Warm-Up Problem

Prove or disprove for a Predicate Logic formula α:

$$\{((\neg\alpha) \to \alpha)\} \vdash_{ND} \alpha$$

Solution

$$\{((\neg\alpha) \rightarrow \alpha)\} \vdash_{ND} \alpha$$

1.	$((\neg\alpha) \rightarrow \alpha)$	Premise
2.	$(\neg\alpha)$	Assumption
3.	α	$\rightarrow$e: 1,2
4.	$\perp$	$\perp$i: 2,3
5.	$(\neg(\neg\alpha))$	$\neg$i: 2-4
6.	α	$\neg\neg$e: 5

Predicate Logic: Natural Deduction: Definability, Soundness and Completeness

Carmen Bruni

Lecture 16

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts (Special Thanks to Collin for a lot of these slides!), R. Trefler, and P. Van Beek

Last Time

- Peano Arithmetic.

Learning Goals

- Definability
- Completeness of Natural Deduction
- Soundness of Natural Deduction

Let formula φ have free variables x_1, ..., x_k.

Given an interpretation $\mathcal{I}$, a formula φ *defines* the k-ary relation of tuples that make φ true — that is, the relation

$$R_\varphi = \left\{ \langle \mathsf{a}_1, \ldots, \mathsf{a}_k \rangle \in dom(I)^k \mid \varphi^{(\mathcal{I}, E[x_1 \mapsto \mathsf{a}_1] \cdots [x_k \mapsto \mathsf{a}_k])} = \mathtt{T} \right\} \ .$$

A relation R is *definable (in $\mathcal{I}$)* iff $R = R_\varphi$ for some formula φ.

Example: in Peano Arithmetic, the relation $\leq$ is defined by the formula

$$(\exists z \left((x_1 + z) = x_2 \right)) \ .$$

Properties of Defined Relations

The PA axioms allow one to show that the defined relation $\leq$ has the expected properties.

- $(x \leq y)$ and $(y \leq z)$ imply $(x \leq z)$ (transitivity).
- If $(x \leq y)$ and $(y \leq x)$ then $(x = y)$.

We can also define further relations using $\leq$; e.g.,

$$(x < y) \quad \text{iff} \quad ((x \leq y) \wedge (x \neq y)) \ .$$

Transitivity of Less-or-Equal

To show: $\{(x \le y),\, (y \le z)\} \vdash (x \le z)$.

Transitivity of Less-or-Equal

To show: $\{(x \leq y), (y \leq z)\} \vdash (x \leq z)$.

1.	$(\exists w\,((x+w)=y))$	Premise
2.	$(\exists w\,((y+w)=z))$	Premise
3.	$((x+u)=y)$, u fresh	Assumption
4.	$((y+v)=z)$, v fresh	Assumption
5.	$((x+(u+v))=((x+u)+v))$	Associativity of $+$
6.	$(((x+u)+v)=(y+v))$	EQsubs$(\cdot+v)$: 3
7.	$((x+(u+v))=z)$	EqTrans(2): 5, 6, 4
8.	$(\exists w\,((x+w)=z)$	$\exists$i: 7
9.	$(\exists w\,((x+w)=z)$	$\exists$e: 2, 4–8
10.	$(\exists w\,((x+w)=z)$	$\exists$e: 1, 3–9

Defining Functions

To define a k-ary function, use its $(k+1)$-ary relation.

Example: Let $\mathscr{R}_{sq}$ ("square-of") be defined by $((x_1 \times x_1) = x_2)$.
Then we can get the effect of having the squaring function:

> if φ contains a free variable x, but u is fresh, then the formula
>
> $$\left(\exists u\,(\mathscr{R}_{sq}(t,u) \wedge \varphi[u/x])\right)$$
>
> expresses "the square of t satisfies φ."

We must, however, ensure that $\mathscr{R}_{sq}$ really does define a function;
that is, that every number has exactly one square:

$$\left(\forall x \left((\exists y\ \mathscr{R}_{sq}(x,y)) \wedge \left(\forall y \left(\forall z\,((\mathscr{R}_{sq}(x,y) \wedge \mathscr{R}_{sq}(x,z)) \rightarrow (y = z))\right)\right)\right)\right)$$

We leave this proof as an exercise.

Theorem.

- Natural Deduction is sound for Predicate Logic: if $\Sigma \vdash_{ND} \alpha$, then $\Sigma \models \alpha$.
- Natural Deduction is complete for Predicate Logic: if $\Sigma \models \alpha$, then $\Sigma \vdash_{ND} \alpha$.

Proof outline:

Soundness: Each application of a rule is sound. By induction, any finite number of rule applications is sound.

Completeness: We shall show the contrapositive:

$$\text{if } \Sigma \nvdash \alpha, \text{ then } \Sigma \not\models \alpha \ .$$

We shall not give the full proof, but we will sketch the main points.

Completeness of ND for Predicate Logic: Getting started

To show: if $\Sigma \nvdash \alpha$, then $\Sigma \nvDash \alpha$.

Lemma I: If $\Sigma \nvdash \alpha$, then $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$.

Lemma II (the big one):

If $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$, then there are $\mathcal{I}$ and E such that $\mathcal{I} \vDash_E \Sigma \cup \{(\neg\alpha)\}$.

