have that

$$\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash \gamma_1$$
$$\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash (\neg\gamma_2)$$

More Cases

Case 2a Continued : The proofs related to these two lines

$$\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash \gamma_1$$
$$\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash (\neg\gamma_2)$$

can be combined using an $\wedge$ i to get a proof of

$$\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash (\gamma_1 \wedge (\neg\gamma_2))$$

By the warm up problem today(!) we have that

$$(\gamma_1 \wedge (\neg\gamma_2)) \vdash (\neg(\gamma_1 \rightarrow \gamma_2))$$

Thus, as $\gamma = (\gamma_1 \rightarrow \gamma_2)$

$$(\gamma_1 \wedge (\neg\gamma_2)) \vdash (\neg\gamma)$$

which is what we wanted for Case 2a.

More of Case 2

Case 2b: Assume that $\gamma^t = \mathtt{T}$ so that one of

- $(\gamma_1)^t = \mathtt{T}$ and $(\gamma_2)^t = \mathtt{T}$
- $(\gamma_1)^t = \mathtt{F}$ and $(\gamma_2)^t = \mathtt{T}$
- $(\gamma_1)^t = \mathtt{F}$ and $(\gamma_2)^t = \mathtt{F}$

holds. You can hopefully see how these cases are similar. We would also have to prove the claim for the other binary connectives. These are left as exercises. This completes the sublemma.

Reminder and Hammer Time!

We want to prove:

Lemma 2: For any well-formed formula γ, if $\emptyset \vDash \gamma$, then $\emptyset \vdash \gamma$.

We have proven that

Sublemma: For any formula γ containing atoms $p_1, ..., p_n$ and any valuation t;

- If $\gamma^t = \texttt{T}$ then $\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash \gamma$
- If $\gamma^t = \texttt{F}$ then $\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash (\neg\gamma)$

To prove Lemma 2, we now use LEM n times for each of $(p_i \vee (\neg p_i))$ and proceed to do 2^n $\vee$e applications to get to a situation where we have some tautology γ consisting of these p_i atoms for which we know that the sublemma case 1 can be applied 2^n times. It will somewhat look like the next slide...

Outline of the Proof of a Tautology

1.	$p_1 \vee (\neg p_1)$	L.E.M.
2.	$p_2 \vee (\neg p_2)$	L.E.M.
$\vdots$	$\vdots$	
n.	$p_n \vee (\neg p_n)$	L.E.M.
$n+1$.	p_1	assumption
	p_2	assumption
	$\vdots$	
	φ	
	$(\neg p_2)$	assumption
	$\vdots$	
	γ	
m.	γ	$\vee$e: 2, ...

$m+1$.	$(\neg p_1)$	assumption
	$\vdots$	
	$\vdots$	
	γ	$\vee$e: $m+1$, ...
ℓ.	γ	$\vee$e: 1, $m-(n+1)$, $\ell-(m+1)$

Once each variable is assumed true or false, the previous sublemma provides a proof.

Proof of Lemma 3

If $\emptyset \vdash (\alpha_0 \rightarrow (\alpha_1 \rightarrow (... \rightarrow (\alpha_n \rightarrow \beta)..)))$, then $\{\alpha_0, \alpha_1, ..., \alpha_n\} \vdash \beta$

Proof:

1.		
$\vdots$	$\vdots$	
k.	$(\alpha_0 \rightarrow (\alpha_1 \rightarrow (... \rightarrow (\alpha_n \rightarrow \beta)..)))$	Some Rule
$k+1$.	α_0	Premise
$k+2$.	$(\alpha_1 \rightarrow (\alpha_2 \rightarrow (... \rightarrow (\alpha_n \rightarrow \beta)..)))$	$\rightarrow$ e: $k, k+1$
$k+2$.	α_1	Premise
$k+3$.	$(\alpha_2 \rightarrow (\alpha_3 \rightarrow (... \rightarrow (\alpha_n \rightarrow \beta)..)))$	$\rightarrow$ e: $k+2, k+3$
$\vdots$	$\vdots$	
$k+(2n+1)$.	α_n	Premise
$k+(2n+2)$.	β	$\rightarrow$ e: $k+(2n+1), k+(2n+2)$

completing the proof. (Should use induction to make this formal).

Summing it All Up

Reminder: Putting it all together:

If $\Sigma \vDash \beta$ holds, then $\sigma \vdash \beta$ is valid.

- **Lemma 1:** If $\Sigma \vDash \beta$, then

$$\emptyset \vDash (\alpha_0 \rightarrow (\alpha_1 \rightarrow (... \rightarrow (\alpha_n \rightarrow \beta)..)))$$

- **Lemma 2:** Tautologies are provable, that is, for any well-formed formula γ, if $\emptyset \vDash \gamma$, then $\emptyset \vdash \gamma$. (This is the key step!)
- **Lemma 3:** If $\emptyset \vdash (\alpha_0 \rightarrow (\alpha_1 \rightarrow (... \rightarrow (\alpha_n \rightarrow \beta)..)))$, then

$$\{\alpha_0, \alpha_1, ..., \alpha_n\} \vdash \beta$$

that is, $\Sigma \vdash \beta$.

Immediate applications of Lemma 1, Lemma 2 (with $\gamma = (\alpha_0 \rightarrow (\alpha_1 \rightarrow (... \rightarrow (\alpha_n \rightarrow \beta)..))))$ and Lemma 3 completes the proof.