# Warm Up Problem

Let $t$ be a truth valuation, $p$ and $q$ propositional variables with $t(p) = \mathtt{T}$, and let $\alpha$ be a well-formed formula. Which of the following are written using the correct notation? What does the equality-type character mean on each line?

1. $\alpha = \mathtt{T}$
2. $\alpha^t = \mathtt{T}$
3. $\alpha^t \equiv \mathtt{T}$
4. $(p \wedge q)^t = (\mathtt{T} \wedge q)$
5. $(p \wedge q)^t \equiv (\mathtt{T} \wedge q^t)$
6. $(p \wedge q) = (q \wedge p)$
7. $(p \wedge q) \equiv (q \wedge p)$

# *Propositional Logic: Semantics*

Carmen Bruni
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Lecture 5

# Outline for today

- How can we prove that two formulas have the same meaning? (Logical equivalence)
- Analyzing dead code.
- Circuit design.

# Learning goals

Logical equivalence of formulas:

- Prove that the logical equivalence of formulas using truth tables and/or logical identities.
- Describe strategies to prove logical equivalence using logical identities.
- Translate a condition in a block of code into a propositional logic formula.
- Simplify code using truth tables and logical identities.
- Determine whether a piece of code is live or dead using truth tables and logical identities.

Dead Code:

- Determine whether or not a specific line of code is reachable.
- Give examples of parameters that reach a specific line of code.

# Definition of logical equivalence

Two formulas $\alpha$ and $\beta$ are logically equivalent if and only if they have the same value under any valuation.

- $\alpha^t = \beta^t$, for every valuation $t$.
- $\alpha$ and $\beta$ must have the same final column in their truth tables.
- $(\alpha \leftrightarrow \beta)$ is a tautology.

# Why do we care about logical equivalence?

- Will I lose marks if I provide a solution that is syntactically different but logically equivalent to the provided solution?
- Do these two circuits behave the same way?
- Do these two pieces of code fragments behave the same way?

You already know one way of proving logical equivalent. What is it?

Theorem: $(((\neg p) \wedge q) \vee p) \equiv (p \vee q)$.

# Logical Identities

Commutativity

$$(\alpha \land \beta) \equiv (\beta \land \alpha)$$
$$(\alpha \lor \beta) \equiv (\beta \lor \alpha)$$

Associativity

$$(\alpha \land (\beta \land \gamma)) \equiv ((\alpha \land \beta) \land \gamma)$$
$$(\alpha \lor (\beta \lor \gamma)) \equiv ((\alpha \lor \beta) \lor \gamma)$$

Distributivity

$$(\alpha \lor (\beta \land \gamma)) \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma))$$
$$(\alpha \land (\beta \lor \gamma)) \equiv ((\alpha \land \beta) \lor (\alpha \land \gamma))$$

Idempotence

$$(\alpha \lor \alpha) \equiv \alpha$$
$$(\alpha \land \alpha) \equiv \alpha$$

Double Negation

$$(\neg(\neg\alpha)) \equiv \alpha$$

De Morgan's Laws

$$(\neg(\alpha \land \beta)) \equiv ((\neg\alpha) \lor (\neg\beta))$$
$$(\neg(\alpha \lor \beta)) \equiv ((\neg\alpha) \land (\neg\beta))$$

# Logical Identities, cont'd

Simplification I (Absorption)

$$(\alpha \wedge \mathtt{T}) \equiv \alpha$$
$$(\alpha \vee \mathtt{T}) \equiv \mathtt{T}$$
$$(\alpha \wedge \mathtt{F}) \equiv \mathtt{F}$$
$$(\alpha \vee \mathtt{F}) \equiv \alpha$$

Simplification II

$$(\alpha \vee (\alpha \wedge \beta)) \equiv \alpha$$
$$(\alpha \wedge (\alpha \vee \beta)) \equiv \alpha$$