Lemma III: If there are $\mathcal{I}$ and E such that $\mathcal{I} \vDash_E \Sigma \cup \{(\neg\alpha)\}$, then $\Sigma \nvDash \alpha$.

We will prove this in the order of Lemma I, Lemma III, then Lemma II (skipping most of the details in the latter).

Proof of Lemma 1

If $\Sigma \nvdash \alpha$, then $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$.

Proof of Lemma 1

If $\Sigma \not\vdash \alpha$, then $\Sigma \cup \{(\neg\alpha)\} \not\vdash \alpha$.

Proof: We prove this by contradiction. Assume that $\Sigma \not\vdash \alpha$ and that $\Sigma \cup \{(\neg\alpha)\} \vdash_{ND} \alpha$.

Proof of Lemma 1

If $\Sigma \nvdash \alpha$, then $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$.

Proof: We prove this by contradiction. Assume that $\Sigma \nvdash \alpha$ and that $\Sigma \cup \{(\neg\alpha)\} \vdash_{ND} \alpha$.

Taking the second statement, by $\rightarrow$i, we see that $\Sigma \vdash_{ND} ((\neg\alpha) \rightarrow \alpha)$.

Proof of Lemma 1

If $\Sigma \nvdash \alpha$, then $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$.

Proof: We prove this by contradiction. Assume that $\Sigma \nvdash \alpha$ and that $\Sigma \cup \{(\neg\alpha)\} \vdash_{ND} \alpha$.

Taking the second statement, by $\rightarrow$i, we see that $\Sigma \vdash_{ND} ((\neg\alpha) \rightarrow \alpha)$.

Hence, by the warmup problem, we have that $\Sigma \vdash_{ND} \alpha$. This contradicts the fact that $\Sigma \nvdash \alpha$. Hence, $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$.

Proof of Lemma III

If there are $\mathcal{I}$ and E such that $\mathcal{I} \vDash_E \Sigma \cup \{(\neg\alpha)\}$, then $\Sigma \nvDash \alpha$.

Proof of Lemma III

If there are $\mathcal{I}$ and E such that $\mathcal{I} \models_E \Sigma \cup \{(\neg\alpha)\}$, then $\Sigma \not\models \alpha$.

Proof: Assume there are $\mathcal{I}$ and E such that $\mathcal{I} \models_E \Sigma \cup \{(\neg\alpha)\}$. Then, $\mathcal{I} \models_E \Sigma$ and $\mathcal{I} \models_E (\neg\alpha)$. The latter means that $\mathcal{I} \not\models_E \alpha$. Thus, there is an interpretation and environment that shows us by definition that $\Sigma \not\models \alpha$.

Beginning of Proof of Lemma II

If $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$, then there are $\mathcal{I}$ and E such that $\mathcal{I} \models_E \Sigma \cup \{(\neg\alpha)\}$.

Proof: Assume that $\Sigma \cup \{\neg\alpha\} \nvdash \alpha$.

We need an interpretation $\mathcal{I}$ and environment E that satisfy $\Sigma \cup \{\neg\alpha\}$.

To start, we need a domain. Where can we get one?

Beginning of Proof of Lemma II

If $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$, then there are $\mathcal{I}$ and E such that $\mathcal{I} \vDash_E \Sigma \cup \{(\neg\alpha)\}$.

Choice of domain: We use the set of terms. That is, let the domain be

$$\mathcal{D} = dom(\mathcal{I}) = \{ \ulcorner t \urcorner \mid t \text{ is a term} \} \ .$$

(The notation '$\ulcorner\ \urcorner$' indicates that we refer to the domain element, rather than to the expression.)

Note that this domain $\mathcal{D}$ consisting of terms will mean that the interpretations of constants, variables and functions all will correspond to term elements in the domain of our interpretation (yes this is very abstract and strange!)

Interpretation of Terms

For a set Σ of premises, we want an interpretation $\mathcal{I}$ and an environment E, over the domain of terms.

Constants, variables, and functions are easy to handle.

- For a constant symbol c, we define $c^{\mathcal{I}} = \ulcorner c \urcorner$.
- For a variable x, we define $x^{E} = \ulcorner x \urcorner$.
- For a k-ary function symbol f, we define $f^{\mathcal{I}}(\ulcorner t_1 \urcorner, \dots, \ulcorner t_k \urcorner) = \ulcorner f(t_1, \dots, t_k) \urcorner$.

Interpretation of Terms (Continued)

Relations pose a problem, since they depend on Σ. For a relation symbol $R^{(k)}$, we must determine, for each tuple $\langle t_1, \ldots, t_k \rangle$, whether to put $\langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner \rangle$ into the set $R^{\mathcal{I}}$.

The basic idea is to consider each possible formula, one by one. For each, restrict the set of interpretations to account for it. Since we assume no proof of contradiction exists, we can guarantee that we always have some interpretations left as possible.

We present some abbreviated details.

