Warm-Up Problem

Show that

$$\{(p \to q), (\neg q)\} \vDash (\neg p)$$

using a truth table and without using a truth table.

Have you started to write a list of definitions? Why not start now?

Propositional Logic: Natural Deduction

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Lecture 7

Based on work by J. Buss, A. Gao, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Learning goals

Natural deduction in propositional logic

- Describe rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

Reflexivity / Premise

If you want to write down a previous formula in the proof again, you can do it by *reflexivity*.

Name	**$\vdash$-notation**	**inference notation**
Reflexivity, or Premise	$\Sigma, \alpha \vdash \alpha$	$\dfrac{\alpha}{\alpha}$

The notation on the right: Given the formulas above the line, we can infer the formula below the line.

The version in the centre reminds us of the role of assumptions in Natural Deduction. Other rules will make more use of it.

An example using reflexivity

Here is a proof of $\{p, q\} \vdash p$.

1.	p	Premise
2.	q	Premise
3.	p	Reflexivity: 1

Alternatively, we could simply write

1.	p	Premise

and be done.

For each symbol, the rules come in pairs.

- An “introduction rule” adds the symbol to the formula.
- An “elimination rule” removes the symbol from the formula.

Rules for Conjunction

Name	**$\vdash$-notation**	**inference notation**
$\wedge$-introduction ($\wedge$i)	If $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$, then $\Sigma \vdash (\alpha \wedge \beta)$	$\dfrac{\alpha \quad \beta}{(\alpha \wedge \beta)}$

Name	**$\vdash$-notation**	**inference notation**
$\wedge$-elimination ($\wedge$e)	If $\Sigma \vdash (\alpha \wedge \beta)$, then $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$	$\dfrac{(\alpha \wedge \beta)}{\alpha} \quad \dfrac{(\alpha \wedge \beta)}{\beta}$

Example: Conjunction Rules

Example. Show that $\{(p \wedge q)\} \vdash (q \wedge p)$.

Example: Conjunction Rules

Example. Show that $\{(p \wedge q)\} \vdash (q \wedge p)$.

1.	$(p \wedge q)$	Premise
2.	q	$\wedge$e: 1
3.	p	$\wedge$e: 1
4.	$(q \wedge p)$	$\wedge$i: 2, 3

Example: Conjunction Rules (2)

Example. Show that $\{(p \wedge q), r\} \vdash (q \wedge r)$.

Example: Conjunction Rules (2)

Example. Show that $\{(p \wedge q), r\} \vdash (q \wedge r)$.

1.	$(p \wedge q)$	Premise
2.	r	Premise
3.	q	$\wedge$e: 1
4.	$(q \wedge r)$	$\wedge$i: 3, 2

Rules for Implication: $\rightarrow$e

Name	$\vdash$-notation	inference notation
$\rightarrow$-elimination ($\rightarrow$e) (*modus ponens*)	If $\Sigma \vdash (\alpha \rightarrow \beta)$ and $\Sigma \vdash \alpha$, then $\Sigma \vdash \beta$	$\dfrac{(\alpha \rightarrow \beta) \quad \alpha}{\beta}$

In words:

If you assume α is true and α implies β, then you may conclude β.

Rules for Implication: $\rightarrow$i

Name	**$\vdash$-notation**	**inference notation**
$\rightarrow$-introduction ($\rightarrow$i)	If $\Sigma, \alpha \vdash \beta$, then $\Sigma \vdash (\alpha \rightarrow \beta)$	$\dfrac{\boxed{\begin{matrix}\alpha \\ \vdots \\ \beta\end{matrix}}}{(\alpha \rightarrow \beta)}$

The "box" denotes a sub-proof. In the sub-proof, we starts by assuming that α is true (a premise of the sub-proof), and we conclude that β is true.

Nothing inside the sub-proof may come out.

Outside of the sub-proof, we could only use the sub-proof as a whole.