Implication

$$(\alpha \rightarrow \beta) \equiv ((\neg \alpha) \vee \beta)$$

Contrapositive

$$(\alpha \rightarrow \beta) \equiv ((\neg \beta) \rightarrow (\neg \alpha))$$

Equivalence

$$(\alpha \leftrightarrow \beta) \equiv ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$$

Excluded Middle

$$(\alpha \vee (\neg \alpha)) \equiv \mathtt{T}$$

Contradiction

$$(\alpha \wedge (\neg \alpha)) \equiv \mathtt{F}$$

# A logical equivalence proof

Theorem: $(((\neg p) \wedge q) \vee p) \equiv (p \vee q)$.

# A logical equivalence proof

Theorem: $(((\neg p) \wedge q) \vee p) \equiv (p \vee q)$.

Proof.

| | |
|---|---|
| $(((\neg p) \wedge q) \vee p)$ | |
| $\equiv (((\neg p) \vee p) \wedge (q \vee p))$ | Distributivity |
| $\equiv (\mathtt{T} \wedge (q \vee p))$ | Excluded Middle |
| $\equiv (q \vee p)$ | Simplification I |
| $\equiv (p \vee q)$ | Commutativity |

□

# A practice problem

"If it is sunny, I will play golf, provided that I am relaxed."

$s$: it is sunny. $g$: I will play golf. $r$: I am relaxed.

A few translations:

1. $(s \rightarrow (r \rightarrow g))$
2. $(r \rightarrow (s \rightarrow g))$
3. $((s \wedge r) \rightarrow g)$

Theorem: All three translations are logically equivalent.
Proof: Done in class.

# How do you prove non-equivalence?

"If it snows then I won't go to class, but I will do my assignment."

$s$: it snows. $c$: I will go to class. $a$: I will do my assignment.

2 possible translations:

1. $((s \rightarrow (\neg c)) \wedge a)$
2. $(s \rightarrow ((\neg c) \wedge a))$

Theorem: $((s \rightarrow (\neg c)) \wedge a)$ and $(s \rightarrow ((\neg c) \wedge a))$ are not logically equivalent.

# How do you prove non-equivalence?

"If it snows then I won't go to class, but I will do my assignment."

$s$: it snows. $c$: I will go to class. $a$: I will do my assignment.

2 possible translations:

1. $((s \to (\neg c)) \wedge a)$
2. $(s \to ((\neg c) \wedge a))$

Theorem: $((s \to (\neg c)) \wedge a)$ and $(s \to ((\neg c) \wedge a))$ are not logically equivalent.

Which valuation $t$ can we use to prove this theorem?

a. $s^t = \mathtt{F}$, $(\neg c)^t = \mathtt{F}$, $a^t = \mathtt{F}$
b. $s^t = \mathtt{F}$, $(\neg c)^t = \mathtt{T}$, $a^t = \mathtt{F}$
c. $s^t = \mathtt{T}$, $(\neg c)^t = \mathtt{T}$, $a^t = \mathtt{T}$
d. Two of these.
e. All of these.

# Collected Wisdom

- Try getting rid of $\rightarrow$ and $\leftrightarrow$.
- Try moving negations inward. $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$.
- Work from the more complex side first, BUT
- Switch to different strategies/sides when you get stuck.
- In the end, write the proof in clean ' one-side-to-the-other' form and double-check steps.

```
if ( (input > 0) or (not output) ) {
    if ( not (output and (queuelength < 100) ) ) {
        P1
    } else if ( output and (not (queuelength < 100)) ) {
        P2
    } else { P3 }
} else { P4 }
```

When does each piece of code get executed?

Let $i$: `input > 0`,
$u$: `output`,
$q$: `queuelength < 100`.

