Warm Up Problem

Translate the following form English to Propositional Logic. Write the following as a well formed formula in as many ways as you can.

I will wake up in the morning and I will drink coffee provided that it is a Monday.

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Let m be "I will wake up in the morning", c be "I will drink coffee" and d be "It is Monday".

Propositional Logic: Semantics

Carmen Bruni

With thanks to A. Gao for these slides!

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Lecture 4

Last Time

- More on structural induction
- Started the proof of Structural Induction (which we will discuss briefly now; rest is left as reading)

Main Theorem

Theorem (Unique Readability Theorem)

There is a unique way to construct every well-formed formula.

Proof

Let $P(\varphi)$ be the property that there is a unique way to construct the well-formed formula φ. We prove this property for all well-formed formulas φ by structural induction.

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Base Case: There is only one way to construct an atom.

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Inductive Hypothesis: Assume that $P(\alpha)$ and $P(\beta)$ are true for some well-formed formulas α and β.

Inductive Conclusion

We now have a few possibilities to consider:

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2. $\varphi = (\alpha \star \beta)$

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Each of these has two subcases...

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Subcase 2: If $\varphi = (\neg\alpha)$, then what if we could write $\varphi = (\alpha' \star \beta')$ for some other well-formed formulas α' and β'?

Then in this case, it must be that $\alpha = \gamma \star \varphi$, $\alpha' = \neg\gamma$ and $\beta' = \varphi$. However, we see that as α is well-formed, we have that γ is a proper prefix of α so by lemma 3, this has more open brackets than closed brackets.. However, α' is also well-formed and so it must have an equal number of open and closed brackets. This is a contradiction.

Subcases

Subcase 3: If $\varphi = (\alpha \star \beta)$, then what if we could write $\varphi = (\alpha' \star \beta')$ for some other well-formed formulas α' and β'?

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If $|\alpha'| = |\alpha|$, then $\alpha' = \alpha$ (both start at the second symbol of φ).
Thus also $\star = \star'$ and $\beta = \beta'$, as required.

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If $0 < |\alpha'| < |\alpha|$, then α' is a proper prefix of α.
Thus, by lemmas 2 and 3 on α, α' is not a formula; we have nothing to prove.

Proof

We have assumed that φ is $(\alpha \star \beta)$, where both α and β have property P.
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If $|\alpha'| > |\alpha|$,then $\alpha' = \alpha \star y$, where y is a proper prefix of β (or is empty).
By lemma 2, α has equally many '(' and ')', while by lemma 3 y has more '(' than ')'. Thus α' has more '(' than ')' and hence again by lemma 3, it is not a formula, and again we have nothing to prove.

Therefore α has a unique derivation in this case.

Subcases

Subcase 4: If $\varphi = (\alpha \star \beta)$, then what if we could write $\varphi = (\neg\alpha')$ for some other well-formed formulas α' and β'?

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Argue as in Subcase 2; Notice that $\alpha' = \gamma \star \varphi$ for some well formed formulas γ and φ which also satisfies $\alpha = \neg\gamma$ and $\beta = \varphi$. Again γ is a proper prefix of α' and so has more open brackets than closed brackets by lemma 3. This contradicts lemma 2 since this is also equal to $\alpha = \neg\gamma$.

Learning goals

By the end of this lecture, you should be able to

- Evaluate the truth value of a formula
  - Define a (truth) valuation.
  - Determine the truth value of a formula by using truth tables.
  - Determine the truth value of a formula by using valuation trees.
- Determine and prove whether a formula has a particular property
  - Define tautology, contradiction, and satisfiable formula.
  - Compare and contrast the three properties (tautology, contradiction, and satisfiable formula).
  - Prove whether a formula is a tautology, a contradiction, or satisfiable, using a truth table and/or a valuation tree.
  - Describe strategies to prove whether a formula is a tautology, a contradiction or a satisfiable formula.

The meaning of well-formed formulas

To interpret a formula, we have to give meanings to the propositional variables and the connectives.

A propositional variable has no intrinsic meaning; it gets a meaning via a valuation.

