Warm-Up Problem

Determine $\alpha[(y-z)/x]$ where

$$\alpha \stackrel{\text{def}}{=} \Bigg(\exists z \Bigg(P(z) \to \Big(Q(x,y) \wedge \big(\forall y \left(P(y) \vee (\exists x\ P(x))\right)\big)\Big)\Bigg)\Bigg)$$

with $P^{(1)}$ and $Q^{(2)}$ predicates and x, y, z variables.

Predicate Logic: Semantics, Interpretations and Environments

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Lecture 11

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Last Time

- Did another substitution example (Please Review!)
- Discussed Interpretations with respect to Predicate Logic

Learning Goals

- Define an **interpretation** and an **environment**.
- Give examples of interpretations and environments in specific situations.
- Define validity, satisfiable and unsatisfiable.

Leading Question

Given a well-formed Predicate logic formula, is it T or F in some context?

- In Propositional logic, a truth valuation was enough to assign a meaning to our atoms (propositional variables)
- In Predicate logic, we need **a lot** more.

Motivating Example

For example, if we consider the formula (for $P^{(2)}$ a predicate symbol and variable x)

$$\alpha \stackrel{\text{def}}{=} (\forall x \; P(x, x))$$

and we use an interpretation $\mathcal{I}$ of $P(x, x)$ to mean x is equal to x and consider a domain $\mathcal{D} = \{1\}$, then indeed, α is true under this interpretation.

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However, if we consider an interpretation $\mathcal{J}$ satisfying $P(x, x)$ is x is greater than x and still consider the domain $\mathcal{D} = \{1\}$, then α is false under this interpretation.

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We will formalize the notation of interpretations (and later environments) and explain what it means for a Predicate logic formula to be **valid**, **satisfiable**, and **unsatisfiable**.

Brief Definition

An interpretation consists of a domain as well as meanings for all of the constant, function and predicate symbols.

Huth and Ryan use the term “model” instead of “interpretation”.

More formally...

Semantics: Interpretations

Definition: Fix a set $\mathcal{L}$ of constant symbols, function symbols, variable symbols and predicate symbols. (The "language" of our formulas.)

An *interpretation* $\mathcal{I}$ (for the set $\mathcal{L}$) consists of

- A non-empty set $dom(\mathcal{I})$ or $\mathcal{D}^{\mathcal{I}}$ or more simply $\mathcal{D}$, called the domain (or universe) of $\mathcal{I}$.
- For each constant symbol c, a member $c^{\mathcal{I}}$ of $dom(\mathcal{I})$.
- For each function symbol $f^{(i)}$, an i-ary function $f^{\mathcal{I}}$.
- For each predicate symbol $P^{(i)}$, an i-ary predicate (relation) $P^{\mathcal{I}}$.

When there are no variables and no quantifiers, this is more than enough to specify meaning to a formula.

High Brow Comment

Technically, our language should have all of the variable symbols we will ever need. In practice this is a bit cumbersome so we will usually forgo including variables in our language explicitly and simply use them as they appear in our formulas.

Values of Variable-Free Terms

Definition: Fix an interpretation $\mathcal{I}$. For each term t containing no variables, the value of t under interpretation $\mathcal{I}$, denoted $t^{\mathcal{I}}$, is as follows.

- If t is a constant c, the value $t^{\mathcal{I}}$ is $c^{\mathcal{I}}$.
- If t is $f(t_1, \dots, t_n)$, the value $t^{\mathcal{I}}$ is $f^{\mathcal{I}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}})$.

The value of a term is always a member of the domain of $\mathcal{I}$.

For example, consider f a unary function and 0 a constant. If we have an interpretation $\mathcal{I}$ with domain $\mathbb{N}$, $0^{\mathcal{I}}$ the usual zero and $f^{\mathcal{I}}$ the usual successor function (increment by 1), then

$$f(0)^{\mathcal{I}} = f^{\mathcal{I}}(0^{\mathcal{I}}) = 1$$

Watch out!

Notice that in the previous example, even though we used the constant symbol 0, we still needed to specify that the interpretation of 0 is indeed the usual zero.

For example, consider f a unary function and 0 a constant. If we have an interpretation $\mathcal{I}$ with domain $\mathbb{N}$, $0^{\mathcal{I}}$ to be the usual number 1 and $f^{\mathcal{I}}$ the usual successor function (increment by 1), then

$$f(0)^{\mathcal{I}} = f^{\mathcal{I}}(0^{\mathcal{I}}) = 2$$

While doable, this is not advised...

Clarity

Most of these issues are taken care of by not using a symbol that could be misinterpreted as being in the domain.

