Warm-Up Problem

Show that $\{((\neg p) \to (\neg q))\} \vdash (q \to p)$

Propositional Logic: Natural Deduction Derived Rules and Examples

Carmen Bruni

Lecture 8

Based on work by J. Buss, A. Gao, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Learning goals

Natural deduction in propositional logic

- Describe the derived rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{(p \to q), (\neg q)\} \vdash (\neg p)$.

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The principle of *modus tollens*: $\{(p \to q), (\neg q)\} \vdash (\neg p)$.

1.	$(p \to q)$	Premise
2.	$(\neg q)$	Premise
	$(\neg p)$	??

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{(p \to q), (\neg q)\} \vdash (\neg p)$.

1.	$(p \to q)$	Premise
2.	$(\neg q)$	Premise
3.	p	Assumption
4.		
5.	$\bot$	
6.	$(\neg p)$	$\neg$i: ??

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{(p \to q), (\neg q)\} \vdash (\neg p)$.

1.	$(p \to q)$	Premise
2.	$(\neg q)$	Premise
3.	p	Assumption
4.	q	$\to$e: 3, 1
5.	$\bot$	
6.	$(\neg p)$	$\neg$i: ??

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{(p \to q), (\neg q)\} \vdash (\neg p)$.

1.	$(p \to q)$	Premise
2.	$(\neg q)$	Premise
3.	p	Assumption
4.	q	$\to$e: 3, 1
5.	$\bot$	
6.	$(\neg p)$	$\neg$i: 3–5

What should go on line 5?

Example: "*Modus tollens*"

Modus tollens is sometimes taken as a "derived rule":

$$\frac{(\alpha \to \beta) \quad (\neg\beta)}{(\neg\alpha)} \text{ MT}$$

Derived Rules

Whenever we have a proof of the form $\Gamma \vdash \alpha$, we can consider it as a derived rule:

$$\frac{\Gamma}{\alpha}$$

If we use this in a proof, it can be replaced by the original proof of $\Gamma \vdash \alpha$. The result is a proof using only the basic rules.

Using derived rules does not expand the things that can be proved. But they can make it easier to find a proof.

More Derived Rules

Double-Negation Introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))}\ \neg\neg\text{i}$$

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1.	α	Premise
	$(\neg(\neg\alpha))$	??

Double-Negation Introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))}\ \neg\neg\text{i}$$

1.	α	Premise
2.	$(\neg\alpha)$	Assumption
3.		
4.	$(\neg(\neg\alpha))$	$\neg$i: ??

More Derived Rules

Double-Negation Introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))}\ \neg\neg\text{i}$$

1.	α	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\bot$	$\neg$e: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: ??

More Derived Rules

Double-Negation Introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))}\ \neg\neg\text{i}$$

1.	α	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\bot$	$\neg$e: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: 2–3

More Derived Rules

*Proof by contradiction (**reductio ad absurdum**):*

$$\dfrac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots\\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

More Derived Rules

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots \\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \to \bot)$	Premise
5.	α	??

More Derived Rules

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots \\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \rightarrow \bot)$	Premise
	$(\neg(\neg\alpha))$	??
5.	α	$\neg\neg$e: ??

More Derived Rules

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots\\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha)\rightarrow\bot)$	Premise
2.	$(\neg\alpha)$	Assumption
3.		
4.	$(\neg(\neg\alpha))$	$\neg$i: ??
5.	α	$\neg\neg$e: 4

More Derived Rules

Proof by contradiction (***reductio ad absurdum***):

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots \\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \rightarrow \bot)$	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\bot$	$\rightarrow$e: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: ??
5.	α	$\neg\neg$e: 4

More Derived Rules

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots \\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \rightarrow \bot)$	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\bot$	$\rightarrow$e: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: 2–3
5.	α	$\neg\neg$e: 4

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg \alpha))}\ \text{LEM}$$

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg \alpha))} \text{ LEM}$$

1.

9. $(\alpha \vee (\neg \alpha))$??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.