Checking a Formula

Suppose we have a list of all atomic formulas over our language $\mathcal{L}$:

$$\varphi_1, \varphi_2, \varphi_3, \cdots.$$

We will create a list of sets: Σ_0, Σ_1, Σ_2, etc., with $\Sigma_0 = \Sigma$ and Σ_i determined from Σ_{i-1} and φ_i:

$$
\begin{array}{l}
\Sigma_0 \leftarrow \Sigma \\
\texttt{for } i \leftarrow 1, 2, 3, \ldots \\
\quad \texttt{if } \Sigma_{i-1} \cup \{\varphi_i\} \nvdash (\neg\varphi_i) \\
\quad\quad \text{set } \Sigma_i \leftarrow \Sigma_{i-1} \cup \{\varphi_i\} \\
\quad \texttt{else} \\
\quad\quad \text{set } \Sigma_i \leftarrow \Sigma_{i-1} \cup \{(\neg\varphi_i)\}
\end{array}
$$

Let β_k denote the formula added at step k; thus β_k is either φ_k or $(\neg\varphi_k)$.

Let Σ_∞ denote the union of all Σ_i: $\Sigma_\infty = \Sigma \cup \{\beta_1, \beta_2, \beta_3, \ldots\}$.

The Satisfying Interpretation—Almost

Once we have Σ_∞, it

- is consistent by construction (there is no γ s.t. $\Sigma_\infty \cup \{\neg\gamma\} \vdash_{\mathit{ND}} \gamma$), and
- contains either $R(t_1, \dots, t_k)$ or $\neg R(t_1, \dots, t_k)$ for every relation symbol R and terms $t_1, \dots, t_k$.

Thus Σ_∞ defines an interpretation and environment:

$$R^{\mathcal{I}} = \left\{ \langle \ulcorner t_1 \urcorner, \dots, \ulcorner t_k \urcorner \rangle \in \mathcal{D}^k \mid R(t_1, \dots, t_k) \in \Sigma_\infty \right\} .$$

This seems good but there a few holes...

Subtleties

Two subtleties:

- We need a list of all formulas, in some order.
- Existential formulas (i.e., $\exists x \ldots$) may never get a term to satisfy them. Thus they evaluate to $\mathtt{F}$ — even if $\left(\neg(\forall x\,(\neg\varphi))\right)$ is true.

One "Piece of the Puzzle": Listing all formulas

Listing all possible formulas:

Since we may have arbitrarily many constant, variable, functions and relation symbols, of any arity, we must take care that everything gets onto the list at some point. For example, if we take the ith formula to be $R(c_i)$, then many formulas ($R(y)$, $Q_2(f_{19}(y_{42}))$, etc.) never appear on the list.

We do the listing "in stages", starting from stage 1. At stage j, consider the first j constants, variables, and function symbols. Form all terms that combine these, using at most j applications of a function. Apply each of the first j relation symbols to each of these terms, and then form all formulas from these that use at most j connectives or quantifiers.

The set of formulas formed this way is large, but finite. After all have been listed, continue to stage $j + 1$.

Ensuring Satisfaction of Existential Formulas

We use the following trick to ensure that each existential formula in Σ_∞ has a term that satisfies it.

Let c_1, c_2, ... be a list of fresh constant symbols, that do not occur in any formula of Σ. For each formula γ_i, add the formula

$$(\exists x\ \gamma_i) \to \gamma_i[c_i/x]$$

to the set Σ_0 at the start of the construction (and include their terms in the domain!)

At the end of the construction, if $\Sigma_\infty \models (\exists x\ \gamma_i)$, then also $\Sigma_\infty \models \gamma_i[c_i/x]$.

Setup for Proof of Soundness

We are trying to prove

if $\Sigma \vdash_{ND} \alpha$, then $\Sigma \models \alpha$.

Let Σ be a set of well-formed Predicate logic formulas, α a well-form Predicate logic formula and suppose that $\Sigma \vdash_{ND} \alpha$.

Natural Deduction for Propositional logic is sound, so we need only to show that our new rules are sound. We go in order of difficulty: $\forall$e and $\exists$i are easier since they don't involve subproofs or free variables. The other two will be more convoluted. We begin with a few lemmata (Special thanks to Collin Roberts for these slides):

Lemma 1:
Let t be any well-formed term of Predicate logic. Let $\mathcal{I}$ be any interpretation, with domain $\mathcal{D}$. Let E be any environment. Then $t^{(\mathcal{I},E)} \in \mathcal{D}$, where $\mathcal{D}$ denotes the domain of $\mathcal{I}$.

Proof. The proof is by structural induction on t and is left as an exercise.

Lemma 2:
Let α be any Predicate formula. Let t be any term. Let x be a variable. Let $\mathcal{I}$ be any interpretation, with domain $\mathcal{D}$. Let E be any environment. Then $\alpha[t/x]^{(\mathcal{I},E)} = \alpha^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}$.

Lemma 2:
Let α be any Predicate formula. Let t be any term. Let x be a variable. Let $\mathcal{I}$ be any interpretation, with domain $\mathcal{D}$. Let E be any environment. Then $\alpha[t/x]^{(\mathcal{I},E)} = \alpha^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}$.

Notice that this is saying that the two formulas on either side both have the same truth value.

Lemma 2:
Let α be any Predicate formula. Let t be any term. Let x be a variable. Let $\mathcal{I}$ be any interpretation, with domain $\mathcal{D}$. Let E be any environment. Then $\alpha[t/x]^{(\mathcal{I},E)} = \alpha^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}$.

