Warm-Up Problem

Write a Resolution Proof for

$$\{((\neg p) \vee q), (p \vee r), ((\neg q) \vee r), ((\neg q) \vee (\neg r)), (q \vee (\neg r))\} \vdash_{\mathsf{Res}} \perp$$

A second 'Rule'

Sometimes throughout we need to also make simplifications:

$$\frac{(p \vee p)}{p}$$

You can do this in line without explicitly mentioning it (just pretend you simplified after applying the resolution rule - this actually makes more sense using the set notation.

Propositional Logic: Natural Deduction Derived Rules and Examples

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Lecture 8

Based on work by J. Buss, A. Gao, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Learning goals

Natural Deduction in Propositional logic

- Describe rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

Derived Rules in Natural Deduction:

- Describe the derived rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

The Natural Deduction Proof System

We will now consider a proof system called Natural Deduction.

- It closely follows how people (mathematicians, at least) normally make formal arguments.
- It extends easily to more-powerful forms of logic.

A proof is syntactic

First, we think about proofs in a purely syntactic way.

A proof

- starts with a set of premises,
- transforms the premises based on a set of inference rules (which form a proof system),
- and reaches a conclusion.

We write

$$\Sigma \vdash_{ND} \varphi \quad \text{or simply if context is clear} \quad \Sigma \vdash \varphi$$

if we can find such a proof in our proof system of Natural Deduction that starts with a set of premises Σ and ends with the conclusion φ.

Revisit Soundness and Completeness

We revisit Soundness and Completeness just like we did for Resolution:

- (Soundness)
  1. Does $\Sigma \vdash_{ND} \varphi$ imply $\Sigma \vDash \varphi$?
  2. Does every proof establish a semantic entailment?
  3. If a proof of Σ to φ exists, is it true that Σ semantically entails φ?
- (Completeness)
  1. Does $\Sigma \vDash \varphi$ imply $\Sigma \vdash_{ND} \varphi$?
  2. For every semantic entailment, can I find a proof for it?
  3. If Σ semantically entails φ, does a proof of Σ to φ exist?

Reflexivity / Premise

If you want to write down a previous formula in the proof again, you can do it by *reflexivity*.

Name	**$\vdash$-notation**	**inference notation**
Reflexivity, or Premise	$\Sigma, \alpha \vdash_{ND} \alpha$	$\dfrac{\alpha}{\alpha}$

The notation on the right: Given the formulas above the line, we can infer the formula below the line.

The version in the centre reminds us of the role of assumptions in Natural Deduction. Other rules will make more use of it.

An example using reflexivity

Here is a proof of $\{p, q\} \vdash_{ND} p$.

1.	p	Premise
2.	q	Premise
3.	p	Reflexivity: 1

Alternatively, we could simply write

1.	p	Premise

and be done.

For each symbol, the rules come in pairs.

- An “introduction rule” adds the symbol to the formula.
- An “elimination rule” removes the symbol from the formula.

Rules for Conjunction

Name	$\vdash$-notation	inference notation
$\wedge$-introduction ($\wedge$i)	If $\Sigma \vdash_{ND} \alpha$ and $\Sigma \vdash_{ND} \beta$, then $\Sigma \vdash_{ND} (\alpha \wedge \beta)$	$\dfrac{\alpha \quad \beta}{(\alpha \wedge \beta)}$

Name	$\vdash$-notation	inference notation
$\wedge$-elimination ($\wedge$e)	If $\Sigma \vdash_{ND} (\alpha \wedge \beta)$, then $\Sigma \vdash_{ND} \alpha$ and $\Sigma \vdash_{ND} \beta$	$\dfrac{(\alpha \wedge \beta)}{\alpha} \quad \dfrac{(\alpha \wedge \beta)}{\beta}$

Example. Show that $\{(p \wedge q)\} \vdash_{ND} (q \wedge p)$.

Example: Conjunction Rules

Example. Show that $\{(p \wedge q)\} \vdash_{ND} (q \wedge p)$.

1.	$(p \wedge q)$	Premise
2.	q	$\wedge$e: 1
3.	p	$\wedge$e: 1
4.	$(q \wedge p)$	$\wedge$i: 2, 3

Example: Conjunction Rules (2)

Example. Show that $\{(p \land q), r\} \vdash_{ND} (q \land r)$.

Example. Show that $\{(p \wedge q), r\} \vdash_{ND} (q \wedge r)$.

