

Warmup Problem

Describe how structural induction differs from our MATH 135 notion of induction.

CS 245: Logic and Computation

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Lecture 3

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Let's begin!

Announcements

1. Assignment 1 due. Submit to Crowdmark! If you're missing a link, please email myself and Ahmed (the ISC)!
2. Office Hours Monday 9-11 and Tuesday 10-11 in DC 3119 (in the bridge between DC and MC)
3. Midterm is June 7th 4:30-6:20.

Previously on CS245

- Give multiple translations of English sentences with ambiguity.
- Describe the three types of symbols in propositional logic.
- Describe the recursive definition of well-formed formulas.
- Write the parse tree for a well-formed formula.
- Determine and give reasons for whether a given formula is well formed or not.
- A template for writing structural induction proofs.

Let's Review

Let's review last week with a few clicker questions!

Plan for today

- Review the definition of well-formed formulas.
- Analyze the recursive structure of the well-formed formula definition.
- Review the template for writing structural induction proofs.
- Prove the unique readability theorem using structural induction.

Proving the unique readability of well-formed formulas

Lemma 1: Every well-formed formula starts with a propositional variable or an opening bracket.

Lemma 2: Every well-formed formula has an equal number of opening and closing brackets.

Lemma 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

Theorem: There is a unique way to construct every well-formed formula.

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Lemma 2: Every well-formed formula has an equal number of opening and closing brackets.

Structural induction (1)

Step 1: Identify the recursive structure in the problem.

Theorem 1: Every *well-formed formula* has an equal number of opening and closing brackets.

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Theorem 1: Every *well-formed formula* has an equal number of opening and closing brackets. Notes:

- “Well-formed formulas” are the recursive structures that we are dealing with.
- “Has an equal number of opening and closing brackets” is the property that we are going to prove that well formed formulas have. Some examples of other properties that we could prove: (1) contains at least one propositional variable, (2) has an even number of brackets, etc.

Structural induction (2)

Step 2: Identify each recursive appearance of the structure inside its definition. (A recursive structure is self-referential. Where in the definition of the object does the object reference itself?)

Let \mathcal{P} be a set of propositional variables. A *well-formed formula* over \mathcal{P} has exactly one of the following forms:

1. A single symbol of \mathcal{P} ,
2. $(\neg\alpha)$ if α is well-formed,
3. One of $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ if α and β are well-formed.

Structural induction (2)

Step 2: Identify each recursive appearance of the structure inside its definition. (A recursive structure is self-referential. Where in the definition of the object does it the object reference itself?)

There are 3 cases in the following definition. The definition references itself in cases 2 and 3, as highlighted below.

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Structural induction (3)

Step 3: Divide the cases into: those without recursive appearances (“base cases”) and those with (“inductive” cases).

Case 1 has no recursive appearance of the definition. Cases 2 and 3 have recursive appearances of the definition.

Let \mathcal{P} be a set of propositional variables. A *well-formed formula* over \mathcal{P} has exactly one of the following forms:

1. A single symbol of \mathcal{P} , (base case)
2. $(\neg\alpha)$ if α is *well-formed*, (inductive case)
3. One of $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ if α and β are *well-formed*. (inductive case)

A structural induction template

Problem: Prove that every recursive structure φ has property P .

Define $P(\varphi)$ to be the property in the problem.

Theorem: For every *recursive structure* φ , $P(\varphi)$ holds.

Proof by structural induction:

Base case: *For every base case you identified, prove that the recursive structure φ has property P .*

(Continued on the next slide)

A structural induction template

Induction step: *For each recursive case you identified, write an induction step*

Recursive case 1:

Induction hypothesis: Assume *each recursive appearance of the structure in this case* has property P .

Prove that the recursive structure φ has property P using the induction hypothesis.

Recursive case 2:

(State the induction hypothesis and use it to prove the theorem.)

(Possibly more recursive cases)

By the principle of structural induction, every recursive structure φ has property P . QED

Unbalanced brackets in a proper prefix of a formula

Lemma 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

A *proper prefix* of φ is a non-empty segment of φ starting from the first symbol of φ and ending before the last symbol of φ . Sometimes, people emphasize **non-empty** in the statement and say this is a *proper non-empty prefix*.

One can reword this as: A *proper prefix* of φ is a non-empty expression, say x , such that there exists a non-empty expression y satisfying $\varphi = xy$.

For example, the formula $(\neg p)$ has as proper prefixes $($, $(\neg$ and $(\neg p$.

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Let $P(\varphi)$ be the property that every proper prefix of the well-formed formula φ has more opening brackets than closing brackets. We prove $P(\varphi)$ is true for all well-formed formulas φ by structural induction.

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Base Case: If φ is an atom, then there are no proper prefixes and the claim is vacuously true.

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Complete the induction step.

Main Theorem

Theorem (Unique Readability Theorem)

There is a unique way to construct every well-formed formula.

Proof

Let $P(\varphi)$ be the property that there is a unique way to construct the well-formed formula φ . We prove this property for all well-formed formulas φ by structural induction.

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Inductive Conclusion

We now have a few possibilities to consider:

1. $\varphi = (\neg\alpha)$
2. $\varphi = (\alpha \star \beta)$

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Each of these has two subcases...

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Then in this case, it must be that $\alpha = \gamma \star \eta$, $\alpha' = \neg\gamma$ and $\beta' = \eta$. However, we see that as α is well-formed, we have that γ is a proper prefix of α so by lemma 3, this has more open brackets than closed brackets. However, α' is also well-formed and so it must have an equal number of open and closed brackets. This is a contradiction.

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If $|\alpha'| = |\alpha|$, then $\alpha' = \alpha$ (both start at the second symbol of φ).

Thus also $\star = \star'$ and $\beta = \beta'$, as required.

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If $0 < |\alpha'| < |\alpha|$, then α' is a proper prefix of α .

Thus, **by lemmas 2 and 3 on α** , α' is not a formula; we have nothing to prove.

Proof

We have assumed that φ is $(\alpha \star \beta)$, where both α and β have property P . To conclude, we must now show that

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If $0 < |\alpha'| < |\alpha|$, then α' is a proper prefix of α .

Thus, **by lemmas 2 and 3 on α** , α' is not a formula; we have nothing to prove.

If $|\alpha'| > |\alpha|$, then $\alpha' = \alpha \star y$, where y is a proper prefix of β (or is empty).

By lemma 2, α has equally many '(' and ')', while **by lemma 3** y has more '(' than ')'. Thus α' has more '(' than ')' and hence again **by lemma 3**, it is not a formula, and again we have nothing to prove.

Therefore α has a unique derivation in this case.

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Subcase 4: If $\varphi = (\alpha \star \beta)$, then what if we could write $\varphi = (\neg\alpha')$ for some other well-formed formulas α' and β' ?

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Argue as in Subcase 2; Notice that $\alpha' = \gamma \star \eta$ for some well formed formulas γ and η which also satisfies $\alpha = \neg\gamma$ and $\beta = \eta$. Again γ is a proper prefix of α' and so has more open brackets than closed brackets by lemma 3. This contradicts lemma 2 since this is also equal to $\alpha = \neg\gamma$.