For example, consider f a unary function and a a constant. If we have an interpretation $\mathcal{I}$ with domain $\mathbb{N}$, $a^{\mathcal{I}}$ to be the usual number 1 and $f^{\mathcal{I}}$ the usual successor function (increment by 1), then

$$f(a)^{\mathcal{I}} = f^{\mathcal{I}}(a^{\mathcal{I}}) = 2$$

Total Function

Another issue arises that your function's interpretation must be defined on the entire domain! for a function with arity k, we need to define an interpretation such that the function $f^{\mathcal{I}}$ satisfies:

$$f^{\mathcal{I}} : \mathcal{D}^k \to \mathcal{D}$$

that is, *every* k-tuple from the domain must map into the domain. Such functions capable of doing this are called **total functions**.

- For example, the usual addition over the natural numbers is total since the sum of any two natural numbers gives another natural number.
- However, the usual subtraction over the natural numbers is not total. For example, we cannot perform $2 - 6$ and get a natural number.
- Similarly, square roots over the integers (or even the real numbers!) is not a total function since the square root of -1 is not an integer (or a real number).

Formulas with Variable-Free Terms

Formulas get values in much the same fashion as terms,
except that values of formulas lie in $\{\mathtt{T}, \mathtt{F}\}$.

Definition: Fix an interpretation $\mathcal{I}$. For each formula α containing no variables, the value of α under interpretation $\mathcal{I}$, denoted $\alpha^{\mathcal{I}}$, is as follows.

- If α is $P(t_1, \dots, t_n)$, then

$$\alpha^{\mathcal{I}} = \begin{cases} \mathtt{T} & \text{if } \langle t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}} \rangle \in P^{\mathcal{I}} \\ \mathtt{F} & \text{otherwise.} \end{cases}$$

- If α is $(\neg\beta)$ or $(\beta \star \gamma)$, then $\alpha^{\mathcal{I}}$ is determined by $\beta^{\mathcal{I}}$ and $\gamma^{\mathcal{I}}$ in the same way as for propositional logic.

Example

Let $f^{(1)}$ and $h^{(2)}$ be function symbols. Let $P^{(1)}$ and $Q^{(2)}$ be predicate symbols and let a, b, c be constant symbols. Define an interpretation by:

- Domain: $\mathcal{D} = \{1, 2, 3\}$
- Constants: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}} : f^{\mathcal{I}}(1) = 2,\ f^{\mathcal{I}}(2) = 3,\ f^{\mathcal{I}}(3) = 1$
- $h^{\mathcal{I}} : (x, y) \mapsto \min\{x, y\}$ (min is the minimum function)
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$
- $Q^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$

What is the meaning of each of these formulas in this interpretation?

- $f(h(f(a), f(c)))^{\mathcal{I}}$
- $f(h(b, f(a)))^{\mathcal{I}}$
- $Q(f(c), a)^{\mathcal{I}}$
- $P(h(f(a), f(c)))^{\mathcal{I}}$

Follow Ups

We saw that under $\mathcal{I}$ we have that $\langle f^{\mathcal{I}}(c^{\mathcal{I}}), a^{\mathcal{I}} \rangle \notin Q^{\mathcal{I}}$. Is there another interpretation $\mathcal{J}$ such that $\langle f^{\mathcal{J}}(c^{\mathcal{J}}), a^{\mathcal{J}} \rangle \in Q^{\mathcal{J}}$?

Follow Ups

We saw that under $\mathcal{I}$ we have that $\langle f^{\mathcal{I}}(c^{\mathcal{I}}), a^{\mathcal{I}}\rangle \notin Q^{\mathcal{I}}$. Is there another interpretation $\mathcal{J}$ such that $\langle f^{\mathcal{J}}(c^{\mathcal{J}}), a^{\mathcal{J}}\rangle \in Q^{\mathcal{J}}$?

Notice that $f(c)^{\mathcal{I}} = 1$ and $a^{\mathcal{I}} = 1$. So if we set $\mathcal{J} = \mathcal{I}$ except we define

$$Q^{\mathcal{J}} = \{\langle 1, 1\rangle\}$$

then we see that $\langle f^{\mathcal{J}}(c^{\mathcal{J}}), a^{\mathcal{J}}\rangle \in Q^{\mathcal{J}}$.

Follow Ups

We saw that under $\mathcal{I}$ we have that $P(h(f(a), f(c)))^{\mathcal{I}} = \texttt{T}$. Is there another interpretation $\mathcal{K}$ such that $P(h(f(a), f(c)))^{\mathcal{K}} = \texttt{F}$?

