Warm-Up Problem

Consider the statement

$$\Big\{\Big(\forall x\,(\exists y\,(P(x) \vee Q(y)))\Big)\Big\} \vdash \Big(\exists y\,(\forall x\,(P(x) \vee Q(y)))\Big)$$

and the following attempted proof:

1.	$\Big(\forall x\,(\exists y\,(P(x) \vee Q(y)))\Big)$	Premise
2.	u fresh	
3.	$(\exists y\,(P(u) \vee Q(y)))$	$\forall$e: 1
4.	$(P(u) \vee Q(z))$, z fresh	Assumption
5.	$(P(u) \vee Q(z))$	Reflexive: 4
6.	$(P(u) \vee Q(z))$	$\exists$e: 3,4-5
7.	$(\forall x\,(P(x) \vee Q(z)))$	$\forall$i: 2-6
8.	$\Big(\exists y\,(\forall x\,(P(x) \vee Q(y)))\Big)$	$\exists$i: 7

Identify the main error.

Predicate Logic: Natural Deduction Continued and Axioms of Equality

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Lecture 14

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Last Time

- Natural Deduction for Predicate Logic

Learning Goals

- State the axioms for Natural Deduction with Equality
- Solve problems in this new proof system.

Natural Deduction Proof Questions (Avoid using derived rules!)

1. $\left\{(\exists x\ P(x)), (\forall x\,(P(x) \rightarrow Q(x)))\right\} \vdash_{ND} (\exists x\ Q(x))$
2. $\left\{(\exists x\,(P(x) \vee Q(x)))\right\} \vdash_{ND} ((\exists x\ P(x)) \vee (\exists x\ Q(x)))$
3. $\left\{((\forall x\ P(x)) \rightarrow (\exists x\ R(x))), (\forall x\,(P(x) \wedge Q(x)))\right\} \vdash_{ND} (\exists x\ R(x))$
4. $\left\{(\forall x\,(P(x) \rightarrow Q(x)))\right\} \vdash_{ND} ((\forall x\ P(x)) \rightarrow (\forall x\ Q(x)))$
5. $\left\{(\forall x\,(Q(x) \rightarrow R(x))), (\exists x\,(P(x) \wedge Q(x)))\right\} \vdash_{ND}$ $(\exists x\,(P(x) \wedge R(x)))$
6. $\left\{(\forall x\,(\forall y\,(P(x) \rightarrow Q(y)))), (\exists x\ P(x))\right\} \vdash_{ND} (\forall z\ Q(z))$
7. $\left\{(\forall x\,(\forall y\,(R(x,y) \rightarrow R(y,x))))\right\} \vdash_{ND}$ $((\forall x\,(\forall y\,(R(x,y) \rightarrow R(y,x)))) \wedge (\forall x\,(\forall y\,(R(y,x) \rightarrow R(x,y)))))$
8. $\left\{(\forall x\,(\exists y\ R(x,y)))\right\} \vdash_{ND} (\forall x\,(\exists y\,(\exists z\,(R(x,y) \wedge R(y,z)))))$
9. $\left\{(\forall x\,(\forall y\,(\forall z\,((R(x,y) \wedge R(x,z)) \rightarrow R(y,z))))), (\forall x\ R(x,x))\right\}$ $\vdash_{ND} (\forall x\,(\forall y\,(R(x,y) \rightarrow R(y,z))))$
10. $\emptyset \vdash_{ND} ((\forall x\,(\exists y\ R(x,y))) \vee (\neg(\forall x\ R(x,x))))$

Natural Deduction Proof Questions (Avoid using derived rules!) De Morgan's Laws in Predicate Logic:

1. $\left\{\left(\neg(\exists x\ P(x))\right)\right\} \vdash_{ND} \left(\forall x\,(\neg P(x))\right)$
2. $\left\{\left(\forall x\,(\neg P(x))\right)\right\} \vdash_{ND} \left(\neg(\exists x\ P(x))\right)$
3. $\left\{\left(\exists x\,(\neg P(x))\right)\right\} \vdash_{ND} \left(\neg(\forall x\ P(x))\right)$
4. (Assignment problem!) $\left\{\left(\neg(\forall x\ P(x))\right)\right\} \vdash_{ND} \left(\exists x\,(\neg P(x))\right)$

Note

On the next few slides, we discuss some space saving techniques like putting two fresh variables in the same subproof box. All proofs can be done without these but using these can sometimes save space.

