Warmup Problem

Describe how structural induction differs from our MATH 135 notion of induction.

The Principle of Structural Induction

Theorem. Let R be a property. Suppose that

1. for each atomic formula p, we have $R(p)$; and
2. for each formula α, if $R(\alpha)$ then $R((\neg\alpha))$; and
3. for each pair of formulas α and β, and each binary connective $\star$, if $R(\alpha)$ and $R(\beta)$ then $R((\alpha \star \beta))$.

Then $R(\alpha)$ for every formula α.

Use of this principle is called *structural induction*.

Structural induction is a special case of mathematical induction.

CS 245: Logic and Computation

Carmen Bruni

Lecture 3

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, A. Gao, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Let's begin!

Announcements

1. Assignment 1 due. Submit to Crowdmark! If you're missing a link, please email myself and the ISC!
2. Reminder: Please attend your own class! Spots in a section are reserved for the people registered in the section. Standing in the back is a fire hazard.
3. Office Hours Monday 9-10 and Tuesday 10-11:20 in DC 3119 (in the bridge between DC and MC)

Previously on CS245

- Give multiple translations of English sentences with ambiguity.
- Describe the three types of symbols in propositional logic.
- Describe the recursive definition of well-formed formulas.
- Write the parse tree for a well-formed formula.
- Determine and give reasons for whether a given formula is well formed or not.
- A template for writing structural induction proofs.

Up next some more sample clicker questions!

Plan for today

By the end of this lecture, you should be able to

- Review the template for writing structural induction proofs.
- Prove the unique readability theorem using structural induction.
- Evaluate the truth value of a formula
  - Define a (truth) valuation.
  - Determine the truth value of a formula by using truth tables.
  - Determine the truth value of a formula by using valuation trees.
- Determine and prove whether a formula has a particular property
  - Define tautology, contradiction, and satisfiable formula.
  - Compare and contrast the three properties (tautology, contradiction, and satisfiable formula).
  - Prove whether a formula is a tautology, a contradiction, or satisfiable, using a truth table and/or a valuation tree.
  - Describe strategies to prove whether a formula is a tautology, a contradiction or a satisfiable formula.

Unbalanced brackets in a proper prefix of a formula

Lemma: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

A *proper prefix* of φ is a non-empty segment of φ starting from the first symbol of φ and ending before the last symbol of φ. Sometimes, people emphasize **non-empty** in the statement and say this is a *proper non-empty prefix*.

One can reword this as: A *proper prefix* of φ is a non-empty expression, say x, such that there exists a non-empty expression y satisfying $\varphi = xy$.

For example, the formula $(\neg p)$ has as proper prefixes $($, $(\neg$ and $(\neg p$.

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Let $P(\varphi)$ be the property that every proper prefix of the well-formed formula φ has more opening brackets than closing brackets. We prove $P(\varphi)$ is true for all well-formed formulas φ by structural induction.

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Base Case: If φ is an atom, then there are no proper prefixes and the claim is vacuously true.

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Inductive Hypothesis: Assume that $P(\alpha)$ and $P(\beta)$ are true for some well-formed formulas α and β.

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Complete the induction step.

Structural induction (1)

Step 1: Identify the recursive structure in the problem.

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Theorem 1: Every *well-formed formula* has an equal number of opening and closing brackets. Notes:

- "Well-formed formulas" are the recursive structures that we are dealing with.
- "Has an equal number of opening and closing brackets" is the property that we are going to prove that well formed formulas have. Some examples of other properties that we could prove: (1) contains at least one propositional variable, (2) has an even number of brackets, etc.

Structural induction (2)

Step 2: Identify each recursive appearance of the structure inside its definition. (A recursive structure is self-referential. Where in the definition of the object does the object reference itself?

Let $\mathcal{P}$ be a set of propositional variables. A *well-formed formula* over $\mathcal{P}$ has exactly one of the following forms:

1. A single symbol of $\mathcal{P}$,
2. $(\neg\alpha)$ if α is well-formed,
3. One of $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta), (\alpha \leftrightarrow \beta)$ if α and β are well-formed.

Structural induction (2)

Step 2: Identify each recursive appearance of the structure inside its definition. (A recursive structure is self-referential. Where in the definition of the object does it the object reference itself?

There are 3 cases in the following definition. The definition references itself in cases 2 and 3, as highlighted below.

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Structural induction (3)

Step 3: Divide the cases into: those without recursive appearances (" base cases") and those with ("inductive" cases).

Case 1 has no recursive appearance of the definition. Cases 2 and 3 have recursive appearances of the definition.

