Warm-Up Problem

Let $dom(\mathcal{I}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{a} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{b}, \mathsf{b} \rangle\}$.

Let $E(x) = a$ and $E(y) = b$.

What is $\left(\forall x \ (\forall y \ P(x, y))\right)^{(\mathcal{I}, E)}$?

Predicate Logic: Semantic Entailment

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Lecture 12

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Last Time

- Discussed Interpretations and Environments and how these model (give meaning to) predicate logic formulas
- Gave meaning to quantifiers

Learning Goals

- Discussed validity, satisfiability, unsatisfiable.
- State and prove the Relevance Lemma.
- Define what it means for a set of [well-formed] Predicate Logic formulas to semantically entail a [well-formed] formula.
- Solve problems using this definition.

Examples: Value of a Quantified Formula

Example. Let $dom(\mathcal{I}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{a}\rangle, \langle \mathsf{a}, \mathsf{b}\rangle, \langle \mathsf{b}, \mathsf{b}\rangle\}$.

Let $E(x) = a$ and $E(y) = b$. We have

- $P(x, x)^{(\mathcal{I}, E)} = \mathtt{T}$, since $\langle E(x), E(x)\rangle = \langle \mathsf{a}, \mathsf{a}\rangle \in P^{\mathcal{I}}$.
- $P(y, x)^{(\mathcal{I}, E)} = \mathtt{F}$, since $\langle E(y), E(x)\rangle = \langle \mathsf{b}, \mathsf{a}\rangle \notin P^{\mathcal{I}}$.
- $\left(\exists y \; P(y, x)\right)^{(\mathcal{I}, E)} = \mathtt{T}$, since $P(y, x)^{(\mathcal{I}, E[y \mapsto \mathsf{a}])} = \mathtt{T}$.

  (That is, $\langle E[y \mapsto \mathsf{a}](y), E[y \mapsto \mathsf{a}](x)\rangle = \langle \mathsf{a}, \mathsf{a}\rangle \in P^{\mathcal{I}}$).
- What is $\left(\forall x \; (\forall y \; P(x, y))\right)^{(\mathcal{I}, E)}$?

Example. Let $dom(\mathcal{I}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{a} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{b}, \mathsf{b} \rangle\}$.

Let $E(x) = \mathsf{a}$ and $E(y) = \mathsf{b}$.

- What is $\big(\forall x \ (\forall y \ P(x, y))\big)^{(\mathcal{I}, E)}$?

  Since $\langle \mathsf{b}, \mathsf{a} \rangle \notin P^{\mathcal{I}}$, we have

  $$P(x, y)^{(\mathcal{I}, E[x \mapsto \mathsf{b}][y \mapsto \mathsf{a}])} = \mathtt{F} \ ,$$

  and thus

  $$\big(\forall x \ (\forall y \ P(x, y))\big)^{(\mathcal{I}, E)} = \mathtt{F} \ .$$

Examples: Continued

Example. Let $dom(\mathcal{I}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{a} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{b}, \mathsf{b} \rangle\}$.

Let $E(x) = \mathsf{a}$ and $E(y) = \mathsf{b}$.

- What is $\big(\forall x\ (\forall y\ P(x,y))\big)^{(\mathcal{I},E)}$?

  Since $\langle \mathsf{b}, \mathsf{a} \rangle \notin P^{\mathcal{I}}$, we have

  $$P(x,y)^{(\mathcal{I},E[x\mapsto \mathsf{b}][y \mapsto \mathsf{a}])} = \mathtt{F} \ ,$$

  and thus

  $$\big(\forall x\ (\forall y\ P(x,y))\big)^{(\mathcal{I},E)} = \mathtt{F} \ .$$

- What about $\big(\forall x\ (\exists y\ P(x,y))\big)^{(\mathcal{I},E)}$?