Repeated Quantifiers

The rules for elimination and introduction of quantifiers can be generalized to multiple quantifiers.

Let $x_1, \ldots, x_n$ be n distinct variables.

- If $\Sigma \vdash_{ND} (\forall x_1 \cdots (\forall x_n\ \alpha) \cdots)$, then $\Sigma \vdash_{ND} \alpha[t_1/x_1] \cdots [t_n/x_n]$.
- If $\Sigma \vdash_{ND} \alpha[t_1/x_1] \cdots [t_n/x_n]$, for terms $t_1, \ldots, t_n$, then $\Sigma \vdash_{ND} (\exists x_1 \cdots (\exists x_n\ \alpha) \cdots)$.
- If $\Sigma \vdash_{ND} \alpha[u_1/x_1] \cdots [u_n/x_n]$, with variables $u_1, \ldots, u_n$ fresh, then $\Sigma \vdash_{ND} (\forall x_1 \cdots (\forall x_n\ \alpha) \cdots)$.
- If $\Sigma \vdash_{ND} (\exists x_1 \cdots (\exists x_n\ \alpha) \cdots)$ and $\Sigma \cup \{\alpha[u_1/x_1] \cdots [u_n/x_n]\} \vdash_{ND} \beta$, with $u_1, \ldots, u_n$ fresh, then $\Sigma \vdash \beta$.

Example: Repeated universal quantifiers

Example. Show that $\{(\forall x\,(\forall y\; A(x,y)))\} \vdash_{ND} (\forall y\,(\forall x\; A(x,y)))$.

Old way:

Example: Repeated universal quantifiers

Example. Show that $\{(\forall x\,(\forall y\; A(x,y)))\} \vdash_{ND} (\forall y\,(\forall x\; A(x,y)))$.

Old way:

1.	$(\forall x\,(\forall y\; A(x,y)))$	Premise
2.	v fresh	
3.	u fresh	
4.	$(\forall y\; A(u,y))$	∀e:1
5.	$A(u,v)$	∀e: 4
6.	$(\forall x\; A(x,y))$	∀i:3–5
7.	$(\forall y\,(\forall x\; A(x,y)))$	∀i: 2–6

Example: Repeated universal quantifiers

Example. Show that $\{(\forall x\,(\forall y\; A(x,y)))\} \vdash_{ND} (\forall y\,(\forall x\; A(x,y)))$.

1.	$(\forall x\,(\forall y\; A(x,y)))$	Premise
2.	u, v fresh	
3.	$A(u,v)$	$\forall$e $(\times 2)$: 1
4.	$(\forall y\,(\forall x\; A(x,y)))$	$\forall$i $(\times 2)$: 3

FOL with Equality

Generally, relation symbols have no mandated interpretation. Sometimes, however, one makes an exception for the symbol $=$.

Definition: First-Order Logic with Equality is
First-Order Logic with the restriction that the symbol "$=$" must be interpreted as equality on the domain:

$$(=)^{\mathcal{I}} = \{ \langle \mathsf{d}, \mathsf{d} \rangle \mid \mathsf{d} \in dom(\mathcal{I}) \} \ .$$

There are two ways to account for this restriction in proofs.

1. Add deduction rules for symbol $=$:

   Equals-Introduction: $$\frac{}{(t = t)} = \mathsf{i}$$

   Equals-Elimination: $$\frac{(t_1 = t_2) \quad \alpha[t_1/x]}{\alpha[t_2/x]} = \mathsf{e}$$

2. Alternatively, use axioms rather than deduction rules....

Axioms

Instead of deduction rules for $=$, we shalll use *axioms* for equality.

> *Definition:* An *axiom* is a formula that is assumed as a premise in any proof. An *axiom schema* is a set of axioms, defined by a pattern or rule.

Axioms often behave like additional "inference rules".

An axiom may be used at any time, just as an explicit premise.

Axioms for Equality

EQ1: $(\forall x\ (x = x))$ is an axiom.

EQ2: For each formula α and variable z,

$$\Big(\ \forall x\ \Big(\ \forall y\ \Big((x = y) \to \Big(\alpha[x/z] \to \alpha[y/z]\Big)\Big)\Big)\Big)$$

is an axiom.