Example: Rule $\rightarrow$i and sub-proofs

Example. Give a proof of $\{(p \rightarrow q), (q \rightarrow r)\} \vdash (p \rightarrow r)$.

To start, we write down the premises at the beginning, and the conclusion at the end.

What next?

1.	$(p \rightarrow q)$	Premise
2.	$(q \rightarrow r)$	Premise
	$(p \rightarrow r)$	???

Example. Give a proof of $\{(p \to q), (q \to r)\} \vdash (p \to r)$.

To start, we write down the premises at the beginning, and the conclusion at the end.

1.	$(p \to q)$	Premise
2.	$(q \to r)$	Premise
3.	p	Assumption
4.		
5.		
6.	$(p \to r)$	$\to$i: ??

What next?

The goal "$(p \to r)$" contains $\to$.
Let's try rule $\to$i....

Example: Rule $\to$i and sub-proofs

Example. Give a proof of $\{(p \to q), (q \to r)\} \vdash (p \to r)$.

To start, we write down the premises at the beginning, and the conclusion at the end.

1.	$(p \to q)$	Premise
2.	$(q \to r)$	Premise
3.	p	Assumption
4.	q	$\to$e: 1, 3
5.	r	$\to$e: 2, 4
6.	$(p \to r)$	$\to$i: ??

(Lines 3–5 are enclosed in a box.)

What next?

The goal "$(p \to r)$" contains $\to$. Let's try rule $\to$i....

Inside the sub-proof, we can use rule $\to$e.

Example: Rule $\rightarrow$i and sub-proofs

Example. Give a proof of $\{(p \rightarrow q), (q \rightarrow r)\} \vdash (p \rightarrow r)$.

To start, we write down the premises at the beginning, and the conclusion at the end.

1.	$(p \rightarrow q)$	Premise	
2.	$(q \rightarrow r)$	Premise	
3.	p	Assumption	(sub-proof begins)
4.	q	$\rightarrow$e: 1, 3	
5.	r	$\rightarrow$e: 2, 4	(sub-proof ends)
6.	$(p \rightarrow r)$	$\rightarrow$i: 3–5	

What next?

The goal "$(p \rightarrow r)$" contains $\rightarrow$. Let's try rule $\rightarrow$i....

Inside the sub-proof, we can use rule $\rightarrow$e.

Done!

Rules of Disjunction: $\lor$i and $\lor$e

Name	**$\vdash$-notation**	**inference notation**
$\lor$-introduction ($\lor$i)	If $\Sigma \vdash \alpha$, then $\Sigma \vdash (\alpha \lor \beta)$ and $\Sigma \vdash (\beta \lor \alpha)$	$\dfrac{\alpha}{(\alpha \lor \beta)}$ $\quad$ $\dfrac{\alpha}{(\beta \lor \alpha)}$
$\lor$-elimination ($\lor$e)	If Σ, $\alpha_1 \vdash \beta$ and Σ, $\alpha_2 \vdash \beta$, then Σ, $(\alpha_1 \lor \alpha_2) \vdash \beta$	$\dfrac{(\alpha_1 \lor \alpha_2) \quad \boxed{\begin{matrix}\alpha_1 \\ \vdots \\ \beta\end{matrix}} \quad \boxed{\begin{matrix}\alpha_2 \\ \vdots \\ \beta\end{matrix}}}{\beta}$

$\lor$e is also known as "proof by cases".

Example: Or-Introduction and -Elimination

Example: Show that $\{p \vee q\} \vdash (p \to q) \vee (q \to p)$.

Example: Or-Introduction and -Elimination

Example: Show that $\{p \lor q\} \vdash (p \to q) \lor (q \to p)$.