Let $\mathcal{P}$ be a set of propositional variables. A *well-formed formula* over $\mathcal{P}$ has exactly one of the following forms:

1. A single symbol of $\mathcal{P}$, (base case)
2. $(\neg\alpha)$ if α *is well-formed*, (inductive case)
3. One of $(\alpha \land \beta), (\alpha \lor \beta), (\alpha \to \beta), (\alpha \leftrightarrow \beta)$ if α *and* β *are well-formed*. (inductive case)

A structural induction template

Problem: Prove that every recursive structure φ has property P.

Define $P(\varphi)$ to be the property in the problem.

Theorem: For every *recursive structure* φ, $P(\varphi)$ holds.

Proof by structural induction:

> Base case: *For every base case you identified, prove that the recursive structure* φ *has property* P*.*

(Continued on the next slide)

A structural induction template

Induction step: *For each recursive case you identified, write an induction step*

Recursive case 1:

Induction hypothesis: Assume *each recursive appearance of the structure in this case* has property P.
Prove that the recursive structure φ has property P using the induction hypothesis.

Recursive case 2:

(State the induction hypothesis and use it to prove the theorem.)

(Possibly more recursive cases)

By the principle of structural induction, every recursive structure φ has property P. QED

Proving the unique readability of well-formed formulas

Lemma 1: Every well-formed formula starts with a propositional variable or an opening bracket.

Lemma 2: Every well-formed formula has an equal number of opening and closing brackets.

Lemma 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

Theorem: There is a unique way to construct every well-formed formula.

Main Theorem

Theorem (Unique Readability Theorem)

There is a unique way to construct every well-formed formula.

Proof

Let $P(\varphi)$ be the property that there is a unique way to construct the well-formed formula φ. We prove this property for all well-formed formulas φ by structural induction.

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Base Case: There is only one way to construct an atom.

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Inductive Hypothesis: Assume that $P(\alpha)$ and $P(\beta)$ are true for some well-formed formulas α and β.

Inductive Conclusion

We now have a few possibilities to consider:

1. $\varphi = (\neg\alpha)$
2. $\varphi = (\alpha \star \beta)$

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Each of these has two subcases...

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Subcase 2: If $\varphi = (\neg\alpha)$, then what if we could write $\varphi = (\alpha' \star \beta')$ for some other well-formed formulas α' and β'?

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Subcase 2: If $\varphi = (\neg\alpha)$, then what if we could write $\varphi = (\alpha' \star \beta')$ for some other well-formed formulas α' and β'?

Then in this case, it must be that $\alpha = (\gamma \star \eta)$, $\alpha' = \neg(\gamma$ and $\beta' = \eta$. This is a contradiction since α' is not well-formed but α' was assumed to be well-formed (could argue formally using bracket argument if desired).

Subcases

Subcase 3: If $\varphi = (\alpha \star \beta)$, then what if we could write $\varphi = (\alpha' \star \beta')$ for some other well-formed formulas α' and β'?

Proof

We have assumed that φ *is* $(\alpha \star \beta)$*, where both* α *and* β *have property* P*.*
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If φ is $(\alpha' \star' \beta')$ for **formulas** α' and β', then $\alpha = \alpha'$, $\star = \star'$ and $\beta = \beta'$.

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We have assumed that φ is $(\alpha \star \beta)$, where both α and β have property P.
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If φ is $(\alpha' \star' \beta')$ for **formulas** α' and β', then $\alpha = \alpha'$, $\star = \star'$ and $\beta = \beta'$.

If $|\alpha'| = |\alpha|$, then $\alpha' = \alpha$ (both start at the second symbol of φ).
Thus also $\star = \star'$ and $\beta = \beta'$, as required.

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Thus also $\star = \star'$ and $\beta = \beta'$, as required.

If $0 < |\alpha'| < |\alpha|$, then α' is a proper prefix of α.
Thus, by lemmas 2 and 3 on α, α' is not a formula; we have nothing to prove.

Proof

We have assumed that φ is $(\alpha \star \beta)$, where both α and β have property P.
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If φ is $(\alpha' \star' \beta')$ for **formulas** α' and β', then $\alpha = \alpha'$, $\star = \star'$ and $\beta = \beta'$.

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Thus, by lemmas 2 and 3 on α, α' is not a formula; we have nothing to prove.