These axioms imply

- Symmetry of $=$: $\emptyset \vdash_{ND=} \Big(\forall x\, (\forall y\, ((x = y) \to (y = x)))\Big)$.
- Transitivity of $=$:
  $\emptyset \vdash_{ND=} \Bigg(\forall w\, \Big(\forall x\, \Big(\forall y\, ((x = y) \to ((y = w) \to (x = w)))\Big)\Big)\Bigg)$.

Lemma (EQsymm): $\emptyset \vdash_{ND=} \left(\forall x\left(\forall y\left((x=y) \to (y=x)\right)\right)\right)$.

1.	u fresh	
2.	v fresh	
	$((u=v) \to (v=u))$	???
	$\left(\forall y\left((x=y) \to (y=x)\right)\right)$	∀i: 2–?
13.	$\left(\forall x\left(\forall y\left((x=y) \to (y=x)\right)\right)\right)$	∀i: 1–?

Symmetry of Equality: Proof

Lemma (EQsymm): $\emptyset \vdash_{ND=} \big(\forall x\, (\forall y\, ((x = y) \rightarrow (y = x)))\big)$.

1.	u fresh	
2.	v fresh	
3.	$(u = v)$	Assumption
	$(v = u)$	??
	$((u = v) \rightarrow (v = u))$	$\rightarrow$i: 3–?
	$\big(\forall y\, ((x = y) \rightarrow (y = x))\big)$	$\forall$i: 2–?
13.	$\Big(\forall x\, \big(\forall y\, ((x = y) \rightarrow (y = x))\big)\Big)$	$\forall$i: 1–?

Lemma (EQsymm): $\emptyset \vdash_{ND=} \big(\forall x\, (\forall y\, ((x = y) \to (y = x)))\big)$.

1.	u fresh	
2.	v fresh	
3.	$(u = v)$	Assumption
4.	$\big(\forall x\, \big(\forall y\, \big((x = y) \to ((x = u) \to (y = u))\big)\big)\big)$	EQ2 $[z = u]$
5.	$\big(\forall y\, ((u = y) \to ((u = u) \to (y = u)))\big)$	$\forall$e: 4
6.	$((u = v) \to ((u = u) \to (v = u)))$	$\forall$e: 5
7.	$((u = u) \to (v = u))$	$\to$e: 3, 6
	$(v = u)$	??
	$((u = v) \to (v = u))$	$\to$i: 3–?
	$\big(\forall y\, ((x = y) \to (y = x))\big)$	$\forall$i: 2–?
13.	$\big(\forall x\, \big(\forall y\, ((x = y) \to (y = x))\big)\big)$	$\forall$i: 1–?

Symmetry of Equality: Proof

Lemma (EQsymm): $\emptyset \vdash_{ND=} \Big(\forall x\, (\forall y\, ((x = y) \to (y = x)))\Big)$.

1.	u fresh	
2.	v fresh	
3.	$(u = v)$	Assumption
4.	$\Big(\forall x\, \Big(\forall y\, \Big((x = y) \to ((x = u) \to (y = u))\Big)\Big)\Big)$	EQ2 $[z = u]$
5.	$\Big(\forall y\, ((u = y) \to ((u = u) \to (y = u)))\Big)$	$\forall$e: 4
6.	$((u = v) \to ((u = u) \to (v = u)))$	$\forall$e: 5
7.	$((u = u) \to (v = u))$	$\to$e: 3, 6
8.	$(\forall x\, (x = x))$	EQ1
9.	$(u = u)$	$\forall$e: 8
10.	$(v = u)$	$\to$e: 7, 9
11.	$((u = v) \to (v = u))$	$\to$i: 3–10
12.	$\Big(\forall y\, ((x = y) \to (y = x))\Big)$	$\forall$i: 2–11
13.	$\Big(\forall x\, \Big(\forall y\, ((x = y) \to (y = x))\Big)\Big)$	$\forall$i: 1–12

Symmetry of Equality: Proof (Short cuts)

Lemma (EqSymm): $\emptyset \vdash_{ND=} (\forall x\, (\forall y\, (x = y \to y = x)))$.

1.	u, v fresh	
	$((u = v) \to (v = u))$	???
	$(\forall x\, (\forall y\, ((x = y) \to (y = x))))$	$\forall$i $(\times 2)$: 1–?