Follow Ups

We saw that under $\mathcal{I}$ we have that $P(h(f(a), f(c)))^{\mathcal{I}} = \texttt{T}$. Is there another interpretation $\mathcal{K}$ such that $P(h(f(a), f(c)))^{\mathcal{K}} = \texttt{F}$?

Notice that $f(c)^{\mathcal{I}} = 1$ and $f(a)^{\mathcal{I}} = 2$ and so $h(f(a), f(c))^{\mathcal{I}} = 1$. Thus, if we set $\mathcal{K} = \mathcal{I}$ except we define

$$P^{\mathcal{K}} = \emptyset$$

so that $1 \notin P^{\mathcal{K}}$, we see that $P(h(f(a), f(c)))^{\mathcal{K}} = \texttt{F}$.

Environments

How do we deal with variables and quantifiers?

Definition: An environment is a function that assigns a value in the domain to every variable symbol in the language.

- An environment needs to be defined on **all** variables [in the language]!
- We will see in practice, environments will only be used to interpret free variables but must nonetheless be defined on all variables including bound variables.
- Bound variables will get their meaning primarily through the corresponding quantifier.

Meaning of Terms

The combination of an interpretation and an environment supplies a value for every term.

Definition: Fix an interpretation $\mathcal{I}$ and environment E. For each term t, the value of t under $\mathcal{I}$ and E, denoted $t^{(\mathcal{I},E)}$, is as follows.

- If t is a constant c, the value $t^{(\mathcal{I},E)}$ is $c^{\mathcal{I}}$.
- If t is a variable x, the value $t^{(\mathcal{I},E)}$ is x^{E}.
- If t is $f(t_1, \ldots, t_n)$, the value $t^{(\mathcal{I},E)}$ is $f^{\mathcal{I}}(t_1^{(\mathcal{I},E)}, \ldots, t_n^{(\mathcal{I},E)})$.

To extend this definition to formulas, we must consider quantifiers.

But first, a few examples.

Example: Let α_1 be $P(c)$ (where c is a constant), and let α_2 be $P(x)$ (where x a variable).

Let $\mathcal{I}$ be the interpretation with domain $\mathbb{N}$, $c^{\mathcal{I}} = 2$ and $P^{\mathcal{I}} =$ "is even". Then $\alpha_1^{\mathcal{I}} = \mathtt{T}$, but $\alpha_2^{\mathcal{I}}$ is undefined.

Constants Vs. Variables

Example: Let α_1 be $P(c)$ (where c is a constant), and let α_2 be $P(x)$ (where x a variable).

Let $\mathcal{I}$ be the interpretation with domain $\mathbb{N}$, $c^{\mathcal{I}} = 2$ and $P^{\mathcal{I}} =$ "is even". Then $\alpha_1^{\mathcal{I}} = \mathtt{T}$, but $\alpha_2^{\mathcal{I}}$ is undefined.

To give α_2 a value, we must also specify an environment. For example, if $E(x) = 2$, then $\alpha_2^{(\mathcal{I},E)} = \mathtt{T}$.

Example

Let $f^{(1)}$ and $h^{(2)}$ be function symbols. Let $P^{(1)}$ and $Q^{(2)}$ be predicate symbols, let a, b, c be constant symbols and let x, y, z be variable symbols. Define an interpretation $\mathcal{I}$ by:

- Domain: $\mathcal{D} = \{1, 2, 3\}$
- Constants: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}} : f^{\mathcal{I}}(1) = 2,\ f^{\mathcal{I}}(2) = 3,\ f^{\mathcal{I}}(3) = 1$
- $h^{\mathcal{I}} : (x, y) \mapsto \min\{x, y\}$ (min is the minimum function)
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$
- $Q^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$

and define an environment E by

$$E(x) = 3, E(y) = 3, E(z) = 1.$$

(see next slide...)

Interpretations

What is the meaning of each of these formulas in the interpretation and environment on the previous slide ?

- $f(h(f(a), z))^{(\mathcal{I}, E)}$
- $f(h(y, c))^{(\mathcal{I}, E)}$
- $Q(x, h(a, b))^{(\mathcal{I}, E)}$
- $P(h(f(a), x))^{(\mathcal{I}, E)}$

Meaning of Terms—Example

Example. Suppose a language has constant symbol 0, a unary function s, and a binary function $+$. We shall write $+$ in infix position: $(x+y)$ instead of $+(x,y)$.

The expressions $s((s(0)+s(x)))$ and $s\big((x+s((x+s(0))))\big)$ are both terms.

The following are examples of interpretations and environments.