Notice that this is saying that the two formulas on either side both have the same truth value.

Proof. Delayed until we have proved a small sublemma first...

Lemma 3: Let s, t be any Predicate terms. Let x be a variable. Let $\mathcal{I}$ be any interpretation. Let E be any environment. Then

$$s[t/x]^{(\mathcal{I},E)} = s^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}.$$

Note: The left hand side and the right hand side are both domain elements. This Lemma 3 asserts that these domain elements are the same.

Lemma 3: Let s, t be any Predicate terms. Let x be a variable. Let $\mathcal{I}$ be any interpretation. Let E be any environment. Then

$$s[t/x]^{(\mathcal{I}, E)} = s^{(\mathcal{I}, E[x \mapsto t^{(\mathcal{I}, E)}])}.$$

Note: The left hand side and the right hand side are both domain elements. This Lemma 3 asserts that these domain elements are the same.

Proof: Let $P(s)$ be the property that for any term t, variable x, interpretation $\mathcal{I}$ and environment E that

$$s[t/x]^{(\mathcal{I}, E)} = s^{(\mathcal{I}, E[x \mapsto t^{(\mathcal{I}, E)}])}.$$

The proof is by structural induction on s.

Proof By Structural Induction - Base Case

Base Case: If s is c, for some constant symbol c, then

$$s[t/x]^{(\mathcal{I},E)} = c[t/x]^{(\mathcal{I},E)} = c^{\mathcal{I}} = c^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} = s^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}.$$

If s is y, for some variable symbol $y \neq x$, then

$$s[t/x]^{(\mathcal{I},E)} = y[t/x]^{(\mathcal{I},E)} = y^{E} = y^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} = s^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}.$$

If s is x, then

$$s[t/x]^{(\mathcal{I},E)} = x[t/x]^{(\mathcal{I},E)} = t^{(\mathcal{I},E)} = x^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} = s^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}.$$

Induction Step: In this case, s is $g(t_1, \ldots, t_\ell)$, for some ℓ-ary function symbol g, and some terms $t_1, \ldots, t_\ell$. The induction hypothesis in this case is that for all $1 \leq i \leq \ell$ that

$$t_i[t/x]^{(\mathcal{I}, E)} = t_i^{(\mathcal{I}, E[x \mapsto t^{(\mathcal{I}, E)}])}.$$

Then

Induction Step: In this case, s is $g(t_1, \dots, t_\ell)$, for some ℓ-ary function symbol g, and some terms $t_1, \dots, t_\ell$. The induction hypothesis in this case is that for all $1 \leq i \leq \ell$ that

$$t_i[t/x]^{(\mathcal{I},E)} = t_i^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}.$$

Then

$$\begin{aligned}
s[t/x]^{(\mathcal{I},E)} &= g(t_1, \dots, t_\ell)[t/x]^{(\mathcal{I},E)} \\
&= g^{\mathcal{I}}\left(t_1[t/x]^{(\mathcal{I},E)}, \dots, t_\ell[t/x]^{(\mathcal{I},E)}\right) \\
&\underset{IH}{=} g^{\mathcal{I}}\left(t_1^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}, \dots, t_\ell^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}\right) \\
&= g(t_1, \dots, t_\ell)^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} \\
&= s^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}.
\end{aligned}$$

This completes the induction step, and the proof.

Lemma 2:

Let α be any Predicate formula. Let t be any term. Let x be a variable. Let $\mathcal{I}$ be any interpretation, with domain $\mathcal{D}$. Let E be any environment. Then $\alpha[t/x]^{(\mathcal{I},E)} = \alpha^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}$.

Proof.

Lemma 2:
Let α be any Predicate formula. Let t be any term. Let x be a variable. Let $\mathcal{I}$ be any interpretation, with domain $\mathcal{D}$. Let E be any environment. Then $\alpha[t/x]^{(\mathcal{I}, E)} = \alpha^{(\mathcal{I}, E[x \mapsto t^{(\mathcal{I}, E)}])}$.

Proof. Let $R(\alpha)$ be the property that for any term t, variable x, interpretation $\mathcal{I}$ and environment E that

$$\alpha[t/x]^{(\mathcal{I}, E)} = \alpha^{(\mathcal{I}, E[x \mapsto t^{(\mathcal{I}, E)}])}.$$

The proof is by structural induction on α. (Again special thanks to Collin Roberts for these slides!)

Base Case: α is $P(t_1, \dots, t_k)$, for some k-ary predicate symbol P, and some terms $t_1, \dots, t_k$. Then

$$\begin{aligned}\alpha[t/x]^{(\mathcal{I},E)} &= P(t_1,\dots,t_k)[t/x]^{(\mathcal{I},E)} \\ &= P(t_1[t/x],\dots,t_k[t/x])^{(\mathcal{I},E)} \\ &= P^{\mathcal{I}}\Big(t_1[t/x]^{(\mathcal{I},E)},\dots,t_k[t/x]^{(\mathcal{I},E)}\Big).\end{aligned}$$

We also have that

$$\begin{aligned}\alpha^{(\mathcal{I},E[x\mapsto t^{\mathcal{I},E}])} &= P(t_1,\dots,t_k)^{(\mathcal{I},E[x\mapsto t^{(\mathcal{I},E)}])} \\ &= P^{\mathcal{I}}\left(t_1^{(\mathcal{I},E[x\mapsto t^{(\mathcal{I},E)}])},\dots,t_k^{(\mathcal{I},E[x\mapsto t^{(\mathcal{I},E)}])}\right).\end{aligned}$$

So we are done with the base case if we can prove that

$$t_i[t/x]^{(\mathcal{I},E)} = t_i^{(\mathcal{I},E[x\mapsto t^{(\mathcal{I},E)}])}, \text{ for all } 1 \le i \le k.$$

which is true by the Lemma 3.

