Warm Up Problem

What is the definition of a truth valuation? How did we extend this to well-formed formulas?

Propositional Logic: Semantics

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Lecture 4

Learning goals

By the end of this lecture, you should be able to

- Evaluate the truth value of a formula
  - Define a (truth) valuation.
  - Determine the truth value of a formula by using truth tables.
  - Determine the truth value of a formula by using valuation trees.
- Determine and prove whether a formula has a particular property
  - Define tautology, contradiction, and satisfiable formula.
  - Compare and contrast the three properties (tautology, contradiction, and satisfiable formula).
  - Prove whether a formula is a tautology, a contradiction, or satisfiable, using a truth table and/or a valuation tree.
  - Describe strategies to prove whether a formula is a tautology, a contradiction or a satisfiable formula.

The meaning of well-formed formulas

To interpret a formula, we have to give meanings to the propositional variables and the connectives.

A propositional variable has no intrinsic meaning; it gets a meaning via a valuation.

A (truth) valuation is a function $t : \mathcal{P} \mapsto \{\mathtt{F}, \mathtt{T}\}$ from the set of all proposition variables $\mathcal{P}$ to $\{\mathtt{F}, \mathtt{T}\}$. It assigns true/false to every propositional variable.

Two notations: $t(p)$ and p^t both denote the truth value of p under the truth valuation t.

Truth value of a formula

Fix a truth valuation t. Every formula α has a value under t, denoted α^t, determined as follows.

1. $p^t = t(p)$.
2. $(\neg\alpha)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \mathtt{F} \\ \mathtt{F} & \text{if } \alpha^t = \mathtt{T} \end{cases}$
3. $(\alpha \wedge \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \beta^t = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$
4. $(\alpha \vee \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \mathtt{T} \text{ or } \beta^t = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$
5. $(\alpha \to \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \mathtt{F} \text{ or } \beta^t = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$
6. $(\alpha \leftrightarrow \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \beta^t \\ \mathtt{F} & \text{otherwise} \end{cases}$

Truth tables for connectives

The unary connective $\neg$:

α	$(\neg\alpha)$
T	F
F	T

The binary connectives $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$:

α	β	$(\alpha \wedge \beta)$	$(\alpha \vee \beta)$	$(\alpha \rightarrow \beta)$	$(\alpha \leftrightarrow \beta)$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

Evaluating a formula using a truth table

Example. The truth table of $((p \vee q) \rightarrow (q \wedge r))$.

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p	q	r	$(p \vee q)$	$(q \wedge r)$	$((p \vee q) \rightarrow (q \wedge r))$
F	F	F	F	F	T
F	F	T	F	F	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	F	F
T	T	F	T	F	F
T	T	T	T	T	T

Evaluating a formula using a truth table

Build the truth table of $\big((p \to (\neg q)) \to (q \vee (\neg p))\big)$.

Understanding the disjunction and the biconditional

α	β	$(\alpha \vee \beta)$	Exclusive OR	Biconditional
F	F	F	F	T
F	T	T	T	F
T	F	T	T	F
T	T	T	F	T

- What is the difference between an inclusive OR (the disjunction) and an exclusive OR?
- What is the relationship between the exclusive OR and the biconditional?

A Small Theorem

Theorem: Fix a truth valuation t. Every formula α has a value α^t in $\{\mathtt{F}, \mathtt{T}\}$.

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1. If α is a propositional variable, then t assigns it a value of T or F (by the definition of a truth valuation).
2. If α has a value in $\{\mathtt{F}, \mathtt{T}\}$, then $(\neg\alpha)$ also does, as shown by the truth table of $(\neg\alpha)$.
3. If α and β each has a value in $\{\mathtt{F}, \mathtt{T}\}$, then $(\alpha \star \beta)$ also does for every binary connective $\star$, as shown by the corresponding truth tables.

By the principle of structural induction, every formula has a value.