1.	$p \lor q$	Premise
2.	p	Assumption
3.	q	Assumption
4.	p	Reflexivity: 2
5.	$q \to p$	$\to$i: 3–4
6.	$(p \to q) \lor (q \to p)$	$\lor$i: 5
7.	q	Assumption
8.	p	Assumption
9.	q	Reflexivity: 7
10.	$p \to q$	$\to$i: 8–9
11.	$(p \to q) \lor (q \to p)$	$\lor$i: 10
12.	$(p \to q) \lor (q \to p)$	$\lor$e: 1, 2–6, 7–11

Negation

We shall treat negation by considering contradictions.

We shall use the notation $\perp$ to represent any contradiction.
It may appear in proofs as if it were a formula.

The elimination rule for negation:

Name	**$\vdash$-notation**	**inference notation**
$\perp$-introduction, or $\neg$-elimination ($\neg$e)	$\Sigma,\ \alpha,\ (\neg\alpha) \vdash \perp$	$\dfrac{\alpha \quad (\neg\alpha)}{\perp}$

If we have both α and $(\neg\alpha)$, then we have a contradiction.

Negation Introduction ($\neg$i)

If an assumption α leads to a contradiction, then derive $(\neg\alpha)$.

Name	**$\vdash$-notation**	**inference notation**
$\neg$-introduction ($\neg$i)	If $\Sigma, \alpha \vdash \bot$, then $\Sigma \vdash (\neg\alpha)$	$\dfrac{\boxed{\begin{matrix}\alpha \\ \vdots \\ \bot\end{matrix}}}{(\neg\alpha)}$

Example: Negation

Example. Show that $\{\alpha \to (\neg\alpha)\} \vdash (\neg\alpha)$.

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1.	$\alpha \to (\neg\alpha)$	Premise
	$(\neg\alpha)$	??

Example: Negation

Example. Show that $\{\alpha \to (\neg\alpha)\} \vdash (\neg\alpha)$.

1.	$\alpha \to (\neg\alpha)$	Premise
2.	α	Assumption
3.		
4.	$\bot$	??
5.	$(\neg\alpha)$	$\neg$i: 2–?

Example. Show that $\{\alpha \to (\neg\alpha)\} \vdash (\neg\alpha)$.

1.	$\alpha \to (\neg\alpha)$	Premise
2.	α	Assumption
3.	$(\neg\alpha)$	$\to$e: 1, 2
4.	$\bot$	??
5.	$(\neg\alpha)$	$\neg$i: 2–?

Example: Negation

Example. Show that $\{\alpha \to (\neg\alpha)\} \vdash (\neg\alpha)$.

1.	$\alpha \to (\neg\alpha)$	Premise
2.	α	Assumption
3.	$(\neg\alpha)$	$\to$e: 1, 2
4.	$\perp$	$\neg$e: 2, 3
5.	$(\neg\alpha)$	$\neg$i: 2–4

Double-Negation Elimination:

Name	**$\vdash$-notation**	**inference notation**
$\neg\neg$-elimination ($\neg\neg$e)	If $\Sigma \vdash (\neg(\neg\alpha))$, then $\Sigma \vdash \alpha$	$\dfrac{(\neg(\neg\alpha))}{\alpha}$

Contradiction Elimination:

Name	**$\vdash$-notation**	**inference notation**
$\bot$-elimination ($\bot$e)	If $\Sigma \vdash \bot$, then $\Sigma \vdash \alpha$	$\dfrac{\bot}{\alpha}$

A Redundant Rule

The rule of $\bot$-elimination is not actually needed.

Suppose a proof has

27.	$\bot$	$\langle$*some rule*$\rangle$
28.	α	$\bot$e: 27.

We can replace these by

27.	$\bot$	$\langle$*some rule*$\rangle$
28.	$(\neg\alpha)$	Assumption
29.	$\bot$	Reflexivity: 27
30.	$(\neg(\neg\alpha))$	$\neg$i: 28–29
31.	α	$\neg\neg$e: 30.

Thus any proof that uses $\bot$e can be modified into a proof that does not.