Induction Step (1 of 5)

Induction Step: Throughout, assume that $R(\beta)$ and $R(\gamma)$ hold (this is our induction hypothesis).

If α is $(\neg\beta)$, for some β, then

Induction Step: Throughout, assume that $R(\beta)$ and $R(\gamma)$ hold (this is our induction hypothesis).

If α is $(\neg\beta)$, for some β, then

$$\begin{aligned}
\alpha[t/x]^{(\boldsymbol{I},E)} &= (\neg\beta)[t/x]^{(\boldsymbol{I},E)} \\
&= (\neg\beta[t/x])^{(\boldsymbol{I},E)} \\
&= \begin{cases} \mathtt{T} & \text{if } \beta[t/x]^{(\boldsymbol{I},E)} = \mathtt{F} \\ \mathtt{F} & \text{if } \beta[t/x]^{(\boldsymbol{I},E)} = \mathtt{T} \end{cases} \\
&\underset{IH}{=} \begin{cases} \mathtt{T} & \text{if } \beta^{(\boldsymbol{I},E[x\mapsto t^{(\boldsymbol{I},E)}])} = \mathtt{F} \\ \mathtt{F} & \text{if } \beta^{(\boldsymbol{I},E[x\mapsto t^{(\boldsymbol{I},E)}])} = \mathtt{T} \end{cases} \\
&= (\neg\beta)^{(\boldsymbol{I},E[x\mapsto t^{(\boldsymbol{I},E)}])} \\
&= \alpha^{(\boldsymbol{I},E[x\mapsto t^{(\boldsymbol{I},E)}])}.
\end{aligned}$$

Induction Step (2 of 5)

If α is $(\beta \wedge \gamma)$, for some β, γ, then

If α is $(\beta \wedge \gamma)$, for some β, γ, then

$$\begin{array}{rcl}
\alpha[t/x]^{(\mathcal{I},E)} & = & (\beta \wedge \gamma)[t/x]^{(\mathcal{I},E)} \\
& = & (\beta[t/x] \wedge \gamma[t/x])^{(\mathcal{I},E)} \\
& = & \begin{cases} \mathtt{T} & \text{if } \beta[t/x]^{(\mathcal{I},E)} = \mathtt{T} \text{ and } \gamma[t/x]^{(\mathcal{I},E)} = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases} \\
& \underset{IH}{=} & \begin{cases} \mathtt{T} & \text{if } \beta^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} = \mathtt{T} \text{ and } \gamma^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases} \\
& = & (\beta \wedge \gamma)^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} \\
& = & \alpha^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])}.
\end{array}$$

The cases for the remaining binary connectives are similar.

Induction Step (3 of 5)

If α is $(\mathcal{Q}x\beta)$, for some β and quantifier $\mathcal{Q}$, then

If α is $(\mathcal{Q}x\beta)$, for some β and quantifier $\mathcal{Q}$, then

$$\begin{aligned}\alpha[t/x]^{(\mathcal{I},E)} &= \alpha^{(\mathcal{I},E)} \\ &= \alpha^{(\mathcal{I},E[x\mapsto t^{(\mathcal{I},E)}])},\end{aligned}$$

since overriding the bound variable x in the environment has no effect.

If α is $(\mathcal{Q}y\beta)$, for some variable $y \neq x$, some β and quantifier $\mathcal{Q}$, then we have two sub-cases, depending on whether y occurs in t or not. We suppose that $\mathcal{Q}$ is $\forall$. The sub-cases for $\exists$ are analogous.

If y does not occur in t, then by rule 5a in substitution, If y does not occur in t, then by rule 5a in substitution,

$$\begin{aligned}
\alpha[t/x]^{(\boldsymbol{I},E)} &= (\forall y\ \beta)[t/x]^{(\boldsymbol{I},E)} \\
&= (\forall y\ \beta[t/x])^{(\boldsymbol{I},E)} \\
&= \begin{cases} \mathtt{T} & \text{if } \beta[t/x]^{(\boldsymbol{I},E[y\mapsto d])} = \mathtt{T} \text{ for all } d \\ \mathtt{F} & \text{otherwise} \end{cases} \\
&\underset{IH}{=} \begin{cases} \mathtt{T} & \text{if } \beta^{(\boldsymbol{I},E[y\mapsto d][x\mapsto t^{(\boldsymbol{I},E)}])} = \mathtt{T} \text{ for all } d \\ \mathtt{F} & \text{otherwise} \end{cases} \\
&\underset{y\neq x}{=} \begin{cases} \mathtt{T} & \text{if } \beta^{(\boldsymbol{I},E[x\mapsto t^{(\boldsymbol{I},E)}][y\mapsto d])} = \mathtt{T} \text{ for all } d \\ \mathtt{F} & \text{otherwise} \end{cases} \\
&= (\forall y\ \beta)^{(\boldsymbol{I},E[x\mapsto t^{(\boldsymbol{I},E)}])} \\
&= \alpha^{(\boldsymbol{I},E[x\mapsto t^{(\boldsymbol{I},E)}])}.
\end{aligned}$$