A (truth) valuation is a function $t : \mathcal{P} \mapsto \{\mathtt{F}, \mathtt{T}\}$ from the set of all proposition variables $\mathcal{P}$ to $\{\mathtt{F}, \mathtt{T}\}$. It assigns true/false to every propositional variable.

Two notations: $t(p)$ and p^t both denote the truth value of p under the truth valuation t.

Truth value of a formula

Fix a truth valuation t. Every formula α has a value under t, denoted α^t, determined as follows.

1. $p^t = t(p)$.
2. $(\neg\alpha)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \mathtt{F} \\ \mathtt{F} & \text{if } \alpha^t = \mathtt{T} \end{cases}$
3. $(\alpha \wedge \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \beta^t = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$
4. $(\alpha \vee \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \mathtt{T} \text{ or } \beta^t = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$
5. $(\alpha \rightarrow \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \mathtt{F} \text{ or } \beta^t = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$
6. $(\alpha \leftrightarrow \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \beta^t \\ \mathtt{F} & \text{otherwise} \end{cases}$

Truth tables for connectives

The unary connective $\neg$:

α	$(\neg\alpha)$
T	F
F	T

The binary connectives $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$:

α	β	$(\alpha \wedge \beta)$	$(\alpha \vee \beta)$	$(\alpha \rightarrow \beta)$	$(\alpha \leftrightarrow \beta)$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

Evaluating a formula using a truth table

Example. The truth table of $((p \vee q) \rightarrow (q \wedge r))$.

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p	q	r	$(p \vee q)$	$(q \wedge r)$	$((p \vee q) \rightarrow (q \wedge r))$
F	F	F	F	F	T
F	F	T	F	F	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	F	F
T	T	F	T	F	F
T	T	T	T	T	T

Evaluating a formula using a truth table

Build the truth table of $\big((p \rightarrow (\neg q)) \rightarrow (q \vee (\neg p))\big)$.

Understanding the disjunction and the biconditional

α	β	$(\alpha \vee \beta)$	Exclusive OR	Biconditional
F	F	F	F	T
F	T	T	T	F
T	F	T	T	F
T	T	T	F	T

- What is the difference between an inclusive OR (the disjunction) and an exclusive OR?
- What is the relationship between the exclusive OR and the biconditional?

A Small Theorem

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Proof: By structural induction. Let $R(\alpha)$ be "α has a value α^t in $\{\mathtt{F}, \mathtt{T}\}$".

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Theorem: Fix a truth valuation t. Every formula α has a value α^t in $\{\mathtt{F}, \mathtt{T}\}$.

Proof: By structural induction. Let $R(\alpha)$ be "α has a value α^t in $\{\mathtt{F}, \mathtt{T}\}$".

1. If α is a propositional variable, then t assigns it a value of $\mathtt{T}$ or $\mathtt{F}$ (by the definition of a truth valuation).
2. If α has a value in $\{\mathtt{F}, \mathtt{T}\}$, then $(\neg\alpha)$ also does, as shown by the truth table of $(\neg\alpha)$.
3. If α and β each has a value in $\{\mathtt{F}, \mathtt{T}\}$, then $(\alpha \star \beta)$ also does for every binary connective $\star$, as shown by the corresponding truth tables.

By the principle of structural induction, every formula has a value.

By the unique readability of formulas, we have proved that a formula has **only one** truth value under any truth valuation t. QED

Tautology, Contradiction, Satisfiable

A formula α is a *tautology* if and only if
for every truth valuation t, $\alpha^t = \texttt{T}$.

A formula α is a *contradiction* if and only if
for every truth valuation t, $\alpha^t = \texttt{F}$.

A formula α is *satisfiable* if and only if
there exists a truth valuation t such that $\alpha^t = \texttt{T}$.

Examples

Let p be a Propositional variable.

- The formula $(p \vee (\neg p))$ is a tautology (and satisfiable)
- The formula $(p \wedge (\neg p))$ is a contradiction (and not satisfiable)
- The formula $(p \rightarrow (\neg p))$ is satisfiable (but not a tautology)

How to determine the properties of a formula

- Truth table
- Valuation tree
- Reasoning

Valuation Tree

Rather than fill out an entire truth table, we can analyze what happens if we plug in a truth value for one variable.