Symmetry of Equality: Proof (Short cuts)

Lemma (EqSymm): $\emptyset \vdash_{ND=} (\forall x\, (\forall y\, (x = y \rightarrow y = x)))$.

1.	u, v fresh	
2.	$u = v$	Assumption
	$(v = u)$	??
	$((u = v) \rightarrow (v = u))$	$\rightarrow$i: 2–?
	$(\forall x\, (\forall y\, ((x = y) \rightarrow (y = x))))$	$\forall$i $(\times 2)$: 1–?

Symmetry of Equality: Proof (Short cuts)

Lemma (EqSymm): $\emptyset \vdash_{ND=} (\forall x\, (\forall y\, (x = y \to y = x)))$.

1.	u, v fresh	
2.	$u = v$	Assumption
3.	$(\forall x\, (\forall y\, (x = y) \to ((x = u) \to (y = u)))))$	EQ2 $[(z = u)]$
4.	$((u = v) \to ((u = u) \to (v = u)))$	$\forall$e $(\times 2)$ $[u,\ v]$: 3
5.	$((u = u) \to (v = u))$	$\to$e: 2, 4
	$(v = u)$	??
	$((u = v) \to (v = u))$	$\to$i: 2–?
	$(\forall x\, (\forall y\, ((x = y) \to (y = x))))$	$\forall$i $(\times 2)$: 1–?

Symmetry of Equality: Proof (Short cuts)

Lemma (EqSymm): $\emptyset \vdash_{ND=} (\forall x\, (\forall y\, (x = y \rightarrow y = x)))$.

1.	u, v fresh	
2.	$u = v$	Assumption
3.	$(\forall x\, (\forall y\, (x = y) \rightarrow ((x = u) \rightarrow (y = u)))))$	EQ2 $[\,(z = u)\,]$
4.	$((u = v) \rightarrow ((u = u) \rightarrow (v = u)))$	$\forall$e $(\times 2)$ $[u,\ v]$: 3
5.	$((u = u) \rightarrow (v = u))$	$\rightarrow$e: 2, 4
6.	$((\forall x\, (x = x)))$	EQ1
7.	$(u = u)$	$\forall$e $[u]$: 6
8.	$(v = u)$	$\rightarrow$e: 7,5
9.	$((u = v) \rightarrow (v = u))$	$\rightarrow$i: 2–8
10.	$(\forall x\, (\forall y\, ((x = y) \rightarrow (y = x))))$	$\forall$i $(\times 2)$: 1–10

Transitivity of Equality: Proof (Try also with EQ2 $u = z$!)

Lemma (EQtrans). $\emptyset \vdash_{ND=} (\forall w\, (\forall x\, (\forall y\, ((x = y) \rightarrow ((y = w) \rightarrow (x = w))))))$

1.	$w,\ u,\ v$ fresh	
2.	$(\forall x\, (\forall y\, ((x = y) \rightarrow ((x = w) \rightarrow (y = w)))))$	EQ2 $[z = w]$
3.	$((v = u) \rightarrow ((v = w) \rightarrow (u = w)))$	$\forall$e $(\times 2)$ $[v,\ u]$: 2
4.	$(u = v)$	Assumption
5.	$(v = u)$	EQsymm: 4
6.	$((u = w) \rightarrow (v = w))$	$\rightarrow$e: 5, 3
7.	$((u = v) \rightarrow ((v = w) \rightarrow (u = w)))$	$\rightarrow$i: 4–6
8.	$(\forall w\, (\forall x\, (\forall y$ $((x = y) \rightarrow ((y = w) \rightarrow (x = w))))))$	$\forall$i $(\times 3)$: 1–7

Derived Proof Rules for Equality

Equality satisfies the following derived rules.

EQtrans(k): $\dfrac{(t_1 = t_2) \quad (t_2 = t_3) \quad \cdots \quad (t_k = t_{k+1})}{(t_1 = t_{k+1})}$ for any t_1, ..., t_{k+1}.

EQtrans(k) results from $k-1$ uses of transitivity.

EQsubs(r): $\dfrac{(t_1 = t_2)}{(r[t_1/z] = r[t_2/z])}$ for any variable z and terms r, t_1 and t_2.

Prove as an exercise.