- $\mathcal{D} = dom\{\mathcal{I}\} = \mathbb{N}$, $0^{\mathcal{I}} = 0$, $s^{\mathcal{I}}$ is the successor function and $+^{\mathcal{I}}$ is the addition operation. Then, if $E(x)=3$, the terms get values $\big(s((s(0)+s(x)))\big)^{(\mathcal{I},E)} = 6$ and $\Big(s\big((x+s((x+s(0))))\big)\Big)^{(\mathcal{I},E)} = 9$.

- $\mathcal{D} = dom\{\mathcal{J}\}$ is the collection of all words over the alphabet $\{a, b\}$,
  $0^{\mathcal{J}} = a$,
  $s^{\mathcal{J}}$ appends a to the end of a string, and
  $+^{\mathcal{J}}$ is concatenation.

  Let $G(x) = aba$. Then

  $$s\big((s(0) + s(x))\big)^{(\mathcal{J},G)} = aaabaaa$$

  and

  $$s\big((x + s((x + s(0))))\big)^{(\mathcal{J},G)} = abaabaaaaa \quad .$$

Quantifiers

- Finally, we can evaluate formulas with free and bound variables.
- How can we evaluate a formula of the form $(\forall x\ \alpha)$ or $(\exists x\ \alpha)$?
  - For $(\forall x\ \alpha)$, we need to verify that α is true for every possible value of x in the domain
  - For $(\exists x\ \alpha)$ we need to verify that α is true for at least one possible value of x in the domain

We formalize this on the next few slides.

Quantifiers Over Finite Domains

The universal and existential quantifiers may be understood respectively as generalizations of conjunction and disjunction. If the domain $D = \{a_1, \dots, a_k\}$ is finite then:

For all x, $P(x)$ iff $P(a_1)$ and ... and $P(a_k)$

There exists x, $P(x)$ iff $P(a_1)$ or ... or $P(a_k)$

where P is a predicate (a property).

Quantified Formulas

Definition: For any environment E and domain element d, the environment "E with x re-assigned to d", denoted $E[x \mapsto \mathsf{d}]$, is given by

$$E[x \mapsto \mathsf{d}](y) = \begin{cases} \mathsf{d} & \text{if } y \text{ is } x \\ E(y) & \text{if } y \text{ is not } x. \end{cases}$$

In other words, $E[x \mapsto \mathsf{d}](x) = d$ and for any other variable y, $E[x \mapsto \mathsf{d}](y) = E(y)$.

Key point: $E[x \mapsto \mathsf{d}]$ is just a new environment!

Example

Let $\mathcal{D} = \{1, 2, 3\}$ for some interpretation $\mathcal{I}$ and consider E as defined by

$$E(x) = 3 \qquad E(y) = 3 \qquad E(z) = 1$$

Then

$$E[x \mapsto 2](x) = 2 \qquad E[x \mapsto 2](y) = 3 \qquad E[x \mapsto 2](z) = 1$$

What about the following?

$$E[x \mapsto 2][y \mapsto 2](x) \qquad E[x \mapsto 2][y \mapsto 2](y) \qquad E[x \mapsto 2][y \mapsto 2](z)$$

Values of Quantified Formulas

Definition: The values of $(\forall x\ \alpha)$ and $(\exists x\ \alpha)$ are given by

- $(\forall x\ \alpha)^{(\mathcal{I},E)} = \begin{cases} \mathtt{T} & \text{if } \alpha^{(\mathcal{I},E[x\mapsto \mathsf{d}])} = \mathtt{T} \text{ for every d in } dom(\mathcal{I}) \\ \mathtt{F} & \text{otherwise} \end{cases}$
- $(\exists x\ \alpha)^{(\mathcal{I},E)} = \begin{cases} \mathtt{T} & \text{if } \alpha^{(\mathcal{I},E[x\mapsto \mathsf{d}])} = \mathtt{T} \text{ for some d in } dom(\mathcal{I}) \\ \mathtt{F} & \text{otherwise} \end{cases}$

Note: The values of $(\forall x\ \alpha)^{(\mathcal{I},E)}$ and $(\exists x\ \alpha)^{(\mathcal{I},E)}$ do not depend on the value of $E(x)$.

The value $E(x)$ only matters for free occurrences of x *but nonetheless environments must be specified for all variables!*

Examples: Value of a Quantified Formula

Example. Let $dom(\mathcal{I}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{a} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{b}, \mathsf{b} \rangle\}$.