# A Code Example, cont'd

| $i$ | $u$ | $q$ | $(i \vee (\neg u))$ | $(\neg(u \wedge q))$ | $(u \wedge (\neg q))$ | |
|---|---|---|---|---|---|---|
| T | T | T | T | | | |
| T | T | F | T | | | |
| T | F | T | T | | | |
| T | F | F | T | | | |
| F | T | T | F | | | $P_4$ |
| F | T | F | F | | | $P_4$ |
| F | F | T | T | | | |
| F | F | F | T | | | |

# A Code Example, cont'd

| $i$ | $u$ | $q$ | $(i \vee (\neg u))$ | $(\neg(u \wedge q))$ | $(u \wedge (\neg q))$ | |
|---|---|---|---|---|---|---|
| T | T | T | T | F | F | $P_3$ |
| T | T | F | T | T | | $P_1$ |
| T | F | T | T | T | | $P_1$ |
| T | F | F | T | T | | $P_1$ |
| F | T | T | F | | | $P_4$ |
| F | T | F | F | | | $P_4$ |
| F | F | T | T | T | | $P_1$ |
| F | F | F | T | T | | $P_1$ |

# Finding Live Code

Prove that $P_3$ is live code. That is, the conditions leading to $P_3$ is satisfiable.

Theorem:
$$\Big((i \vee (\neg u)) \wedge \big((\neg(\neg(u \wedge q))) \wedge (\neg(u \wedge (\neg q)))\big)\Big) \equiv ((i \wedge u) \wedge q)$$

Proof: In class

Prove that the two code fragments are equivalent.

Listing 1: Your code

```
if (i || !u) {
  if (!(u && q)) {
    P1
  } else if (u && !q) {
    P2
  } else { P3 }
} else { P4 }
```

Listing 2: Your friend's code

```
if ((i && u) && q) {
  P3
} else if (!i && u) {
  P4
} else {
  P1
}
```

# Simplifying Code

To prove that the two fragments are equivalent, show that each block of code $P_1$, $P_2$, $P_3$, and $P_4$ is executed under equivalent conditions.

| Block | Fragment 1 | Fragment 2 |
|---|---|---|
| $P_1$ | $\big(i \vee (\neg u)\big) \wedge \big(\neg(u \wedge q)\big)$ | $\big(\neg(i \wedge u \wedge q)\big) \wedge \big(\neg((\neg i) \wedge u)\big)$ |
| $P_2$ | $\big(i \vee (\neg u)\big) \wedge \big(\neg(\neg(u \wedge q))\big) \wedge \big(u \wedge (\neg q)\big)$ | F |
| $P_3$ | $\big(i \vee (\neg u)\big) \wedge \big(\neg(\neg(u \wedge q))\big) \wedge \big(\neg(u \wedge (\neg q))\big)$ | $(i \wedge u \wedge q)$ |
| $P_4$ | $\big(\neg(i \vee (\neg u))\big)$ | $\big(\neg(i \wedge u \wedge q)\big) \wedge \big((\neg i) \wedge u\big)$ |

# Another logic puzzle

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie.

You meet three inhabitants: Alice, Rex and Bob.

Alice says, "Rex is a knave."

Rex says, "It is false that Bob is a knave."

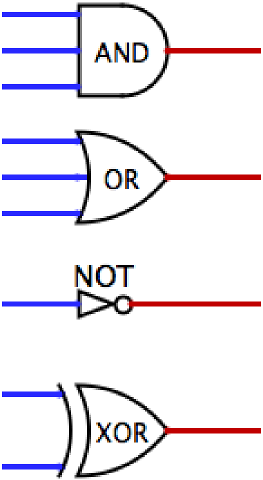
Bob claims, "I am a knight or Alice is a knight."

Can you determine who is a knight and who is a knave?

# Digital Circuits

- An electronic computer is made up of a number of circuits.
- The basic elements of circuits are called logic gates.
- A gate is an electronic device that operates on a collection of binary inputs and produces a binary output.

# Logical Gates



# A circuit design problem

Your instructors, Alice, Carmen, and Collin, are choosing questions to be put on the midterm. For each problem, each instructor votes either yes or not. A question is chosen if it receives two or more yes votes. Design a circuit, which outputs yes whenever a question is chosen.