1.	$(p \wedge q)$	Premise
2.	r	Premise
3.	q	$\wedge$e: 1
4.	$(q \wedge r)$	$\wedge$i: 3, 2

Rules for Implication: $\rightarrow$e

Name	**$\vdash$-notation**	**inference notation**
$\rightarrow$-elimination ($\rightarrow$e) (*modus ponens*)	If $\Sigma \vdash_{ND} (\alpha \rightarrow \beta)$ and $\Sigma \vdash_{ND} \alpha$, then $\Sigma \vdash_{ND} \beta$	$\dfrac{(\alpha \rightarrow \beta) \quad \alpha}{\beta}$

In words:

> If you assume α is true and α implies β, then you may conclude β.

Rules for Implication: $\rightarrow$i

Name	**$\vdash$-notation**	**inference notation**
$\rightarrow$-introduction ($\rightarrow$i)	If $\Sigma, \alpha \vdash_{ND} \beta$, then $\Sigma \vdash_{ND} (\alpha \rightarrow \beta)$	$$\frac{\boxed{\begin{array}{c}\alpha\\ \vdots\\ \beta\end{array}}}{(\alpha \rightarrow \beta)}$$

The "box" denotes a sub-proof. In the sub-proof, we starts by assuming that α is true (a premise of the sub-proof), and we conclude that β is true.

Nothing inside the sub-proof may come out.

Outside of the sub-proof, we could only use the sub-proof as a whole.

Example: Rule $\rightarrow$i and sub-proofs

Example. Give a proof of $\{(p \rightarrow q), (q \rightarrow r)\} \vdash_{ND} (p \rightarrow r)$.

To start, we write down the premises at the beginning, and the conclusion at the end.

1.	$(p \rightarrow q)$	Premise
2.	$(q \rightarrow r)$	Premise
	$(p \rightarrow r)$	???

What next?

Example. Give a proof of $\{(p \rightarrow q), (q \rightarrow r)\} \vdash_{ND} (p \rightarrow r)$.

To start, we write down the premises at the beginning, and the conclusion at the end.

1.	$(p \rightarrow q)$	Premise
2.	$(q \rightarrow r)$	Premise
3.	p	Assumption
4.		
5.		
6.	$(p \rightarrow r)$	$\rightarrow$i: ??

What next?

The goal "$(p \rightarrow r)$" contains $\rightarrow$.
Let's try rule $\rightarrow$i....

Example: Rule $\rightarrow$i and sub-proofs

Example. Give a proof of $\{(p \rightarrow q), (q \rightarrow r)\} \vdash_{ND} (p \rightarrow r)$.

To start, we write down the premises at the beginning, and the conclusion at the end.

1.	$(p \rightarrow q)$	Premise
2.	$(q \rightarrow r)$	Premise
3.	p	Assumption
4.	q	$\rightarrow$e: 1, 3
5.	r	$\rightarrow$e: 2, 4
6.	$(p \rightarrow r)$	$\rightarrow$i: ??

What next?

The goal "$(p \rightarrow r)$" contains $\rightarrow$. Let's try rule $\rightarrow$i....

Inside the sub-proof, we can use rule $\rightarrow$e.

Example: Rule $\to$i and sub-proofs

Example. Give a proof of $\{(p \to q), (q \to r)\} \vdash_{ND} (p \to r)$.

To start, we write down the premises at the beginning, and the conclusion at the end.

1.	$(p \to q)$	Premise
2.	$(q \to r)$	Premise
3.	p	Assumption
4.	q	$\to$e: 1, 3
5.	r	$\to$e: 2, 4
6.	$(p \to r)$	$\to$i: 3–5

What next?

The goal "$(p \to r)$" contains $\to$. Let's try rule $\to$i....

Inside the sub-proof, we can use rule $\to$e.

Done!

Name	**$\vdash$-notation**	**inference notation**
$\vee$-introduction ($\vee$i)	If $\Sigma \vdash_{ND} \alpha$, then $\Sigma \vdash_{ND} (\alpha \vee \beta)$ and $\Sigma \vdash_{ND} (\beta \vee \alpha)$	$\dfrac{\alpha}{(\alpha \vee \beta)}$ $\quad$ $\dfrac{\alpha}{(\beta \vee \alpha)}$
$\vee$-elimination ($\vee$e)	If Σ, $\alpha_1 \vdash_{ND} \beta$ and Σ, $\alpha_2 \vdash_{ND} \beta$, then Σ, $(\alpha_1 \vee \alpha_2) \vdash_{ND} \beta$	$\dfrac{(\alpha_1 \vee \alpha_2) \quad \boxed{\begin{array}{c}\alpha_1 \\ \vdots \\ \beta\end{array}} \quad \boxed{\begin{array}{c}\alpha_2 \\ \vdots \\ \beta\end{array}}}{\beta}$

$\vee$e is also known as "proof by cases".