Induction Step (5 of 5)

If y occurs in t, then, letting z occur nowhere in α or t, we have by substitution rule 5b, If y occurs in t, then, letting z occur nowhere in α or t, we have by substitution rule 5b,

$$
\begin{aligned}
\alpha[t/x]^{(\mathcal{I},E)} &= (\forall z\,(\beta[z/y])[t/x])^{(\mathcal{I},E)} \\
&= \begin{cases} \mathtt{T} & \text{if } (\beta[z/y])[t/x]^{(\mathcal{I},E[z\mapsto d])} = \mathtt{T} \text{ for all } d \\ \mathtt{F} & \text{otherwise} \end{cases} \\
&\underset{IH}{=} \begin{cases} \mathtt{T} & \text{if } \beta[z/y]^{(\mathcal{I},E[z\mapsto d][x\mapsto t^{(\mathcal{I},E)}])} = \mathtt{T} \text{ for all } d \\ \mathtt{F} & \text{otherwise} \end{cases} \\
&\underset{x\neq z}{=} \begin{cases} \mathtt{T} & \text{if } \beta[z/y]^{(\mathcal{I},E[x\mapsto t^{(\mathcal{I},E)}][z\mapsto d])} = \mathtt{T} \text{ for all } d \\ \mathtt{F} & \text{otherwise} \end{cases} \\
&= (\forall z\ \beta[z/y])^{(\mathcal{I},E[x\mapsto t^{(\mathcal{I},E)}])} \\
&= \alpha^{(\mathcal{I},E[x\mapsto t^{(\mathcal{I},E)}])}.
\end{aligned}
$$

All cases have been proved. This completes the induction step and the proof.

Soundness of Natural Deduction

Again, a reminder, we are proving:

if $\Sigma \vdash_{ND} \gamma$, then $\Sigma \models \gamma$.

Each of the four new rules for soundness fall into one of the following four statements which we will prove:

- $\{(\forall x\ \alpha)\} \models \alpha[t/x]$
- $\{\alpha[t/x]\} \models (\exists x\ \alpha)$
- If $\Sigma \models \alpha[y/x]$ where y is not free in Σ, then $\Sigma \models (\forall x\ \alpha)$.
- If $\Sigma \models (\exists x\ \alpha)$ and $\Sigma \cup \{\alpha[u/x]\} \models \beta$ with u fresh and where u is not free in $\Sigma,\ \alpha$ or β, then $\Sigma \models \beta$.

Inductive Conclusion

We go through the steps explicitly for $\forall$e and then just provide the part of the proof that isn't similar to this for the other rules. Suppose that the last rule came from a $\forall$e. Notice that $\gamma = \alpha[t/x]$ in this case.

The last line of $\Sigma \vdash_{ND} \alpha[t/x]$ looks something like the following:

1.		
$\vdots$	$\vdots$	
k.	$(\forall x\ \alpha)$	(Some rule)
$k+1$.	$\alpha[t/x]$	$\forall$e: k

(Notice that the line at line k might be earlier in the proof).

Then, from this, we can see that $\Sigma \vdash_{ND} (\forall x\ \alpha)$ in at most k lines. Thus, by the induction hypothesis, $\Sigma \vDash (\forall x\ \alpha)$. As $\{(\forall x\ \alpha)\} \vdash_{ND} \alpha[t/x]$, it suffices to show that $\{(\forall x\ \alpha)\} \vDash \alpha[t/x]$.

The $\forall$e Inference Rule is Sound.

Claim: $\{(\forall x\ \alpha)\} \vDash \alpha[t/x]$.
Proof: Let α be any Predicate formula. Let x be a variable. Let t be a Predicate term.

The $\forall$e Inference Rule is Sound.

Claim: $\{(\forall x\ \alpha)\} \vDash \alpha[t/x]$.
Proof: Let α be any Predicate formula. Let x be a variable. Let t be a Predicate term.

Let $(\boldsymbol{\mathcal{I}}, E)$ be an interpretation and environment such that $\boldsymbol{\mathcal{I}} \vDash_{E} (\forall x\ \alpha)$. Let $\boldsymbol{\mathcal{I}}$ have domain $\boldsymbol{D}$. By definition of satisfaction for a $\forall$ formula, this means that $\alpha^{(\boldsymbol{\mathcal{I}}, E[x \mapsto d])} = \mathtt{T}$, for all domain elements d. We are done if we can argue that $\boldsymbol{\mathcal{I}} \vDash_{E} \alpha[t/x]$, equivalently that $\alpha[t/x]^{(\boldsymbol{\mathcal{I}}, E)} = \mathtt{T}$.

The ∀e Inference Rule is Sound.

Claim: $\{(\forall x\ \alpha)\} \vDash \alpha[t/x]$.
Proof: Let α be any Predicate formula. Let x be a variable. Let t be a Predicate term.