A Question of Syntax

In the previous example, we wrote

$$P(x,y)^{(\mathcal{I},E[x\mapsto \mathsf{b}][y\mapsto \mathsf{a}])} = \mathtt{F} \ \ .$$

Why did we not write simply

$$P(\mathsf{b},\mathsf{a}) = \mathtt{F}$$

or perhaps

$$P(\mathsf{b},\mathsf{a})^{(\mathcal{I},E)} = \mathtt{F} \ \ ?$$

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Why did we not write simply

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or perhaps

$$P(\mathsf{b}, \mathsf{a})^{(\mathcal{I}, E)} = \mathtt{F} \ ?$$

Because "$P(\mathsf{b}, \mathsf{a})$" is not a formula. The elements a and b of $dom(\mathcal{I})$ are not symbols in the language; they cannot appear in a formula.

Satisfaction of Formulas

An interpretation $\mathcal{I}$ and environment E *satisfy* a formula α, denoted $\mathcal{I} \vDash_E \alpha$, iff $\alpha^{(\mathcal{I},E)} = \mathtt{T}$;
they do not satisfy α, denoted $\mathcal{I} \nvDash_E \alpha$, if $\alpha^{(\mathcal{I},E)} = \mathtt{F}$.

Form of α	*Condition for $\mathcal{I} \vDash_E \alpha$*
$P(t_1, \dots, t_k)$	$\left\langle t_1^{(\mathcal{I},E)}, \dots, t_k^{(\mathcal{I},E)} \right\rangle \in P^{\mathcal{I}}$
$(\neg\beta)$	$\mathcal{I} \nvDash_E \beta$
$(\beta \wedge \gamma)$	both $\mathcal{I} \vDash_E \beta$ and $\mathcal{I} \vDash_E \gamma$
$(\beta \vee \gamma)$	either $\mathcal{I} \vDash_E \beta$ or $\mathcal{I} \vDash_E \gamma$ (or both)
$(\beta \to \gamma)$	either $\mathcal{I} \nvDash_E \beta$ or $\mathcal{I} \vDash_E \gamma$ (or both)
$(\forall x\ \beta)$	for every $\mathsf{a} \in dom(\mathcal{I})$, $\mathcal{I} \vDash_{E[x \mapsto \mathsf{a}]} \beta$
$(\exists x\ \beta)$	there is some $\mathsf{a} \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x \mapsto \mathsf{a}]} \beta$

If $\mathcal{I} \vDash_E \alpha$ for every E, then $\mathcal{I}$ *satisfies* α, denoted $\mathcal{I} \vDash \alpha$.

Example. Consider the formula $(\exists y\ P(x, y \oplus y))$.

(For P a binary predicate and $\oplus$ a binary function.)

Suppose $dom(\mathcal{I}) = \{1, 2, 3, ...\}$,
$\oplus^{\mathcal{I}}$ is the addition operation, and
$P^{\mathcal{I}}$ is the equality predicate.

Give a simple condition that determines when
$\mathcal{I} \vDash_E (\exists y\ P(x, (y \oplus y)))$ holds.

Example: Satisfaction

Example. Consider the formula $(\exists y\ P(x, y \oplus y))$.

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$\mathcal{I} \vDash_E (\exists y\ P(x, (y \oplus y)))$ holds.

$\mathcal{I} \vDash_E (\exists y\ P(x, (y \oplus y)))$ iff $E(x)$ is an even number.

Validity and Satisfiability

Validity and satisfiability of formulas have definitions analogous to the ones for propositional logic.

Definition: A formula α is

- *valid* if every interpretation and environment satisfy α; that is, if $\mathcal{I} \models_E \alpha$ for every $\mathcal{I}$ and E,
- *satisfiable* if some interpretation and environment satisfy α; that is, if $\mathcal{I} \models_E \alpha$ for some $\mathcal{I}$ and E, and
- *unsatisfiable* if no interpretation and environment satisfy α; that is, if $\mathcal{I} \not\models_E \alpha$ for every $\mathcal{I}$ and E.