	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg \alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg \alpha)))$	Assumption
		??
6.	??	??
7.	$\perp$	$\neg$e: ??
8.	$(\neg(\neg(\alpha \vee (\neg \alpha))))$	$\neg$i: ??
9.	$(\alpha \vee (\neg \alpha))$	$\neg\neg$e: ??

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.		??
4.		??
5.	$(\neg\alpha)$	¬i: ??
6.	??	??
7.	$\bot$	¬e: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	¬i: ??
9.	$(\alpha \vee (\neg\alpha))$	¬¬e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	??
4.		??
5.	$(\neg\alpha)$	$\neg$i: ??
6.	??	??
7.	$\bot$	$\neg$e: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.		??
5.	$(\neg\alpha)$	$\neg$i: ??
6.	??	??
7.	$\bot$	$\neg$e: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	??
5.	$(\neg\alpha)$	$\neg$i: ??
6.	??	??
7.	$\bot$	$\neg$e: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\neg$e: 1,3
5.	$(\neg\alpha)$	$\neg$i: ??
6.	??	??
7.	$\bot$	$\neg$e: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\neg$e: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	??	??
7.	$\bot$	$\neg$e: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

The "Law of Excluded Middle" ***(tertiam non datur)****:*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\perp$	$\neg$e: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	??
7.	$\perp$	$\neg$e: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\neg$e: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\bot$	$\neg$e: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\neg$e: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\bot$	$\neg$e: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\perp$	$\neg$e: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\perp$	$\neg$e: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha))))$	$\neg$i: 1–7
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\perp$	$\neg$e: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\perp$	$\neg$e: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: 1–7
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: 8

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\perp$	$\neg$e: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\perp$	$\neg$e: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha))))$	$\neg$i: 1–7
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: 8

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\neg$e: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\bot$	$\neg$e: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha))))$	$\neg$i: 1–7
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: 8

Strategies for natural deduction proofs

1. Work forward from the premises. Can you apply an elimination rule?
2. Work backwards from the conclusion. What introduction rule do you need to use at the end?
3. Stare at the formula. Notice its structure. Use it to guide your proof.
4. If a direct proof doesn't work, try a proof by contradiction.

Further Examples of Natural Deduction

Example. Show that $\{(p \to q)\} \vdash ((r \vee p) \to (r \vee q))$.

Write down premises and conclusion (step 1).
No elimination applies (step 2). Thus try $\to$i (step 3).

1.	$(p \to q)$	Premise
	$((r \vee p)) \to ((r \vee q))$	??

Example. Show that $\{(p \to q)\} \vdash ((r \lor p) \to (r \lor q))$.

In the sub-proof, try $\lor$-elimination on the assumption (step 2).

1.	$(p \to q)$	Premise
2.	$(r \lor p)$	Assumption
	$(r \lor q)$	??
9.	$((r \lor p)) \to ((r \lor q))$	??

Further Examples of Natural Deduction

Example. Show that $\{(p \to q)\} \vdash ((r \vee p) \to (r \vee q))$.

To justify lines 4 and 7:
No elimination applies from the assumptions (step 2).
What about $\vee$-introduction for the conclusion (step 3)?

1.	$(p \to q)$	Premise
2.	$(r \vee p)$	Assumption
3.	r	Assumption
4.	$(r \vee q)$	??
5.	p	Assumption
6.		
7.	$(r \vee q)$	??
8.	$(r \vee q)$	$\vee$e: ??
9.	$((r \vee p)) \to ((r \vee q))$	$\to$i: 2–8

Example. Show that $\{(p \to q)\} \vdash ((r \lor p) \to (r \lor q))$.

It works!

1.	$(p \to q)$	Premise
2.	$(r \lor p)$	Assumption
3.	r	Assumption
4.	$(r \lor q)$	$\lor$i: 3
5.	p	Assumption
6.	q	$\to$e: 5, 1
7.	$(r \lor q)$	$\lor$i: 6
8.	$(r \lor q)$	$\lor$e: 2, 3–4, 5–7
9.	$((r \lor p)) \to ((r \lor q))$	$\to$i: 2–8

Handout

Try some of the problems on the handout! You may work in small groups if you wish or on your own.