Let $(\mathcal{I}, E)$ be an interpretation and environment such that $\mathcal{I} \vDash_E (\forall x\ \alpha)$. Let $\mathcal{I}$ have domain $\mathcal{D}$. By definition of satisfaction for a $\forall$ formula, this means that $\alpha^{(\mathcal{I}, E[x \mapsto d])} = \mathtt{T}$, for all domain elements d. We are done if we can argue that $\mathcal{I} \vDash_E \alpha[t/x]$, equivalently that $\alpha[t/x]^{(\mathcal{I}, E)} = \mathtt{T}$.

By the Lemma 3, this is equivalent to $\alpha^{(\mathcal{I}, E[x \mapsto t^{(\mathcal{I}, E)}])} = \mathtt{T}$. Lemma 1 guarantees that $t^{(\mathcal{I}, E)}$ is some domain element. Hence by the above fact about satisfaction of a $\forall$ formula, the desired result is clear.

The ∃i Inference Rule is Sound.

Proof: Let α be any Predicate formula. Let x be a variable. Let t be a Predicate term. Since the ∃i inference rule asserts $\{\alpha[t/x]\} \vdash_{ND} (\exists x\ \alpha)$, to establish soundness we must prove $\{\alpha[t/x]\} \vDash (\exists x\ \alpha)$.

The ∃i Inference Rule is Sound.

Proof: Let α be any Predicate formula. Let x be a variable. Let t be a Predicate term. Since the ∃i inference rule asserts $\{\alpha[t/x]\} \vdash_{ND} (\exists x\ \alpha)$, to establish soundness we must prove $\{\alpha[t/x]\} \vDash (\exists x\ \alpha)$.

Let $(\mathcal{I}, E)$ be an interpretation and environment such that $\mathcal{I} \vDash_E \alpha[t/x]$, equivalently $\alpha[t/x]^{(\mathcal{I},E)} = \mathtt{T}$. By the Lemma 3, this is equivalent to $\alpha^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} = \mathtt{T}$. Let $\mathcal{I}$ have domain $\mathcal{D}$. We are done if we can argue that $\mathcal{I} \vDash_E (\exists x\ \alpha)$, equivalently that $\alpha^{(\mathcal{I},E[x \mapsto d])} = \mathtt{T}$, for some domain element d.

The ∃i Inference Rule is Sound.

Proof: Let α be any Predicate formula. Let x be a variable. Let t be a Predicate term. Since the ∃i inference rule asserts $\{\alpha[t/x]\} \vdash_{ND} (\exists x\ \alpha)$, to establish soundness we must prove $\{\alpha[t/x]\} \vDash (\exists x\ \alpha)$.

Let $(\mathcal{I}, E)$ be an interpretation and environment such that $\mathcal{I} \vDash_E \alpha[t/x]$, equivalently $\alpha[t/x]^{(\mathcal{I},E)} = \mathtt{T}$. By the Lemma 3, this is equivalent to $\alpha^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} = \mathtt{T}$. Let $\mathcal{I}$ have domain $\mathcal{D}$. We are done if we can argue that $\mathcal{I} \vDash_E (\exists x\ \alpha)$, equivalently that $\alpha^{(\mathcal{I},E[x \mapsto d])} = \mathtt{T}$, for some domain element d.

Lemma 1 guarantees that $t^{(\mathcal{I},E)}$ is some domain element. Hence $\alpha^{(\mathcal{I},E[x \mapsto t^{(\mathcal{I},E)}])} = \mathtt{T}$ witnesses the desired result.

The ∀i Inference Rule is Sound.

Recall that soundness means that a statement written with "$\vdash$" implies the same statement written with "$\vDash$". The statement of the ∀i rule is

If $\Sigma \vdash_{ND} \alpha[y/x]$ where y is not free in Σ, then $\Sigma \vdash_{ND} (\forall x\ \alpha)$.

So to establish the soundness of this result, we need to argue that

If $\Sigma \vDash \alpha[y/x]$ where y is not free in Σ, then $\Sigma \vDash (\forall x\ \alpha)$.

The ∀i Inference Rule is Sound (1 of 2).

Claim: Let α be an arbitrary well-formed formula of Predicate logic. Let Σ be an arbitrary set of well-formed formulæ of Predicate logic. Suppose that $\Sigma \vDash \alpha[y/x]$, and y is not free in Σ. Then $\Sigma \vDash (\forall x\ \alpha)$.

Proof: Fix an arbitrary interpretation $\mathcal{I}$ and environment E satisfying $\mathcal{I} \vDash_E \Sigma$. Our goal is to prove that $\mathcal{I} \vDash_E (\forall x\ \alpha)$.

By our hypotheses, $\mathcal{I} \vDash_E \alpha[y/x]$. Since y is not free in Σ, I claim that the Relevance Lemma implies that for every $a \in dom(\mathcal{I})$,

$$\mathcal{I} \vDash_{E[y \mapsto a]} \Sigma \text{ if and only if } \mathcal{I} \vDash_E \Sigma.$$

Let $\psi \in \Sigma$ be arbitrary. By definition, it suffices to prove that

$$\mathcal{I} \vDash_{E[y \mapsto a]} \psi \text{ if and only if } \mathcal{I} \vDash_E \psi.$$

To apply the Relevance Lemma for formula ψ, we need to verify that

$$E(z) = E[y \mapsto a](z), \text{ for any free variable } z \text{ in } \psi.$$

The ∀i Inference Rule is Sound (2 of 2).