(The term "tautology" is not used in predicate logic.)

Note

If there is no need to specify an environment, then simply defining an interpretation $\mathcal{I}$ and writing $\mathcal{I} \vDash \alpha$ will suffice.

Revisit Example

Let $f^{(1)}$ and $h^{(2)}$ be function symbols. Let $P^{(1)}$ and $Q^{(2)}$ be predicate symbols, let a, b, c be constant symbols and let x, y, z be variable symbols. Define an interpretation by:

- Domain: $\mathcal{D} = \{1, 2, 3\}$
- Constants: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}} : f^{\mathcal{I}}(1) = 2,\ f^{\mathcal{I}}(2) = 3,\ f^{\mathcal{I}}(3) = 1$
- $h^{\mathcal{I}} : (x, y) \mapsto \min\{x, y\}$ (min is the minimum function)
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$
- $Q^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$

and define an environment E by

$$E(x) = 3, E(y) = 3, E(z) = 1.$$

- Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.
- Give a new interpretation $\mathcal{J}_2$ and environment G_2 satisfying $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution: Set $G_1 = E$ and $\mathcal{J}_1$ to be $\mathcal{I}$ except, let $P^{\mathcal{J}_1} = \emptyset$. Then $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution: Set $G_1 = E$ and $\mathcal{J}_1$ to be $\mathcal{I}$ except, let $P^{\mathcal{J}_1} = \emptyset$. Then $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Give a new interpretation $\mathcal{J}_2$ and environment G_2 satisfying $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution: Set $G_1 = E$ and $\mathcal{J}_1$ to be $\mathcal{I}$ except, let $P^{\mathcal{J}_1} = \emptyset$. Then $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Give a new interpretation $\mathcal{J}_2$ and environment G_2 satisfying $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$.

Solution: Set $G_2 = E$ and $\mathcal{J}_2$ to be $\mathcal{I}$ except, let $Q^{\mathcal{J}_2} = \emptyset$. Then $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$..

Relevance Lemma

Lemma:

Let α be a first-order formula, $\mathcal{I}$ be an interpretation, and E_1 and E_2 be two environments such that

$$E_1(x) = E_2(x) \text{ for every } x \text{ that occurs free in } \alpha.$$

Then

$$\mathcal{I} \vDash_{E_1} \alpha \text{ if and only if } \mathcal{I} \vDash_{E_2} \alpha \ .$$

Proof by induction on the structure of α.

Example: Satisfiability and Validity

Let $\mathcal{L}$ be a language consisting of variables x, y, z, function symbols $f^{(2)}$, $g^{(1)}$ and predicate symbol $P^{(2)}$. Let α be the formula $P\big(f(g(x), g(y)), g(z)\big)$. The formula is satisfiable:

- $dom(\boldsymbol{\mathcal{I}})$: $\mathbb{N}$
- $f^{\boldsymbol{\mathcal{I}}}$: summation
- $g^{\boldsymbol{\mathcal{I}}}$: squaring
- $P^{\boldsymbol{\mathcal{I}}}$: equality
- $E(x) = 3$, $E(y) = 4$ and $E(z) = 5$.

α is not valid. (Why?)

Another Example

Let $\mathcal{L}$ be a language consisting of variables x, y, predicate symbol $Q^{(2)}$.
Let $\mathcal{I}$ be the interpretation defined by

- Domain: $\mathcal{D} = \{1, 2\}$ $\qquad Q^{\mathcal{I}} = \emptyset$

and environment E given by:

- $E(x) = 1 \qquad E(y) = 2$

1. Show that $\mathcal{I} \not\models_E \alpha$ where $\alpha \stackrel{\text{def}}{=} \left(\exists x \left(\forall y\ Q(x, y)\right)\right)$.
2. Give an interpretation $\mathcal{J}$ and an environment G such that $\mathcal{J} \models_G (\exists x\ Q(x, y))$
3. Give an interpretation $\mathcal{J}$ and an environment G such that $\mathcal{J} \models_G (\forall x\ Q(x, y))$
4. Give an interpretation $\mathcal{J}$ and an environment G such that $\mathcal{J} \models_G \left(\exists x \left(\forall y\ Q(x, y)\right)\right)$
5. Is $\left(\forall x \left(\exists y\ Q(x, y)\right)\right)$ valid? Why or why not?

