Warm Up Problem

Write a DFA over $\Sigma = \{a, b\}$ that...

- Accepts only words with an even number of as
- Accepts only words with an odd number of as and an even number of bs
- Accepts only words where the parity of the number of as is equal to the parity of the number of bs

If you did the homework above, try these problems!

- What is the definition of a DFA? (Try it without looking!)
- Write a DFA over $\Sigma = \{a, b\}$ that accepts all words ending with bba.

CS 241 Lecture 7

Non-Deterministic Finite Automata

With thanks to Brad Lushman, Troy Vasiga and Kevin Lanctot

Recall Regular Language

Definition

A **regular language** over an alphabet Σ consists of one of the following:

1. The empty language and the language consisting of the empty word are regular
2. All languages $\{a\}$ for all $a \in \Sigma$ are regular.
3. The union, concatenation or Kleene star of any two regular languages are regular.
4. Nothing else.

Recall: Deterministic Finite Automata

Definition

A **DFA** is a 5-tuple $(\Sigma, Q, q_0, A, \delta)$:

- Σ is a finite non-empty set (alphabet).
- Q is a finite non-empty set of states.
- $q_0 \in Q$ is a start state
- $A \subseteq Q$ is a set of accepting states
- $\delta : (Q \times \Sigma) \to Q$ is our [total] transition function (given a state and a symbol of our alphabet, what state should we go to?).

Extending δ

We can extend the definition of $\delta : (Q \times \Sigma) \to Q$ to a function defined over $(Q \times \Sigma^*)$ via:

$$\begin{aligned} \delta^* : (Q \times \Sigma^*) &\to Q \\ (q, \epsilon) &\mapsto q \\ (q, aw) &\mapsto \delta^*(\delta(q, a), w) \end{aligned}$$

where $a \in \Sigma$ and $w \in \Sigma^*$ (aw is concatenation). Basically, if processing a string, process a letter first then process the rest of a string. In this way...

Definition

A DFA given by $M = (\Sigma, Q, q_0, A, \delta)$ **accepts a string** w if and only if $\delta^*(q_0, w) \in A$.

Language of a DFA

With the previous slide we can make one more definition.

Definition

Denote the **language of a DFA** M to be the set of all strings accepted by M, that is:

$$L(M) = \{w : M \text{ accepts } w\}$$

A Beautiful Result

In a future course (CS 360/365), you will prove the following beautiful result:

Theorem (Kleene)

L is regular if and only if $L = L(M)$ for some DFA M. That is, the regular languages are precisely the languages accepted by DFAs.

Implementing a DFA

Algorithm 1 DFA algorithm

```
s = q_0
while not EOF do
    read character ch
    switch (s)
    case q_0:
        switch (ch)
        case ch = a_0:
            s = new_state_a_0
        case ch = a_1:
            s = new_state_a_1
⋮

        case ch = a_|Σ|:
            s = new_state_a_sigma
        end switch
    case q_1:
⋮

    end switch
end while
```

Alternatively

You could also use a lookup table:

	q_0	q_1	$\cdots$	$q_{\|Q\|}$
a_0				
a_1				
$\vdots$				
$a_{\|\Sigma\|}$				

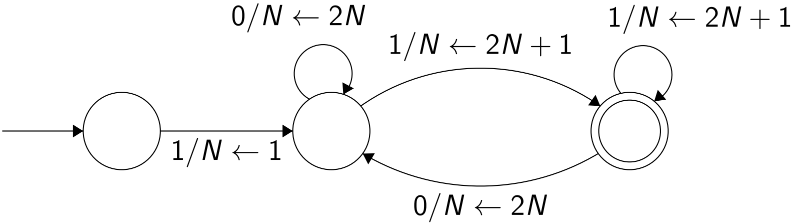
where above, the blank table entries would be the next states.

Check out the provided assembler starter code in your assignment!

Extension to DFAs

We could also have DFAs where we attach actions to arcs.

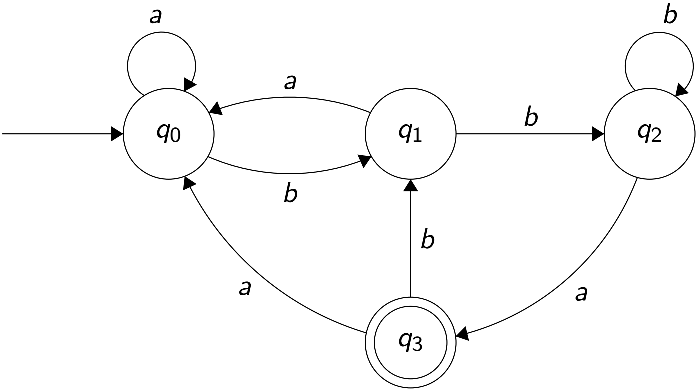
- For example, consider a subset of the language of binary numbers without leading zeroes described below.
- We'll create a DFA where we also compute the decimal value of the number simultaneously. Could then print the value.
- Look at the DFA corresponding to $1(0 \mid 1)^*1$.
- In what follows, you should read $1/N \leftarrow 2N+1$ as the leftmost 1 corresponds to a DFA transition, the / has no meaning and the $N \leftarrow 2N+1$ changes N to be $2N+1$.