Example: Or-Introduction and -Elimination

Example: Show that $\{(p \vee q)\} \vdash_{ND} ((p \to q) \vee (q \to p))$.

Example: Or-Introduction and -Elimination

Example: Show that $\{(p \lor q)\} \vdash_{ND} ((p \to q) \lor (q \to p))$.

1.	$(p \lor q)$	Premise
2.	p	Assumption
3.	q	Assumption
4.	p	Reflexivity: 2
5.	$(q \to p)$	$\to$i: 3–4
6.	$((p \to q) \lor (q \to p))$	$\lor$i: 5
7.	q	Assumption
8.	p	Assumption
9.	q	Reflexivity: 7
10.	$(p \to q)$	$\to$i: 8–9
11.	$((p \to q) \lor (q \to p))$	$\lor$i: 10
12.	$((p \to q) \lor (q \to p))$	$\lor$e: 1, 2–6, 7–11

Negation

We shall treat negation by considering contradictions.

We shall use the notation $\perp$ to represent any contradiction.
It may appear in proofs as if it were a formula.

Do not confuse this with the $\perp$ from resolution!

Negation Rules

The elimination rule for double negations:

Name	$\vdash$-notation	inference notation
$\neg\neg$-elimination ($\neg\neg$e)	If $\Sigma \vdash_{ND} (\neg(\neg\alpha))$, then $\Sigma \vdash_{ND} \alpha$	$\dfrac{(\neg(\neg\alpha))}{\alpha}$

If an assumption α leads to a contradiction, then derive $(\neg\alpha)$.

Name	$\vdash$-notation	inference notation
$\neg$-introduction ($\neg$i)	If $\Sigma, \alpha \vdash_{ND} \bot$, then $\Sigma \vdash_{ND} (\neg\alpha)$	$\dfrac{\boxed{\alpha \; \vdots \; \bot}}{(\neg\alpha)}$

The Last Two Basic Rules for $\bot$

Contradiction Introduction (also known as $\neg$-elimination ($\neg$e)):

Name	**$\vdash$-notation**	**inference notation**
$\bot$-introduction	$\Sigma, \alpha, (\neg\alpha) \vdash_{ND} \bot$	$\dfrac{\alpha \quad (\neg\alpha)}{\bot}$

If we have both α and $(\neg\alpha)$, then we have a contradiction.

Contradiction Elimination:

Name	**$\vdash$-notation**	**inference notation**
$\bot$-elimination ($\bot$e)	If $\Sigma \vdash_{ND} \bot$, then $\Sigma \vdash_{ND} \alpha$	$\dfrac{\bot}{\alpha}$

Example: Negation

Example. Show that $\{(\alpha \to (\neg\alpha))\} \vdash_{ND} (\neg\alpha)$.

Example. Show that $\{(\alpha \to (\neg\alpha))\} \vdash_{ND} (\neg\alpha)$.

1.	$(\alpha \to (\neg\alpha))$	Premise
	$(\neg\alpha)$	??

Example: Negation

Example. Show that $\{(\alpha \to (\neg\alpha))\} \vdash_{ND} (\neg\alpha)$.

1.	$(\alpha \to (\neg\alpha))$	Premise
2.	α	Assumption
3.		
4.	$\bot$	??
5.	$(\neg\alpha)$	$\neg$i: 2–?

Example: Negation

Example. Show that $\{(\alpha \to (\neg\alpha))\} \vdash_{ND} (\neg\alpha)$.

1.	$(\alpha \to (\neg\alpha))$	Premise
2.	α	Assumption
3.	$(\neg\alpha)$	$\to$e: 1, 2
4.	$\perp$	??
5.	$(\neg\alpha)$	$\neg$i: 2–?

Example: Negation

Example. Show that $\{(\alpha \to (\neg\alpha))\} \vdash_{ND} (\neg\alpha)$.

1.	$(\alpha \to (\neg\alpha))$	Premise
2.	α	Assumption
3.	$(\neg\alpha)$	$\to$e: 1, 2
4.	$\bot$	$\bot$i: 2, 3
5.	$(\neg\alpha)$	$\neg$i: 2–4

A Redundant Rule

The rule of $\perp$-elimination is not actually needed.

Suppose a proof has

27.	$\perp$	$\langle$*some rule*$\rangle$
28.	α	$\perp$e: 27.