By the unique readability of formulas, we have proved that a formula has **only one** truth value under any truth valuation t. QED

Tautology, Contradiction, Satisfiable

A formula α is a *tautology* if and only if
for every truth valuation t, $\alpha^t = \mathtt{T}$.

A formula α is a *contradiction* if and only if
for every truth valuation t, $\alpha^t = \mathtt{F}$.

A formula α is *satisfiable* if and only if
there exists a truth valuation t such that $\alpha^t = \mathtt{T}$.

Examples

Let p be a Propositional variable.

- The formula $(p \vee (\neg p))$ is a tautology (and satisfiable)
- The formula $(p \wedge (\neg p))$ is a contradiction (and not satisfiable)
- The formula $(p \to (\neg p))$ is satisfiable (but not a tautology)

How to determine the properties of a formula

- Truth table
- Valuation tree
- Reasoning

Valuation Tree

Rather than fill out an entire truth table, we can analyze what happens if we plug in a truth value for one variable.

$\neg T$	F
$\neg F$	T

$(p \wedge T)$	p
$(p \wedge F)$	F
$(p \wedge p)$	p

$(p \vee T)$	T
$(p \vee F)$	p
$(p \vee p)$	p

$(p \rightarrow T)$	T
$(p \rightarrow F)$	$(\neg p)$
$(T \rightarrow p)$	p
$(F \rightarrow p)$	T
$(p \rightarrow p)$	T

We can evaluate a formula by using these rules to construct a *valuation tree*.

Example of a valuation tree

Example. Show that $\Big(\big(((p \wedge q) \to (\neg r)) \wedge (p \to q)\big) \to (p \to (\neg r))\Big)$ is a tautology by using a valuation tree.

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Suppose $t(p) = \mathtt{T}$. We put $\mathtt{T}$ in for p:

$$\big(((\mathtt{T} \wedge q) \to (\neg r)) \wedge (\mathtt{T} \to q)\big) \to (\mathtt{T} \to (\neg r)) \ .$$

Based on the truth tables for the connectives, the formula becomes $\Big(((q \to (\neg r)) \wedge q) \to (\neg r)\Big)$.

If $t(q) = \mathtt{T}$, this yields $((\neg r) \to (\neg r))$ and then $\mathtt{T}$. (Check!).
If $t(q) = \mathtt{F}$, it yields $(\mathtt{F} \to (\neg r))$ and then $\mathtt{T}$. (Check!).

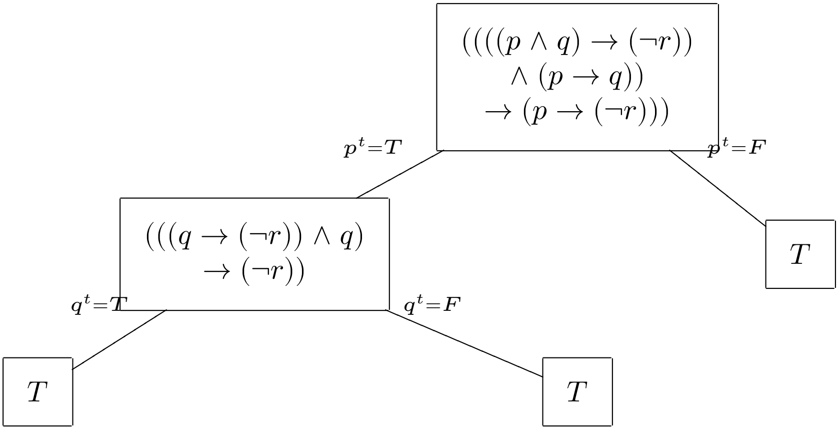
Suppose $t(p) = \mathtt{F}$. We get

$$\big(((\mathtt{F} \wedge q) \to (\neg r)) \wedge (\mathtt{F} \to q)\big) \to (\mathtt{F} \to (\neg r)) \ ,$$

Simplification yields $\big((\mathtt{F} \to (\neg r)) \wedge \mathtt{T}\big) \to \mathtt{T}$ and eventually $\mathtt{T}$.