Semantic Entailment

Let Σ be a set of well-formed Predicate logic formulas and α is a well-formed predicate logic formula.

For interpretation $\mathcal{I}$ and environment E, we write $\mathcal{I} \vDash_E \Sigma$ if and only if for every $\varphi \in \Sigma$, we have that $\mathcal{I} \vDash_E \varphi$.

We say that Σ is a *semantically entails* α (or that α is a *logical consequence* of Σ), written as $\Sigma \vDash \alpha$, if and only if for any interpretation $\mathcal{I}$ and environment E, we have $\mathcal{I} \vDash_E \Sigma$ implies $\mathcal{I} \vDash_E \alpha$. This can also be written as $\alpha^{(\mathcal{I}, E)} = \mathtt{T}$.

$\emptyset \vDash \alpha$ means that α is valid.

Notes

Suppose $\Sigma = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ Then $\Sigma \vDash \beta$ means...

- ...that every pair of interpretation and environment that makes Σ true must also make β true.
- ...that $\emptyset \vDash ((\alpha_1 \wedge (\alpha_2 \wedge (... \wedge \alpha_n))) \to \beta)$
- ...that $((\alpha_1 \wedge (\alpha_2 \wedge (... \wedge \alpha_n))) \to \beta)$ is valid

To prove these, take an arbitrary interpretation $\mathcal{I}$ and environment E and show that if this satisfied Σ then it must also satisfy β. You may also assume towards a contradiction that $\mathcal{I} \nvDash_E \beta$ and proceed from there if this helps.

To prove that $\Sigma \nvDash \beta$ find an interpretation $\mathcal{I}$ and environment E that satisfies Σ but that doesn't satisfy β, that is, show that $\mathcal{I} \nvDash_E \beta$.

Example: Semantic Entailment

Example: Show that for any well-formed Predicate formulas α and β:

$$\emptyset \vDash \big((\forall x\,(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))\big)\ .$$

Example: Semantic Entailment

Example: Show that for any well-formed Predicate formulas α and β:

$$\emptyset \vDash \big((\forall x\,(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))\big)\ .$$

Proof by contradiction. Suppose there are $\mathcal{I}$ and E such that

$$\mathcal{I} \nvDash_E \big((\forall x\,(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))\big)\ .$$

Then we must have $\mathcal{I} \vDash_E (\forall x\,(\alpha \to \beta))$ and $\mathcal{I} \nvDash_E \big((\forall x\ \alpha) \to (\forall x\ \beta)\big)$;

the second gives $\mathcal{I} \vDash_E (\forall x\ \alpha)$ and $\mathcal{I} \nvDash_E (\forall x\ \beta)$.

Using the definition of $\vDash$ for formulas with $\forall$, we have
for every a $\in dom(\mathcal{I})$, $\mathcal{I} \vDash_{E[x \mapsto \mathsf{a}]} (\alpha \to \beta)$ and $\mathcal{I} \vDash_{E[x \mapsto \mathsf{a}]} \alpha$.
Thus also $\mathcal{I} \vDash_{E[x \mapsto \mathsf{a}]} \beta$ for every a $\in dom(\mathcal{I})$.

Thus $\mathcal{I} \vDash_E (\forall x\ \beta)$, a contradiction.

Example

Example. Show that $\{(\forall x\,(\neg\gamma))\} \models (\neg(\exists x\ \gamma))$.

Example

Example. Show that $\{(\forall x\,(\neg\gamma))\} \models (\neg(\exists x\ \gamma))$.