We can replace these by

27.	$\perp$	$\langle$*some rule*$\rangle$
28.	$(\neg\alpha)$	Assumption
29.	$\perp$	Reflexivity: 27
30.	$(\neg(\neg\alpha))$	$\neg$i: 28–29
31.	α	$\neg\neg$e: 30.

Thus any proof that uses $\perp$e can be modified into a proof that does not.

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{(p \to q), (\neg q)\} \vdash_{ND} (\neg p)$.

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{(p \to q), (\neg q)\} \vdash_{ND} (\neg p)$.

1.	$(p \to q)$	Premise
2.	$(\neg q)$	Premise
	$(\neg p)$	??

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{(p \rightarrow q), (\neg q)\} \vdash_{ND} (\neg p)$.

1.	$(p \rightarrow q)$	Premise
2.	$(\neg q)$	Premise
3.	p	Assumption
4.		
5.	$\perp$	
6.	$(\neg p)$	$\neg$i: ??

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{(p \to q), (\neg q)\} \vdash_{ND} (\neg p)$.

1.	$(p \to q)$	Premise
2.	$(\neg q)$	Premise
3.	p	Assumption
4.	q	$\to$e: 3, 1
5.	$\perp$	
6.	$(\neg p)$	$\neg$i: ??

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{(p \to q), (\neg q)\} \vdash_{ND} (\neg p)$.

1.	$(p \to q)$	Premise
2.	$(\neg q)$	Premise
3.	p	Assumption
4.	q	$\to$e: 3, 1
5.	$\bot$	
6.	$(\neg p)$	$\neg$i: 3–5

What should go on line 5?

Example: "*Modus tollens*"

Modus tollens is sometimes taken as a "derived rule":

$$\frac{(\alpha \to \beta) \quad (\neg\beta)}{(\neg\alpha)} \text{ MT}$$

Derived Rules

Whenever we have a proof of the form $\Gamma \vdash_{ND} \alpha$, we can consider it as a derived rule:

$$\frac{\Gamma}{\alpha}$$

If we use this in a proof, it can be replaced by the original proof of $\Gamma \vdash_{ND} \alpha$. The result is a proof using only the basic rules.

Using derived rules does not expand the things that can be proved. But they can make it easier to find a proof.

More Derived Rules

Double-Negation Introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))}\ \neg\neg\mathsf{i}$$

More Derived Rules

Double-Negation Introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))} \ \neg\neg\text{i}$$

1.	α	Premise
	$(\neg(\neg\alpha))$	??

More Derived Rules

Double-Negation Introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))}\ \neg\neg\text{i}$$

1.	α	Premise
2.	$(\neg\alpha)$	Assumption
3.		
4.	$(\neg(\neg\alpha))$	$\neg$i: ??

More Derived Rules

Double-Negation Introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))}\ \neg\neg\text{i}$$

1.	α	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\bot$	$\bot$i: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: ??

More Derived Rules

Double-Negation Introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))}\ \neg\neg\text{i}$$

1.	α	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\bot$	$\bot$i: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: 2–3

More Derived Rules

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots\\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

More Derived Rules

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots \\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \rightarrow \bot)$	Premise
5.	α	??

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots\\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \rightarrow \bot)$	Premise
	$(\neg(\neg\alpha))$	??
5.	α	$\neg\neg$e: ??

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots \\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \rightarrow \bot)$	Premise
2.	$(\neg\alpha)$	Assumption
3.		
4.	$(\neg(\neg\alpha))$	¬i: ??
5.	α	¬¬e: 4

More Derived Rules

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots\\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \rightarrow \bot)$	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\bot$	$\rightarrow$e: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: ??
5.	α	$\neg\neg$e: 4

*Proof by contradiction (**reductio ad absurdum**):*

$$\frac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots \\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \to \bot)$	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\bot$	$\to$e: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: 2–3
5.	α	$\neg\neg$e: 4

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg \alpha))}\ \mathsf{LEM}$$

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.

9. $(\alpha \vee (\neg\alpha))$??

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.

$(\neg(\neg(\alpha \vee (\neg\alpha)))$??