Let $E(x) = a$ and $E(y) = b$. We have

- $P(x,x)^{(\mathcal{I},E)} = \mathtt{T}$, since $\big\langle E(x), E(x) \big\rangle = \langle \mathsf{a}, \mathsf{a} \rangle \in P^{\mathcal{I}}$.
- $P(y,x)^{(\mathcal{I},E)} = \mathtt{F}$, since $\big\langle E(y), E(x) \big\rangle = \langle \mathsf{b}, \mathsf{a} \rangle \notin P^{\mathcal{I}}$.
- $\Big(\exists y \; P(y,x)\Big)^{(\mathcal{I},E)} = \mathtt{T}$, since $P(y,x)^{(\mathcal{I},E[y \mapsto \mathsf{a}])} = \mathtt{T}$.

  (That is, $\big\langle E[y \mapsto \mathsf{a}](y), E[y \mapsto \mathsf{a}](x) \big\rangle = \langle \mathsf{a}, \mathsf{a} \rangle \in P^{\mathcal{I}}$).
- What is $\Big(\forall x \; (\forall y \; P(x,y))\Big)^{(\mathcal{I},E)}$?

Examples: Continued

Example. Let $dom(\mathcal{I}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{a} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{b}, \mathsf{b} \rangle\}$.

Let $E(x) = \mathsf{a}$ and $E(y) = \mathsf{b}$.

- What is $\big(\forall x\ (\forall y\ P(x, y))\big)^{(\mathcal{I}, E)}$?

  Since $\langle \mathsf{b}, \mathsf{a} \rangle \notin P^{\mathcal{I}}$, we have

  $$P(x, y)^{(\mathcal{I}, E[x \mapsto \mathsf{b}][y \mapsto \mathsf{a}])} = \mathtt{F} \ ,$$

  and thus

  $$\big(\forall x\ (\forall y\ P(x, y))\big)^{(\mathcal{I}, E)} = \mathtt{F} \ .$$

Example. Let $dom(\mathcal{I}) = \{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{a} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{b}, \mathsf{b} \rangle\}$.

Let $E(x) = \mathsf{a}$ and $E(y) = \mathsf{b}$.

- What is $\big(\forall x\ (\forall y\ P(x, y))\big)^{(\mathcal{I}, E)}$?

  Since $\langle \mathsf{b}, \mathsf{a} \rangle \notin P^{\mathcal{I}}$, we have

  $$P(x, y)^{(\mathcal{I}, E[x \mapsto \mathsf{b}][y \mapsto \mathsf{a}])} = \mathtt{F} \ ,$$

  and thus

  $$\big(\forall x\ (\forall y\ P(x, y))\big)^{(\mathcal{I}, E)} = \mathtt{F} \ .$$

- What about $\big(\forall x\ (\exists y\ P(x, y))\big)^{(\mathcal{I}, E)}$?

A Question of Syntax

In the previous example, we wrote

$$P(x, y)^{(\mathcal{I}, E[x \mapsto \mathsf{b}][y \mapsto \mathsf{a}])} = \mathtt{F} \ .$$

Why did we not write simply

$$P(\mathsf{b}, \mathsf{a}) = \mathtt{F}$$

or perhaps

$$P(\mathsf{b}, \mathsf{a})^{(\mathcal{I}, E)} = \mathtt{F} \ ?$$

A Question of Syntax

In the previous example, we wrote

$$P(x, y)^{(\mathcal{I}, E[x \mapsto \mathsf{b}][y \mapsto \mathsf{a}])} = \mathtt{F} \ .$$

Why did we not write simply

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or perhaps

$$P(\mathsf{b}, \mathsf{a})^{(\mathcal{I}, E)} = \mathtt{F} \ ?$$

Because "$P(\mathsf{b}, \mathsf{a})$" is not a formula. The elements a and b of $dom(\mathcal{I})$ are not symbols in the language; they cannot appear in a formula.

Satisfaction of Formulas

An interpretation $\mathcal{I}$ and environment E *satisfy* a formula α, denoted $\mathcal{I} \vDash_E \alpha$, if $\alpha^{(\mathcal{I},E)} = \mathtt{T}$;
they do not satisfy α, denoted $\mathcal{I} \nvDash_E \alpha$, if $\alpha^{(\mathcal{I},E)} = \mathtt{F}$.

Form of α	*Condition for $\mathcal{I} \vDash_E \alpha$*
$P(t_1, \ldots, t_k)$	$\left\langle t_1^{(\mathcal{I},E)}, \ldots, t_k^{(\mathcal{I},E)} \right\rangle \in P^{\mathcal{I}}$
$(\neg\beta)$	$\mathcal{I} \nvDash_E \beta$
$(\beta \wedge \gamma)$	both $\mathcal{I} \vDash_E \beta$ and $\mathcal{I} \vDash_E \gamma$
$(\beta \vee \gamma)$	either $\mathcal{I} \vDash_E \beta$ or $\mathcal{I} \vDash_E \gamma$ (or both)
$(\beta \rightarrow \gamma)$	either $\mathcal{I} \nvDash_E \beta$ or $\mathcal{I} \vDash_E \gamma$ (or both)
$(\forall x\ \beta)$	for every a $\in dom(\mathcal{I})$, $\mathcal{I} \vDash_{E[x \mapsto \mathsf{a}]} \beta$
$(\exists x\ \beta)$	there is some a $\in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x \mapsto \mathsf{a}]} \beta$

If $\mathcal{I} \vDash_E \alpha$ for every E, then $\mathcal{I}$ *satisfies* α, denoted $\mathcal{I} \vDash \alpha$.