Proof: Suppose that $\mathcal{I} \models_E (\forall x\,(\neg\gamma))$. By definition, this means

> for every a $\in dom(\mathcal{I})$, $\mathcal{I} \models_{E[x\mapsto \mathsf{a}]} (\neg\gamma)$. that is, $(\neg\gamma)^{(\mathcal{I}, E[x\mapsto \mathsf{a}])} = \mathtt{T}$.

Again by definition (for a formula with $\neg$), this is equivalent to

> for every a $\in dom(\mathcal{I})$, $\mathcal{I} \not\models_{E[x\mapsto \mathsf{a}]} \gamma$ that is, $\gamma^{(\mathcal{I}, E[x\mapsto \mathsf{a}])} = \mathtt{F}$.

and also

> there is no a $\in dom(\mathcal{I})$ such that $\mathcal{I} \models_{E[x\mapsto \mathsf{a}]} \gamma$.

Assuming towards a contradiction that $(\exists x\ \gamma)^{(\mathcal{I}, E)} = \mathtt{T}$, this would mean that there is an b $\in dom(\mathcal{I})$ such that $\mathcal{I} \models_{E[x\mapsto \mathsf{b}]} \gamma$ which contradicts the previous line. Hence $\mathcal{I} \models_E (\neg(\exists x\ \gamma))$ holds as required.

Example

Example: Find well-formed Predicate formulas α and β such that

$$\{((\forall x\ \alpha) \to (\forall x\ \beta))\} \not\models (\forall x\,(\alpha \to \beta))\ .$$

Example

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$$\{((\forall x\ \alpha) \to (\forall x\ \beta))\} \not\models (\forall x\, (\alpha \to \beta))\ .$$

Key idea: $(\varphi_1 \to \varphi_2)$ yields true whenever φ_1 is false.

Let α be $P(x)$. Let $\mathcal{I}$ have domain $\{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\mathsf{a}\}$. Then $\mathcal{I} \models ((\forall x\ \alpha) \to (\forall x\ \beta))$ for any β. (Why?)

Example

Example: Find well-formed Predicate formulas α and β such that

$$\{((\forall x\ \alpha) \rightarrow (\forall x\ \beta))\} \not\models (\forall x\,(\alpha \rightarrow \beta)) \ .$$

Key idea: $(\varphi_1 \rightarrow \varphi_2)$ yields true whenever φ_1 is false.

Let α be $P(x)$. Let $\mathcal{I}$ have domain $\{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\mathsf{a}\}$. Then $\mathcal{I} \models ((\forall x\ \alpha) \rightarrow (\forall x\ \beta))$ for any β. (Why?)

To obtain $\mathcal{I} \not\models \forall x\,(\alpha \rightarrow \beta)$, we can use $\neg P(x)$ for β. (Why?)

Thus $((\forall x\ \alpha) \rightarrow (\forall x\ \beta)) \not\models (\forall x\,(\alpha \rightarrow \beta))$, as required. (Why?)

Example: For any formula α and term t, show that

$$\emptyset \vDash ((\forall x\ \alpha) \to \alpha[t/x]) \ .$$

Recall that functions must be total!

Another Example

Let α be any well-formed Predicate formula **without** a free variable x. Let $\mathcal{I}$ be any interpretation and let E be any environment. Then

$$\alpha^{(\mathcal{I},E)} = (\forall x\ \alpha)^{(\mathcal{I},E)}.$$

Another Example

Let α be any well-formed Predicate formula **without** a free variable x. Let $\mathcal{I}$ be any interpretation and let E be any environment. Then

$$\alpha^{(\mathcal{I},E)} = (\forall x\ \alpha)^{(\mathcal{I},E)}.$$

Proof. Let $\mathcal{D}$ be the domain of $\mathcal{I}$. Since x is not free in α, therefore $E(y) = E[x \mapsto a](y)$, for every $a \in \mathcal{D}$ and for every y that occurs free in α.