Thus every valuation makes the formula true, as required.

Valuation Tree (Do not break on multiple lines!)



Determine which of the following are satisfiable, form a tautology or form a contradiction. Use a truth table. Repeat using a valuation tree.

1. $\Big((((p \vee q) \leftrightarrow (\neg r)) \wedge (p \to q)\big) \wedge (p \to (\neg r))\Big)$
2. $\Big((((p \wedge q) \to r) \wedge (p \to q)\big) \to (p \to r)\Big)$
3. $((p \vee q) \leftrightarrow ((p \wedge (\neg q)) \vee ((\neg p) \to q)))$
4. $(p \wedge (\neg p))$

Definition of logical equivalence

Two formulas α and β are logically equivalent if and only if they have the same value under any valuation.

- $\alpha^t = \beta^t$, for every valuation t.
- α and β must have the same final column in their truth tables.
- $(\alpha \leftrightarrow \beta)$ is a tautology.

Why do we care about logical equivalence?

- Will I lose marks if I provide a solution that is syntactically different but logically equivalent to the provided solution?
- Do these two circuits behave the same way?
- Do these two pieces of code fragments behave the same way?

You already know one way of proving logical equivalent. What is it?

Theorem: $(((\neg p) \wedge q) \vee p) \equiv (p \vee q)$.

Logical Identities

Commutativity

$$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$$
$$(\alpha \vee \beta) \equiv (\beta \vee \alpha)$$

Associativity

$$\big(\alpha \wedge (\beta \wedge \gamma)\big) \equiv \big((\alpha \wedge \beta) \wedge \gamma\big)$$
$$\big(\alpha \vee (\beta \vee \gamma)\big) \equiv \big((\alpha \vee \beta) \vee \gamma\big)$$

Distributivity

$$\big(\alpha \vee (\beta \wedge \gamma)\big) \equiv \big((\alpha \vee \beta) \wedge (\alpha \vee \gamma)\big)$$
$$\big(\alpha \wedge (\beta \vee \gamma)\big) \equiv \big((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\big)$$

Idempotence

$$(\alpha \vee \alpha) \equiv \alpha$$
$$(\alpha \wedge \alpha) \equiv \alpha$$

Double Negation

$$\big(\neg(\neg\alpha)\big) \equiv \alpha$$

De Morgan's Laws

$$\big(\neg(\alpha \wedge \beta)\big) \equiv \big((\neg\alpha) \vee (\neg\beta)\big)$$
$$\big(\neg(\alpha \vee \beta)\big) \equiv \big((\neg\alpha) \wedge (\neg\beta)\big)$$

Logical Identities, cont'd

Simplification I (Absorption)

$$(\alpha \wedge \texttt{T}) \equiv \alpha$$
$$(\alpha \vee \texttt{T}) \equiv \texttt{T}$$
$$(\alpha \wedge \texttt{F}) \equiv \texttt{F}$$
$$(\alpha \vee \texttt{F}) \equiv \alpha$$

Simplification II

$$(\alpha \vee (\alpha \wedge \beta)) \equiv \alpha$$
$$(\alpha \wedge (\alpha \vee \beta)) \equiv \alpha$$

Implication

$$(\alpha \to \beta) \equiv ((\neg\alpha) \vee \beta)$$

Contrapositive

$$(\alpha \to \beta) \equiv ((\neg\beta) \to (\neg\alpha))$$

Equivalence

$$(\alpha \leftrightarrow \beta) \equiv ((\alpha \to \beta) \wedge (\beta \to \alpha))$$

Excluded Middle

$$(\alpha \vee (\neg\alpha)) \equiv \texttt{T}$$

Contradiction

$$(\alpha \wedge (\neg\alpha)) \equiv \texttt{F}$$

A logical equivalence proof

Theorem: $(((\neg p) \wedge q) \vee p) \equiv (p \vee q)$.