If $|\alpha'| > |\alpha|$,then $\alpha' = \alpha \star y$, where y is a proper prefix of β (or is empty).
By lemma 2, α has equally many '(' and ')', while by lemma 3 y has more '(' than ')'. Thus α' has more '(' than ')' and hence again by lemma 3, it is not a formula, and again we have nothing to prove.

Therefore α has a unique derivation in this case.

Subcases

Subcase 4: If $\varphi = (\alpha \star \beta)$, then what if we could write $\varphi = (\neg\alpha')$ for some other well-formed formulas α' and β'?

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Argue as in Subcase 2; Notice that $\alpha' = (\gamma \star \eta)$ for some well-formed formulas γ and η which also satisfies $\alpha = \neg(\gamma$ and $\beta = \eta$. Once again α is not well-formed which is a contradiction.

The meaning of well-formed formulas

To interpret a formula, we have to give meanings to the propositional variables and the connectives.

A propositional variable has no intrinsic meaning; it gets a meaning via a valuation.

A (truth) valuation is a function $t : \mathcal{P} \mapsto \{\mathtt{F}, \mathtt{T}\}$ from the set of all proposition variables $\mathcal{P}$ to $\{\mathtt{F}, \mathtt{T}\}$. It assigns true/false to every propositional variable.

Two notations: $t(p)$ and p^t both denote the truth value of p under the truth valuation t.

Truth value of a formula

Fix a truth valuation t. Every formula α has a value under t, denoted α^t, determined as follows.

1. $p^t = t(p)$.
2. $(\neg\alpha)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \mathtt{F} \\ \mathtt{F} & \text{if } \alpha^t = \mathtt{T} \end{cases}$
3. $(\alpha \wedge \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \beta^t = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$
4. $(\alpha \vee \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \mathtt{T} \text{ or } \beta^t = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$
5. $(\alpha \to \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \mathtt{F} \text{ or } \beta^t = \mathtt{T} \\ \mathtt{F} & \text{otherwise} \end{cases}$
6. $(\alpha \leftrightarrow \beta)^t = \begin{cases} \mathtt{T} & \text{if } \alpha^t = \beta^t \\ \mathtt{F} & \text{otherwise} \end{cases}$

Truth tables for connectives

The unary connective $\neg$:

α	$(\neg\alpha)$
T	F
F	T

The binary connectives $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$:

α	β	$(\alpha \wedge \beta)$	$(\alpha \vee \beta)$	$(\alpha \rightarrow \beta)$	$(\alpha \leftrightarrow \beta)$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

Evaluating a formula using a truth table

Example. The truth table of $((p \vee q) \rightarrow (q \wedge r))$.

Evaluating a formula using a truth table

Example. The truth table of $((p \vee q) \to (q \wedge r))$.

p	q	r	$(p \vee q)$	$(q \wedge r)$	$((p \vee q) \to (q \wedge r))$
F	F	F	F	F	T
F	F	T	F	F	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	F	F
T	T	F	T	F	F
T	T	T	T	T	T

Build the truth table of $\big((p \to (\neg q)) \to (q \vee (\neg p))\big)$.

Understanding the disjunction and the biconditional

α	β	$(\alpha \vee \beta)$	Exclusive OR	Biconditional
F	F	F	F	T
F	T	T	T	F
T	F	T	T	F
T	T	T	F	T

- What is the difference between an inclusive OR (the disjunction) and an exclusive OR?
- What is the relationship between the exclusive OR and the biconditional?

A Small Theorem

Theorem: Fix a truth valuation t. Every formula α has a value α^t in $\{\mathtt{F}, \mathtt{T}\}$.

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Proof: By structural induction. Let $R(\alpha)$ be "α has a value α^t in $\{\mathtt{F}, \mathtt{T}\}$".

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Theorem: Fix a truth valuation t. Every formula α has a value α^t in $\{\texttt{F}, \texttt{T}\}$.

Proof: By structural induction. Let $R(\alpha)$ be "α has a value α^t in $\{\texttt{F}, \texttt{T}\}$".

1. If α is a propositional variable, then t assigns it a value of T or F (by the definition of a truth valuation).
2. If α has a value in $\{\texttt{F}, \texttt{T}\}$, then $(\neg\alpha)$ also does, as shown by the truth table of $(\neg\alpha)$.
3. If α and β each has a value in $\{\texttt{F}, \texttt{T}\}$, then $(\alpha \star \beta)$ also does for every binary connective $\star$, as shown by the corresponding truth tables.

By the principle of structural induction, every formula has a value.

By the unique readability of formulas, we have proved that a formula has **only one** truth value under any truth valuation t. QED

Tautology, Contradiction, Satisfiable

A formula α is a *tautology* if and only if
for every truth valuation t, $\alpha^t = \mathtt{T}$.