# Draw the truth table

| x | y | z | output |
|---|---|---|---|
| T | T | T | |
| T | T | F | |
| T | F | T | |
| T | F | F | |
| F | T | T | |
| F | T | F | |
| F | F | T | |
| F | F | F | |

# Draw the truth table

| x | y | z | output |
|---|---|---|---|
| T | T | T | T |
| T | T | F | T |
| T | F | T | T |
| T | F | F | F |
| F | T | T | T |
| F | T | F | F |
| F | F | T | F |
| F | F | F | F |

1. Convert the truth table a propositional formula.

   For convenience, we will use the symbol $\oplus$ to represent an exclusive OR connective. This is a temporary convenience only. You are not allowed to use this connective unless otherwise specified in the problem.

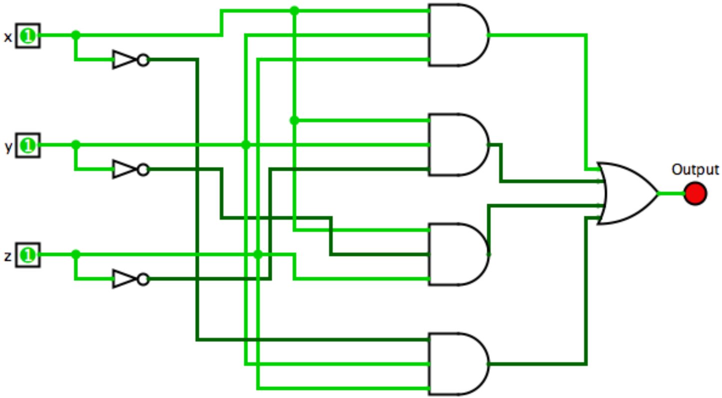
2. Then, convert the formula to a circuit.

# Solution 1

1. Convert each row of the truth table to a conjunction.
   - $((x \wedge y) \wedge z)$
   - $((x \wedge y) \wedge (\neg z))$
   - $((x \wedge (\neg y)) \wedge z)$
   - $(((\neg x) \wedge y) \wedge z)$
2. Connect all formulas to form a disjunction. (Below is not well formed to save confusion)

   $$((x \wedge y) \wedge z) \vee ((x \wedge y) \wedge (\neg z)) \vee ((x \wedge (\neg y)) \wedge z) \vee (((\neg x) \wedge y) \wedge z)$$
3. Draw the circuit.

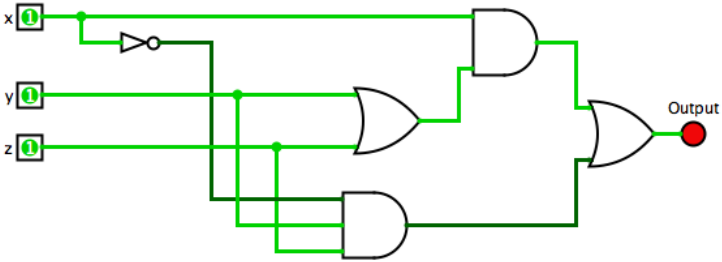
# Circuit 1



# Solution 2

1. Converts rows 1-3 to a propositional formula.
   $(x \wedge (y \vee z))$
2. Convert row 5 to a propositional formula.
   $(((\neg x) \wedge y) \wedge z)$
3. Connect all formulas into a disjunction.
   $((x \wedge (y \vee z)) \vee (((\neg x) \wedge y) \wedge z))$
4. Draw the circuit.

# Circuit 2



# Solution 3

1. Convert rows 1 and 5 into a propositional formula.
   $(y \wedge z)$
2. Convert rows 2 and 3 into a propositional formula.
   $(x \wedge (y \oplus z))$
3. Connect all formulas into a disjunction.
   $((y \wedge z) \vee (x \wedge (y \oplus z)))$
4. Draw the circuit.

# Circuit 3