Claim: Let α be an arbitrary well-formed formula of Predicate logic. Let Σ be an arbitrary set of well-formed formulæ of Predicate logic. Suppose that $\Sigma \vDash \alpha[y/x]$, and y is not free in Σ. Then $\Sigma \vDash (\forall x\ \alpha)$.

Proof (Continued): We claim that:

$$E(z) = E[y \mapsto a](z), \text{ for any free variable } z \text{ in } \psi.$$

Proof:

The ∀i Inference Rule is Sound (2 of 2).

Claim: Let α be an arbitrary well-formed formula of Predicate logic. Let Σ be an arbitrary set of well-formed formulæ of Predicate logic. Suppose that $\Sigma \vDash \alpha[y/x]$, and y is not free in Σ. Then $\Sigma \vDash (\forall x\ \alpha)$.

Proof (Continued): We claim that:

$$E(z) = E[y \mapsto a](z), \text{ for any free variable } z \text{ in } \psi.$$

Proof: Our hypothesis that y is not free in Σ and this gives us that y is not free in ψ. Hence E and $E[y \mapsto a](z)$ must coincide on every free variable z in ψ. □

Thus for every $a \in dom(\mathcal{I})$ by the Relevance Lemma, we have $\mathcal{I} \vDash_{E[y \mapsto a]} \alpha[y/x]$ and thus $\mathcal{I} \vDash_{E[x \mapsto a]} \alpha$. By definition, this is the required $\mathcal{I} \vDash_{E} (\forall x\ \alpha)$. Since $\mathcal{I}$ and E were arbitrary, therefore we have satisfied the definition. In other words we have $\Sigma \vDash (\forall x\ \alpha)$, as required.

The ∃e Inference Rule is Sound (1 of 2).

Recall the definition of the ∃e inference rule:

$$\frac{\Sigma \vdash_{ND} (\exists x\ \alpha) \quad \Sigma \cup \{\alpha[u/x]\} \vdash_{ND} \beta, u \text{ fresh}}{\Sigma \vdash_{ND} \beta}$$

As earlier, soundness means that a statement written with "$\vdash$" implies the same statement written with "$\vDash$". The statement of the ∃e rule is

If $\Sigma \vdash_{ND} (\exists x\ \ \alpha)$ and $\Sigma \cup \{\alpha[u/x]\} \vdash_{ND} \beta$ with u fresh and where u is not free in $\Sigma,\ \alpha$ or β, then $\Sigma \vdash_{ND} \beta$.

So to establish the soundness of this result, we need to argue that

If $\Sigma \vDash (\exists x\ \ \alpha)$ and $\Sigma \cup \{\alpha[u/x]\} \vDash \beta$ with u fresh and where u is not free in $\Sigma,\ \alpha$ or β, then $\Sigma \vDash \beta$.

This is exactly the statement of the next result.

The ∃e Inference Rule is Sound (2 of 2).

Claim: Let α, β be arbitrary well-formed formulæ of Predicate logic. Let Σ be an arbitrary set of well-formed formulæ of Predicate logic. Suppose that $\Sigma \vDash (\exists x \;\; \alpha)$ and $\Sigma \cup \{\alpha[u/x]\} \vDash \beta$ with u fresh and where u is not free in $\Sigma,\ \alpha$ or β. Then $\Sigma \vDash \beta$.

Proof: Fix an arbitrary interpretation $\mathcal{I}$ and environment E satisfying $\mathcal{I} \vDash_{E} \Sigma$. Our goal is to prove that $\mathcal{I} \vDash_{E} \beta$.

The ∃e Inference Rule is Sound (2 of 2).

Claim: Let α, β be arbitrary well-formed formulæ of Predicate logic. Let Σ be an arbitrary set of well-formed formulæ of Predicate logic. Suppose that $\Sigma \vDash (\exists x \ \ \alpha)$ and $\Sigma \cup \{\alpha[u/x]\} \vDash \beta$ with u fresh and where u is not free in $\Sigma,\ \alpha$ or β. Then $\Sigma \vDash \beta$.

Proof: Fix an arbitrary interpretation $\mathcal{I}$ and environment E satisfying $\mathcal{I} \vDash_E \Sigma$. Our goal is to prove that $\mathcal{I} \vDash_E \beta$.

Since $\mathcal{I} \vDash_E \Sigma$ and $\Sigma \vDash (\exists x\ \alpha)$, by the first entailment we have $\mathcal{I} \vDash_E (\exists x\ \alpha)$. By the definition of satisfaction for existential formulæ , this says that there exists a domain element $a \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x \mapsto a]} \alpha$. In other words, there exists a fresh u which is not free in $\Sigma,\ \alpha$ or β, such that $\mathcal{I} \vDash_E \alpha[u/x]$.

We have $\mathcal{I} \vDash_E \Sigma$ and $\mathcal{I} \vDash_E \alpha[u/x]$. Therefore, by the second entailment, we have $\mathcal{I} \vDash_E \beta$.

This completes the proof.