Revisiting our Warm Up

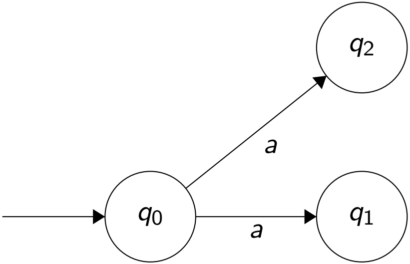
What happens if we make out DFAs more complex? Let's revisit our warmup example from today over the alphabet $\Sigma = \{a, b\}$:

$$L = \{w : w \text{ ends with } bba\}$$



Imagine

But what if we allowed more than one transition from a state?



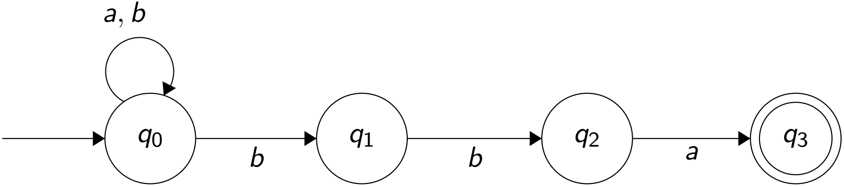
Does such a thing make sense? Do we gain any computability power from this?

Multiple Transitions

- When we allow for a state to have multiple branches given the same input, we say that the machine *chooses* which path to go on.
- This is called *non-determinism*.
- We then say that a machine accepts a word w if and only if there exists *some* path that leads to an accepting state!
- We can then simplify the previous example to an NFA as defined on the next slide:

Simplified NFA

$$L = \{w : w \text{ ends with } bba\}$$



Machine "guesses" to stay in first state until *bba* is seen. How does a machine do this?

Language of a NFA

Similar to before, we have the following definition:

Definition

Let M be an NFA. We say that M **accepts** w if and only if there exists *some* path through M that leads to an accepting state.

Denote the **language of an NFA** M to be the set of all strings accepted by M, that is:

$$L(M) = \{w : M \text{ accepts } w\}$$

Non-Deterministic Finite Automata

The above idea can be mathematically described as follows:

Definition

An **NFA** is a 5-tuple $(\Sigma, Q, q_0, A, \delta)$:

- Σ is a finite non-empty set (alphabet).
- Q is a finite non-empty set of states.
- $q_0 \in Q$ is a start state
- $A \subseteq Q$ is a set of accepting states
- $\delta : (Q \times \Sigma) \to 2^Q$ is our [total] transition function. Note that 2^Q denotes the *power set* of Q, that is, the set of all subsets of Q. This allows us to go to multiple states at once!

Extending δ For an NFA

Again we can extend the definition of $\delta : (Q \times \Sigma) \to 2^Q$ to a function $\delta^* : (2^Q \times \Sigma^*) \to 2^Q$ via:

$$\begin{aligned}\delta^* : (2^Q \times \Sigma^*) &\to 2^Q \\ (S, \epsilon) &\mapsto S \\ (S, aw) &\mapsto \delta^*\left(\bigcup_{q \in S} \delta(q, a), w\right)\end{aligned}$$

where $a \in \Sigma$. Analogously, we also have:

Definition

An NFA given by $M = (\Sigma, Q, q_0, A, \delta)$ **accepts a string** w if and only if...

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Definition

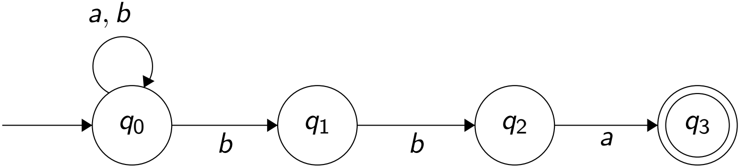
An NFA given by $M = (\Sigma, Q, q_0, A, \delta)$ **accepts a string** w if and only if $\delta^*(\{q_0\}, w) \cap A \neq \emptyset$.

Simulating an NFA

Algorithm 2 Algorithm to Simulate an NFA

$S = \{q_0\}$
while not EOF **do**
    c = read_char()
    $S = \bigcup_{q \in S} \delta(q, c)$
end while
if $S \cap A \neq \emptyset$ **then**
    Accept
else
    Reject
end if

Practice Simulating $w = abbba$



Processed	Remaining	S
ϵ	$abbba$	$\{q_0\}$
a	$bbba$	$\{q_0\}$
ab	bba	$\{q_0, q_1\}$
abb	ba	$\{q_0, q_1, q_2\}$
$abbb$	a	$\{q_0, q_1, q_2\}$
$abbba$	ϵ	$\{q_0, q_3\}$

Since $\{q_0, q_3\} \cap \{q_3\} \neq \emptyset$, accept.