

**Natural Deduction Proof Questions (Try to avoid using derived rules!)**

1.  $\{(\exists x P(x)), (\forall x (P(x) \rightarrow Q(x)))\} \vdash (\exists x Q(x))$
2.  $\{(\exists x (P(x) \vee Q(x)))\} \vdash ((\exists x P(x)) \vee (\exists x Q(x)))$
3.  $\{((\forall x P(x)) \rightarrow (\exists x R(x))), (\forall x (P(x) \wedge Q(x)))\} \vdash (\exists x R(x))$
4.  $\{(\forall x (P(x) \rightarrow Q(x)))\} \vdash ((\forall x P(x)) \rightarrow (\forall x Q(x)))$
5.  $\{(\forall x (Q(x) \rightarrow R(x))), (\exists x (P(x) \wedge Q(x)))\} \vdash (\exists x (P(x) \wedge R(x)))$
6.  $\{(\forall x (\forall y (P(x) \rightarrow Q(y))))\}, (\exists x P(x))\} \vdash (\forall z Q(z))$
7.  $\{(\forall x (\forall y (R(x, y) \rightarrow R(y, x))))\} \vdash ((\forall x (\forall y (R(x, y) \rightarrow R(y, x)))) \wedge (\forall x (\forall y (R(y, x) \rightarrow R(x, y)))))$
8.  $\{(\forall x (\exists y R(x, y)))\} \vdash (\forall x (\exists y (\exists z (R(x, y) \wedge R(y, z))))$
9.  $\left\{ \left( \forall x \left( \forall y \left( \forall z \left( (R(x, y) \wedge R(x, z)) \rightarrow R(y, z) \right) \right) \right) \right), (\forall x R(x, x)) \right\} \vdash (\forall x (\forall y (R(x, y) \rightarrow R(y, z))))$
10.  $\emptyset \vdash ((\forall x (\exists y R(x, y))) \vee (\neg(\forall x R(x, x))))$

**De Morgan's Laws in Predicate Logic:**

11.  $\{(\neg(\exists x P(x)))\} \vdash (\forall x (\neg P(x)))$
12.  $\{(\forall x (\neg P(x)))\} \vdash (\neg(\exists x P(x)))$
13.  $\{(\exists x (\neg P(x)))\} \vdash (\neg(\forall x P(x)))$
14. (Assignment problem!)  $\{(\neg(\forall x P(x)))\} \vdash (\exists x (\neg P(x)))$

## Rules of Natural Deduction

*Connective  
or Quantifier*

*Introduction Rule(s)*

*Elimination Rule(s)*

$\wedge$

$$\frac{\alpha \quad \beta}{(\alpha \wedge \beta)} \wedge i$$

$$\frac{(\alpha \wedge \beta)}{\alpha} \wedge e \quad \frac{(\alpha \wedge \beta)}{\beta} \wedge e$$

$\vee$

$$\frac{\alpha}{(\alpha \vee \beta)} \vee i \quad \frac{\alpha}{(\beta \vee \alpha)} \vee i$$

$$\frac{(\alpha \vee \beta) \quad \begin{array}{|c|} \hline \alpha \\ \vdots \\ \gamma \\ \hline \end{array} \quad \begin{array}{|c|} \hline \beta \\ \vdots \\ \gamma \\ \hline \end{array}}{\gamma} \vee e$$

$\rightarrow$

$$\frac{\begin{array}{|c|} \hline \alpha \\ \vdots \\ \beta \\ \hline \end{array}}{(\alpha \rightarrow \beta)} \rightarrow i$$

$$\frac{\alpha \quad \begin{array}{c} \gamma \\ (\alpha \rightarrow \beta) \end{array}}{\beta} \rightarrow e$$

$\neg$

$$\frac{\begin{array}{|c|} \hline \alpha \\ \vdots \\ \perp \\ \hline \end{array}}{(\neg \alpha)} \neg i$$

(same as  $\perp i$ )

$\perp$

$$\frac{\alpha \quad (\neg \alpha)}{\perp} \perp i$$

$$\frac{\perp}{\alpha} \perp e$$

$\neg\neg$

(derived)

$$\frac{(\neg(\neg \alpha))}{\alpha} \neg\neg e$$

$\forall$

$$\frac{\begin{array}{|c|} \hline u \text{ fresh} \\ \vdots \\ \alpha[u/x] \\ \hline \end{array}}{(\forall x \alpha)} \forall i$$

$$\frac{(\forall x \alpha)}{\alpha[t/x]} \forall e$$

$\exists$

$$\frac{\alpha[t/x]}{(\exists x \alpha)} \exists i$$

$$\frac{(\exists x \alpha) \quad \begin{array}{|c|} \hline \alpha[u/x], u \text{ fresh} \\ \vdots \\ \beta \\ \hline \end{array}}{\beta} \exists e$$

## Derived Rules

$$\frac{(\alpha \rightarrow \beta) \quad (\neg \beta)}{(\neg \alpha)} \text{MT}$$

$$\frac{\alpha}{(\neg(\neg \alpha))} \neg\neg i$$

$$\frac{\begin{array}{|c|} \hline (\neg \alpha) \\ \vdots \\ \perp \\ \hline \end{array}}{\alpha} \text{PBC}$$

$$\frac{}{(\alpha \vee (\neg \alpha))} \text{LEM}$$