9. $(\alpha \vee (\neg\alpha))$ $\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
		??
6.	??	??
7.	$\bot$	$\bot$i: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.		??
4.		??
5.	$(\neg\alpha)$	$\neg$i: ??
6.	??	??
7.	$\perp$	$\perp$i: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	??
4.		??
5.	$(\neg\alpha)$	$\neg$i: ??
6.	??	??
7.	$\bot$	$\bot$i: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.		??
5.	$(\neg\alpha)$	$\neg$i: ??
6.	??	??
7.	$\bot$	$\bot$i: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha))))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	??
5.	$(\neg\alpha)$	$\neg$i: ??
6.	??	??
7.	$\bot$	$\bot$i: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha))))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\bot$i: 1,3
5.	$(\neg\alpha)$	$\neg$i: ??
6.	??	??
7.	$\bot$	$\bot$i: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\bot$i: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	??	??
7.	$\bot$	$\bot$i: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \lor (\neg\alpha))} \text{ LEM}$$

1.	$(\neg(\alpha \lor (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \lor (\neg\alpha))$	$\lor$i: 2
4.	$\bot$	$\bot$i: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \lor (\neg\alpha))$	??
7.	$\bot$	$\bot$i: ??
8.	$(\neg(\neg(\alpha \lor (\neg\alpha))))$	$\neg$i: ??
9.	$(\alpha \lor (\neg\alpha))$	$\neg\neg$e: ??

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\bot$i: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\bot$	$\bot$i: ??
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\bot$i: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\bot$	$\bot$i: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: ??
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\bot$i: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\bot$	$\bot$i: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: 1–7
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: ??

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\bot$i: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\bot$	$\bot$i: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: 1–7
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: 8

More Derived Rules

*The "Law of Excluded Middle" (**tertiam non datur**):*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\bot$i: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\bot$	$\bot$i: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha)))$	$\neg$i: 1–7
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: 8

More Derived Rules

The "Law of Excluded Middle" ***(tertiam non datur)****:*

$$\frac{}{(\alpha \vee (\neg\alpha))}\ \text{LEM}$$

1.	$(\neg(\alpha \vee (\neg\alpha)))$	Assumption
2.	α	Assumption
3.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 2
4.	$\bot$	$\bot$i: 1,3
5.	$(\neg\alpha)$	$\neg$i: 2-4
6.	$(\alpha \vee (\neg\alpha))$	$\vee$i: 5
7.	$\bot$	$\bot$i: 1,6
8.	$(\neg(\neg(\alpha \vee (\neg\alpha))))$	$\neg$i: 1–7
9.	$(\alpha \vee (\neg\alpha))$	$\neg\neg$e: 8

Strategies for natural deduction proofs

1. Work forward from the premises. Can you apply an elimination rule?
2. Work backwards from the conclusion. What introduction rule do you need to use at the end?
3. Stare at the formula. Notice its structure. Use it to guide your proof.
4. If a direct proof doesn't work, try a proof by contradiction.

Further Examples of Natural Deduction

Example. Show that $\{(p \to q)\} \vdash_{ND} ((r \lor p) \to (r \lor q))$.

Write down premises and conclusion (step 1).
No elimination applies (step 2). Thus try $\to$i (step 3).

1.	$(p \to q)$	Premise
	$((r \lor p)) \to ((r \lor q))$	??

Further Examples of Natural Deduction

Example. Show that $\{(p \to q)\} \vdash_{ND} ((r \lor p) \to (r \lor q))$.

In the sub-proof, try $\lor$-elimination on the assumption (step 2).

1.	$(p \to q)$	Premise
2.	$(r \lor p)$	Assumption
	$(r \lor q)$	??
9.	$((r \lor p)) \to ((r \lor q))$	??

Further Examples of Natural Deduction

Example. Show that $\{(p \to q)\} \vdash_{ND} ((r \lor p) \to (r \lor q))$.

To justify lines 4 and 7:
No elimination applies from the assumptions (step 2).
What about $\lor$-introduction for the conclusion (step 3)?

1.	$(p \to q)$	Premise
2.	$(r \lor p)$	Assumption
3.	r	Assumption
4.	$(r \lor q)$	??
5.	p	Assumption
6.		
7.	$(r \lor q)$	??
8.	$(r \lor q)$	$\lor$e: ??
9.	$((r \lor p)) \to ((r \lor q))$	$\to$i: 2–8

Example. Show that $\{(p \to q)\} \vdash_{ND} ((r \lor p) \to (r \lor q))$.

It works!

1.	$(p \to q)$	Premise
2.	$(r \lor p)$	Assumption
3.	r	Assumption
4.	$(r \lor q)$	$\lor$i: 3
5.	p	Assumption
6.	q	$\to$e: 5, 1
7.	$(r \lor q)$	$\lor$i: 6
8.	$(r \lor q)$	$\lor$e: 2, 3–4, 5–7
9.	$((r \lor p)) \to ((r \lor q))$	$\to$i: 2–8

Handout

Try some of the problems on the handout! You may work in small groups if you wish or on your own.