A formula α is a *contradiction* if and only if
for every truth valuation t, $\alpha^t = \mathtt{F}$.

A formula α is *satisfiable* if and only if
there exists a truth valuation t such that $\alpha^t = \mathtt{T}$.

Examples

Let p be a Propositional variable.

- The formula $(p \vee (\neg p))$ is a tautology (and satisfiable)
- The formula $(p \wedge (\neg p))$ is a contradiction (and not satisfiable)
- The formula $(p \rightarrow (\neg p))$ is satisfiable (but not a tautology)

How to determine the properties of a formula

- Truth table
- Valuation tree
- Reasoning

Valuation Tree

Rather than fill out an entire truth table, we can analyze what happens if we plug in a truth value for one variable.

$\neg T$	F
$\neg F$	T

$p \wedge T$	p
$p \wedge F$	F
$p \wedge p$	p

$p \vee T$	T
$p \vee F$	p
$p \vee p$	p

$p \rightarrow T$	T
$p \rightarrow F$	$\neg p$
$T \rightarrow p$	p
$F \rightarrow p$	T
$p \rightarrow p$	T

We can evaluate a formula by using these rules to construct a *valuation tree*.

Example of a valuation tree

Example. Show that $\big(((p \wedge q) \to (\neg r)) \wedge (p \to q)\big) \to \big(p \to (\neg r)\big)$ is a tautology by using a valuation tree.

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Suppose $t(p) = \mathtt{T}$. We put $\mathtt{T}$ in for p:

$$\big(((\mathtt{T} \wedge q) \to (\neg r)) \wedge (\mathtt{T} \to q)\big) \to \big(\mathtt{T} \to (\neg r)\big)\ .$$

Based on the truth tables for the connectives, the formula becomes $\big(((q \to (\neg r)) \wedge q) \to (\neg r)\big)$.

If $t(q) = \mathtt{T}$, this yields $((\neg r) \to (\neg r))$ and then $\mathtt{T}$. (Check!).
If $t(q) = \mathtt{F}$, it yields $(\mathtt{F} \to (\neg r))$ and then $\mathtt{T}$. (Check!).

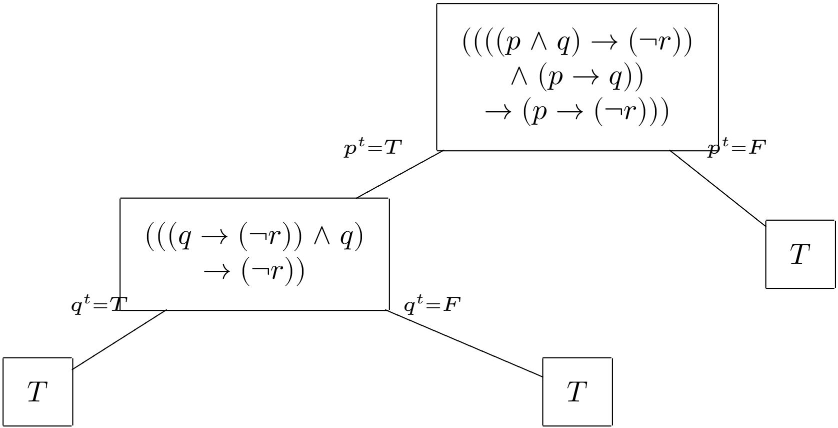
Suppose $t(p) = \mathtt{F}$. We get

$$\big(((\mathtt{F} \wedge q) \to (\neg r)) \wedge (\mathtt{F} \to q)\big) \to \big(\mathtt{F} \to (\neg r)\big)\ ,$$

Simplification yields $\big((\mathtt{F} \to (\neg r)) \wedge \mathtt{T}\big) \to \mathtt{T}$ and eventually $\mathtt{T}$.

Thus every valuation makes the formula true, as required.

Valuation Tree (Do not break on multiple lines!)



Examples

Determine which of the following are satisfiable, form a tautology or form a contradiction. Use a truth table. Repeat using a valuation tree.

1. $\Big(\big(((p \lor q) \leftrightarrow (\neg r)) \land (p \to q)\big) \land (p \to (\neg r))\Big)$
2. $\Big(\big(((p \land q) \to r) \land (p \to q)\big) \to (p \to r)\Big)$
3. $((p \lor q) \leftrightarrow ((p \land (\neg q)) \lor ((\neg p) \to q)))$
4. $(p \land (\neg p))$