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Proof.

$(((\neg p) \wedge q) \vee p)$	
$\equiv (((\neg p) \vee p) \wedge (q \vee p))$	Distributivity
$\equiv (\mathtt{T} \wedge (q \vee p))$	Excluded Middle
$\equiv (q \vee p)$	Simplification I
$\equiv (p \vee q)$	Commutativity

□

A practice problem

"If it is sunny, I will play golf, provided that I am relaxed."

s: it is sunny. g: I will play golf. r: I am relaxed.

A few translations:

1. $(s \to (r \to g))$
2. $(r \to (s \to g))$
3. $((s \land r) \to g)$

Theorem: All three translations are logically equivalent.
Proof: Done in class.

Collected Wisdom

- Try getting rid of $\rightarrow$ and $\leftrightarrow$.
- Try moving negations inward. $(\neg(p \vee q)) \equiv ((\neg p) \wedge (\neg q))$.
- Work from the more complex side first, BUT
- Switch to different strategies/sides when you get stuck.
- In the end, write the proof in clean ' one-side-to-the-other' form and double-check steps.

```
if ( (input > 0) or (not output) ) {
    if ( not (output and (queuelength < 100) ) ) {
        P1
    } else if ( output and (not (queuelength < 100)) ) {
        P2
    } else { P3 }
} else { P4 }
```

When does each piece of code get executed?

Let i: `input > 0`,
u: `output`,
q: `queuelength < 100`.

A Code Example, cont'd

i	u	q	$(i \vee (\neg u))$	$(\neg(u \wedge q))$	$(u \wedge (\neg q))$	
T	T	T	T			
T	T	F	T			
T	F	T	T			
T	F	F	T			
F	T	T	F			P_4
F	T	F	F			P_4
F	F	T	T			
F	F	F	T			

A Code Example, cont'd

i	u	q	$(i \vee (\neg u))$	$(\neg(u \wedge q))$	$(u \wedge (\neg q))$	
T	T	T	T	F	F	P_3
T	T	F	T	T		P_1
T	F	T	T	T		P_1
T	F	F	T	T		P_1
F	T	T	F			P_4
F	T	F	F			P_4
F	F	T	T	T		P_1
F	F	F	T	T		P_1

Finding Live Code

Prove that P_3 is live code. That is, the conditions leading to P_3 is satisfiable.

Theorem:
$$\Big((i \vee (\neg u)) \wedge \Big((\neg(\neg(u \wedge q))) \wedge \big(\neg(u \wedge (\neg q))\big)\Big)\Big) \equiv ((i \wedge u) \wedge q)$$

Proof: In class

Two pieces of code: Are they equivalent?

Prove that the two code fragments are equivalent.

Listing 1: Your code

```
if (i || !u) {
  if (!(u && q)) {
    P1
  } else if (u && !q) {
    P2
  } else { P3 }
} else { P4 }
```

Listing 2: Your friend's code

```
if ((i && u) && q) {
  P3
} else if (!i && u) {
  P4
} else {
  P1
}
```

Simplifying Code

To prove that the two fragments are equivalent, show that each block of code P_1, P_2, P_3, and P_4 is executed under equivalent conditions.

Block	Fragment 1	Fragment 2
P_1	$(i \lor (\neg u)) \land (\neg(u \land q))$	$(\neg(i \land u \land q)) \land (\neg((\neg i) \land u))$
P_2	$(i \lor (\neg u)) \land (\neg(\neg(u \land q))) \land (u \land (\neg q))$	$\mathtt{F}$
P_3	$(i \lor (\neg u)) \land (\neg(\neg(u \land q))) \land (\neg(u \land (\neg q)))$	$(i \land u \land q)$
P_4	$(\neg(i \lor (\neg u)))$	$(\neg(i \land u \land q)) \land ((\neg i) \land u)$