Example: Satisfaction

Example. Consider the formula $(\exists y\ P(x, y \oplus y))$.

(For P a binary predicate and $\oplus$ a binary function.)

Suppose $dom(\boldsymbol{\mathcal{I}}) = \{1, 2, 3, ...\}$,
$\oplus^{\boldsymbol{\mathcal{I}}}$ is the addition operation, and
$P^{\boldsymbol{\mathcal{I}}}$ is the equality predicate.

Give a simple condition that determines when
$\boldsymbol{\mathcal{I}} \vDash_{E} (\exists y\ P(x, (y \oplus y)))$ holds.

Example: Satisfaction

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Suppose $dom(\mathcal{I}) = \{1, 2, 3, ...\}$,
$\oplus^{\mathcal{I}}$ is the addition operation, and
$P^{\mathcal{I}}$ is the equality predicate.

Give a simple condition that determines when
$\mathcal{I} \models_E (\exists y\ P(x, (y \oplus y)))$ holds.

$\mathcal{I} \models_E (\exists y\ P(x, (y \oplus y)))$ iff $E(x)$ is an even number.

Validity and Satisfiability

Validity and satisfiability of formulas have definitions analogous to the ones for propositional logic.

Definition: A formula α is

- *valid* if every interpretation and environment satisfy α; that is, if $\mathcal{I} \vDash_E \alpha$ for every $\mathcal{I}$ and E,
- *satisfiable* if some interpretation and environment satisfy α; that is, if $\mathcal{I} \vDash_E \alpha$ for some $\mathcal{I}$ and E, and
- *unsatisfiable* if no interpretation and environment satisfy α; that is, if $\mathcal{I} \nvDash_E \alpha$ for every $\mathcal{I}$ and E.

(The term "tautology" is not used in predicate logic.)

Note

If there is no need to specify an environment, then simply defining an interpretation $\mathcal{I}$ and writing $\mathcal{I} \models \alpha$ will suffice.

Revisit Example

Let $f^{(1)}$ and $h^{(2)}$ be function symbols. Let $P^{(1)}$ and $Q^{(2)}$ be predicate symbols, let a, b, c be constant symbols and let x, y, z be variable symbols. Define an interpretation by:

- Domain: $\mathcal{D} = \{1, 2, 3\}$
- Constants: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}} : f^{\mathcal{I}}(1) = 2,\ f^{\mathcal{I}}(2) = 3,\ f^{\mathcal{I}}(3) = 1$
- $h^{\mathcal{I}} : (x, y) \mapsto \min\{x, y\}$ (min is the minimum function)
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$
- $Q^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$

and define an environment E by

$$E(x) = 3, E(y) = 3, E(z) = 1.$$

- Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.
- Give a new interpretation $\mathcal{J}_2$ and environment G_2 satisfying $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution: Set $G_1 = E$ and $\mathcal{J}_1$ to be $\mathcal{I}$ except, let $P^{\mathcal{J}_1} = \emptyset$. Then $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution: Set $G_1 = E$ and $\mathcal{J}_1$ to be $\mathcal{I}$ except, let $P^{\mathcal{J}_1} = \emptyset$. Then $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Give a new interpretation $\mathcal{J}_2$ and environment G_2 satisfying $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$.

Solution

Give a new interpretation $\mathcal{J}_1$ and environment G_1 satisfying $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Solution: Set $G_1 = E$ and $\mathcal{J}_1$ to be $\mathcal{I}$ except, let $P^{\mathcal{J}_1} = \emptyset$. Then $\mathcal{J}_1 \not\models_{G_1} P(h(f(a), z))$.

Give a new interpretation $\mathcal{J}_2$ and environment G_2 satisfying $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$.

Solution: Set $G_2 = E$ and $\mathcal{J}_2$ to be $\mathcal{I}$ except, let $Q^{\mathcal{J}_2} = \emptyset$. Then $\mathcal{J}_2 \not\models_{G_2} Q(y, h(a, b))$..