$\neg \mathtt{T}$	$\mathtt{F}$
$\neg \mathtt{F}$	$\mathtt{T}$

$p \wedge \mathtt{T}$	p
$p \wedge \mathtt{F}$	$\mathtt{F}$
$p \wedge p$	p

$p \vee \mathtt{T}$	$\mathtt{T}$
$p \vee \mathtt{F}$	p
$p \vee p$	p

$p \rightarrow \mathtt{T}$	$\mathtt{T}$
$p \rightarrow \mathtt{F}$	$\neg p$
$\mathtt{T} \rightarrow p$	p
$\mathtt{F} \rightarrow p$	$\mathtt{T}$
$p \rightarrow p$	$\mathtt{T}$

We can evaluate a formula by using these rules to construct a *valuation tree*.

Example of a valuation tree

Example. Show that $\big(((p \wedge q) \to (\neg r)) \wedge (p \to q)\big) \to \big(p \to (\neg r)\big)$ is a tautology by using a valuation tree.

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Suppose $t(p) = \texttt{T}$. We put T in for p:

$$\big(((\texttt{T} \wedge q) \to (\neg r)) \wedge (\texttt{T} \to q)\big) \to \big(\texttt{T} \to (\neg r)\big) \; .$$

Based on the truth tables for the connectives, the formula becomes $\big(((q \to (\neg r)) \wedge q) \to (\neg r)\big)$.

If $t(q) = \texttt{T}$, this yields $((\neg r) \to (\neg r))$ and then T. (Check!).
If $t(q) = \texttt{F}$, it yields $(\texttt{F} \to (\neg r))$ and then T. (Check!).

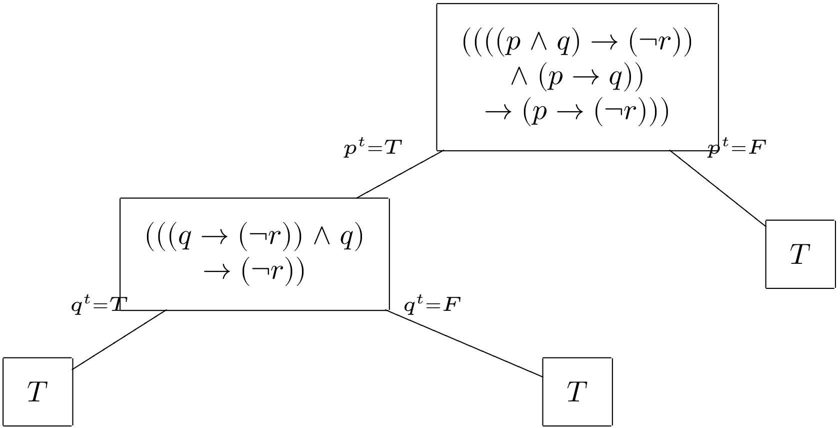
Suppose $t(p) = \texttt{F}$. We get

$$\big(((\texttt{F} \wedge q) \to (\neg r)) \wedge (\texttt{F} \to q)\big) \to \big(\texttt{F} \to (\neg r)\big) \; ,$$

Simplification yields $\big((\texttt{F} \to (\neg r)) \wedge \texttt{T}\big) \to \texttt{T}$ and eventually T.

Thus every valuation makes the formula true, as required.

Valuation Tree (Do not break on multiple lines!)



Examples

Determine which of the following are satisfiable, form a tautology or form a contradiction. Use a truth table. Repeat using a valuation tree.

1. $\Big(\big(((p \vee q) \leftrightarrow (\neg r)) \wedge (p \rightarrow q)\big) \wedge (p \rightarrow (\neg r))\Big)$
2. $\Big(\big(((p \wedge q) \rightarrow r) \wedge (p \rightarrow q)\big) \rightarrow (p \rightarrow r)\Big)$
3. $((p \vee q) \leftrightarrow ((p \wedge (\neg q)) \vee ((\neg p) \rightarrow q)))$
4. $(p \wedge (\neg p))$