Then by the Relevance Lemma, we have that

- $\mathcal{I} \vDash_E \alpha$
- if and only if $\mathcal{I} \vDash_{E[x \mapsto a]} \alpha$, for any $a \in \mathcal{D}$.
- if and only if $\mathcal{I} \vDash_E (\forall x\ \alpha)$,

which establishes the desired result.

Example

Example: Show that

$$\{(\forall x\,(\exists y\ P(x,y)))\} \not\models (\exists y\,(\forall x\ P(x,y)))$$

Example

Example: Show that

$$\{(\forall x\,(\exists y\ P(x,y)))\} \not\models (\exists y\,(\forall x\ P(x,y)))$$

Solution: We simply need to come up with an interpretation and an environment satisfying

$$\mathcal{I} \models_E \big((\forall x\,(\exists y\ P(x,y)))\big) \qquad \text{and} \qquad \mathcal{I} \not\models_E \big((\exists y\,(\forall x\ P(x,y)))\big)$$

Define an interpretation $\mathcal{I}$ with $dom(\mathcal{I}) = \{1,2,3\}$ and predicate $P^{\mathcal{I}} = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 3,1\rangle\}$. Further, let E be an arbitrary environment. Now, we need to show that

$$(\forall x\,(\exists y\ P(x,y)))^{(\mathcal{I},E)} = \mathtt{T} \qquad \text{and} \qquad (\exists y\,(\forall x\ P(x,y)))^{(\mathcal{I},E)} = \mathtt{F}$$

Example Continued (2 of 3)

With $dom(\mathcal{I}) = \{1, 2, 3\}$ and predicate $P^{\mathcal{I}} = \{\langle 1, 1\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$, we have

$$\big(\forall x\, (\exists y\ P(x, y))\big)^{(\mathcal{I}, E)} = \mathtt{T}$$

because

$$P(x, y)^{(\mathcal{I}, E[x\mapsto 1][y\mapsto 1])} = \mathtt{T} \text{ since } \langle 1, 1\rangle \in P^{\mathcal{I}}$$
$$P(x, y)^{(\mathcal{I}, E[x\mapsto 2][y\mapsto 3])} = \mathtt{T} \text{ since } \langle 2, 3\rangle \in P^{\mathcal{I}}$$
$$P(x, y)^{(\mathcal{I}, E[x\mapsto 3][y\mapsto 1])} = \mathtt{T} \text{ since } \langle 3, 1\rangle \in P^{\mathcal{I}}$$

With $dom(\mathcal{I}) = \{1, 2, 3\}$ and predicate $P^{\mathcal{I}} = \{\langle 1, 1\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$, we have

$$\left(\exists y \left(\forall x \; P(x, y)\right)\right)^{(\mathcal{I}, E)} = \mathtt{F}$$

because

$$P(x, y)^{(\mathcal{I}, E[y \mapsto 1][x \mapsto 2])} = \mathtt{F} \text{ since } \langle 2, 1\rangle \notin P^{\mathcal{I}}$$
$$P(x, y)^{(\mathcal{I}, E[y \mapsto 2][x \mapsto 1])} = \mathtt{F} \text{ since } \langle 1, 2\rangle \notin P^{\mathcal{I}}$$
$$P(x, y)^{(\mathcal{I}, E[y \mapsto 3][x \mapsto 1])} = \mathtt{F} \text{ since } \langle 1, 3\rangle \notin P^{\mathcal{I}}$$

(Think above, we are really proving that $\left(\forall y \left(\exists x \left(\neg P(x, y)\right)\right)\right)^{(\mathcal{I}, E)} = \mathtt{F}$).
Thus,

$$\{\left(\forall x \left(\exists y \; P(x, y)\right)\right)\} \not\models \left(\exists y \left(\forall x \; P(x, y)\right)\right)$$