Lemma:

Let α be a first-order formula, $\mathcal{I}$ be an interpretation, and E_1 and E_2 be two environments such that

$$E_1(x) = E_2(x) \text{ for every } x \text{ that occurs free in } \alpha.$$

Then

$$\mathcal{I} \models_{E_1} \alpha \text{ if and only if } \mathcal{I} \models_{E_2} \alpha\ .$$

Proof by induction on the structure of α.

Example: Satisfiability and Validity

Let $\mathcal{L}$ be a language consisting of variables x, y, z, function symbols $f^{(2)}$, $g^{(1)}$ and predicate symbol $P^{(2)}$. Let α be the formula $P\big(f(g(x), g(y)), g(z)\big)$. The formula is satisfiable:

- $dom(\mathcal{I})$: $\mathbb{N}$
- $f^{\mathcal{I}}$: summation
- $g^{\mathcal{I}}$: squaring
- $P^{\mathcal{I}}$: equality
- $E(x) = 3$, $E(y) = 4$ and $E(z) = 5$.

α is not valid. (Why?)

Another Example

Let $\mathcal{L}$ be a language consisting of variables x, y, predicate symbol $Q^{(2)}$.
Let $\mathcal{I}$ be the interpretation defined by

- Domain: $\mathcal{D} = \{1, 2\}$ $\qquad Q^{\mathcal{I}} = \emptyset$

and environment E given by:

- $E(x) = 1 \qquad E(y) = 2$

1. Show that $\mathcal{I} \not\models_E \alpha$ where $\alpha \stackrel{\text{def}}{=} \big(\exists x\, (\forall y\; Q(x, y))\big)$.
2. Give an interpretation $\mathcal{J}$ and an environment G such that $\mathcal{J} \models_G (\exists x\; Q(x, y))$
3. Give an interpretation $\mathcal{J}$ and an environment G such that $\mathcal{J} \models_G (\forall x\; Q(x, y))$
4. Give an interpretation $\mathcal{J}$ and an environment G such that $\mathcal{J} \models_G \big(\exists x\, (\forall y\; Q(x, y))\big)$
5. Is $\big(\forall x\, (\exists y\; Q(x, y))\big)$ valid? Why or why not?

Semantic Entailment

Let Σ be a set of well-formed Predicate logic formulas and α is a well-formed predicate logic formula.

For interpretation $\mathcal{I}$ and environment E, we write $\mathcal{I} \vDash_E \Sigma$ if and only if for every $\varphi \in \Sigma$, we have that $\mathcal{I} \vDash_E \varphi$.

We say that Σ is a *semantically entails* α, written as $\Sigma \vDash \alpha$, if and only if for any interpretation $\mathcal{I}$ and environment E, we have $\mathcal{I} \vDash_E \Sigma$ implies $\mathcal{I} \vDash_E \alpha$. This can also be written as $\alpha^{(\mathcal{I}, E)} = \mathtt{T}$.

$\emptyset \vDash \alpha$ means that α is valid.

Notes

Suppose $\Sigma = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ Then $\Sigma \models \beta$ means...

- ...that every pair of interpretation and environment that makes Σ true must also make β true.
- ...that $\emptyset \models ((\alpha_1 \wedge (\alpha_2 \wedge (... \wedge \alpha_n))) \rightarrow \beta)$
- ...that $((\alpha_1 \wedge (\alpha_2 \wedge (... \wedge \alpha_n))) \rightarrow \beta)$ is valid

To prove these, take an arbitrary interpretation $\mathcal{I}$ and environment E and show that if this satisfied Σ then it must also satisfy β. You may also assume towards a contradiction that $\mathcal{I} \not\models_E \beta$ and proceed from there if this helps.

To prove that $\Sigma \not\models \beta$ find an interpretation $\mathcal{I}$ and environment E that satisfies Σ but that doesn't satisfy β, that is, show that $\mathcal{I} \not\models_E \beta$.

Example: Semantic Entailment

Example: Show that for any well-formed Predicate formulas α and β:

$$\emptyset \vDash \big((\forall x\,(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))\big)\ .$$

Example: Semantic Entailment

Example: Show that for any well-formed Predicate formulas α and β:

$$\emptyset \models \big((\forall x\,(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))\big)\ .$$

Proof by contradiction. Suppose there are $\mathcal{I}$ and E such that

$$\mathcal{I} \not\models_E \big((\forall x\,(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))\big)\ .$$

Then we must have $\mathcal{I} \models_E (\forall x\,(\alpha \to \beta))$ and $\mathcal{I} \not\models_E ((\forall x\ \alpha) \to (\forall x\ \beta))$;

the second gives $\mathcal{I} \models_E (\forall x\ \alpha)$ and $\mathcal{I} \not\models_E (\forall x\ \beta)$.

Using the definition of $\models$ for formulas with $\forall$, we have
for every a $\in dom(\mathcal{I})$, $\mathcal{I} \models_{E[x\mapsto \mathsf{a}]} (\alpha \to \beta)$ and $\mathcal{I} \models_{E[x\mapsto \mathsf{a}]} \alpha$.
Thus also $\mathcal{I} \models_{E[x\mapsto \mathsf{a}]} \beta$ for every a $\in dom(\mathcal{I})$.

Thus $\mathcal{I} \models_E (\forall x\ \beta)$, a contradiction.

Example. Show that $\{(\forall x\,(\neg\gamma))\} \vDash (\neg(\exists x\ \gamma))$.

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Proof: Suppose that $\mathcal{I} \vDash_E (\forall x\,(\neg\gamma))$. By definition, this means

for every $\mathsf{a} \in dom(\mathcal{I})$, $\mathcal{I} \vDash_{E[x\mapsto \mathsf{a}]} (\neg\gamma)$. that is, $(\neg\gamma)^{(\mathcal{I},E[x\mapsto \mathsf{a}])} = \mathtt{T}$.

Again by definition (for a formula with $\neg$), this is equivalent to

for every $\mathsf{a} \in dom(\mathcal{I})$, $\mathcal{I} \nvDash_{E[x\mapsto \mathsf{a}]} \gamma$ that is, $\gamma^{(\mathcal{I},E[x\mapsto \mathsf{a}])} = \mathtt{F}$.

and also

there is no $\mathsf{a} \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x\mapsto \mathsf{a}]} \gamma$.

Assuming towards a contradiction that $(\exists x\;\gamma)^{(\mathcal{I},E)} = \mathtt{T}$, this would mean that there is an $\mathsf{b} \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x\mapsto \mathsf{b}]} \gamma$ which contradicts the previous line. Hence $\mathcal{I} \vDash_E (\neg(\exists x\;\gamma))$ holds as required.

Example

Example: Find well-formed Predicate formulas α and β such that

$$\{((\forall x\ \alpha) \to (\forall x\ \beta))\} \not\models (\forall x\,(\alpha \to \beta))\ .$$

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Key idea: $\varphi_1 \to \varphi_2$ yields true whenever φ_1 is false.

Let α be $P(x)$. Let $\mathcal{I}$ have domain $\{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\mathsf{a}\}$. Then $\mathcal{I} \models (\forall x\ \alpha) \to (\forall x\ \beta)$ for any β. (Why?)

Example: Find well-formed Predicate formulas α and β such that

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Key idea: $\varphi_1 \to \varphi_2$ yields true whenever φ_1 is false.

Let α be $P(x)$. Let $\mathcal{I}$ have domain $\{\mathsf{a}, \mathsf{b}\}$ and $P^{\mathcal{I}} = \{\mathsf{a}\}$. Then $\mathcal{I} \models (\forall x\ \alpha) \to (\forall x\ \beta)$ for any β. (Why?)

To obtain $\mathcal{I} \not\models \forall x\,(\alpha \to \beta)$, we can use $\neg P(x)$ for β. (Why?)

Thus $((\forall x\ \alpha) \to (\forall x\ \beta)) \not\models (\forall x\,(\alpha \to \beta))$, as required. (Why?)

Example

Example: For any formula α and term t, show that

$$\emptyset \models ((\forall x\ \alpha) \rightarrow \alpha[t/x])\ .$$

Recall that functions must be total!

Another Example

Let α be any well-formed Predicate formula **without** a free variable x. Let $\mathcal{I}$ be any interpretation and let E be any environment. Then

$$\alpha^{(\mathcal{I},E)} = (\forall x\ \alpha)^{(\mathcal{I},E)}.$$

Let α be any well-formed Predicate formula **without** a free variable x. Let $\mathcal{I}$ be any interpretation and let E be any environment. Then

$$\alpha^{(\mathcal{I},E)} = (\forall x\ \alpha)^{(\mathcal{I},E)}.$$

Proof. Let $\mathcal{D}$ be the domain of $\mathcal{I}$. Since x is not free in α, therefore $E(y) = E[x \mapsto a](y)$, for every $a \in \mathcal{D}$ and for every y that occurs free in α.

Then by the Relevance Lemma, we have that

- $\mathcal{I} \vDash_E \alpha$
- if and only if $\mathcal{I} \vDash_{E[x \mapsto a]} \alpha$, for any $a \in \mathcal{D}$.
- if and only if $\mathcal{I} \vDash_E (\forall x\ \alpha)$,

which establishes the desired result.