CO 480 Lecture 20

Gauss and Quadratic Reciprocity

July 13th, 2017

July 13th, 1832

Charles Babbage received a Gold Medal. Babbage was the first recipient of the Royal Astronomical Society's Gold Medal. He earned it for his work "Observations on the Application of Machinery to the Computation of Numerical Tables."

Announcements

- Assignment 4 is due next week at 2:30pm on Crowdmark
- Final edition is due Tuesday July 25th on LEARN.
- Please put your name on your final edition! (In case your project should get used as exemplars in future years!)
- Check your inboxes for an announcement on Monday evening about your final editions.
- Final Quiz is on Tuesday July 25th in M3 at 2:30pm.
- Quiz will cover from post Quiz 1 material. Math will cover topics after Alain (history will include Alain's topics).

Exploratory Question Part 3

Examine when the following equations are solvable. Can you make any conjectures about the relationships between $\left(\frac{p}{q}\right)$ and $\left(\frac{q}{p}\right)$?

$$x^2 \equiv 3 \mod 5$$

$$x^2 \equiv 5 \mod 3$$

$$x^2 \equiv 3 \mod 7$$

$$x^2 \equiv 7 \mod 3$$

$$x^2 \equiv 7 \mod 5$$

$$x^2 \equiv 5 \mod 7$$

$$x^2 \equiv 5 \mod 13$$

$$x^2 \equiv 13 \mod 5$$

Quadratic Reciprocity

Quadratic Reciprocity

Let p and q be two odd distinct primes. Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)/2\cdot(q-1)/2}$$

Examples Using Quadratic Reciprocity

Compute $\left(\frac{19}{31}\right)$

Examples Using Quadratic Reciprocity

Compute $\left(\frac{19}{31}\right)$

Solution: Notice that both 19 and 31 are congruent to 3 modulo 4 and so

$$\left(\frac{19}{31}\right) = -\left(\frac{31}{19}\right) = -\left(\frac{12}{19}\right) = -\left(\frac{4}{19}\right)\left(\frac{3}{19}\right) = -\left(\frac{3}{19}\right)$$

Once again we use quadratic reciprocity to see that

$$\left(\frac{19}{31}\right) = -\left(\frac{3}{19}\right) = \left(\frac{19}{3}\right) = \left(\frac{1}{3}\right) = 1$$

Quadratic Reciprocity

- Euler conjectures this result and Legendre attempted a proof in his book *Théorie des Nombres* [Leg09, p. 214-226].
- Proof was corrected by Gauss in his book *Disquisitiones Arithmeticae* (See section 4, particularly around pages 88-89 in [Gau86]).
- Gauss also mentions why Legendre's proof is inadequate [Gau86, p. 104-106, 349-352, Art. 151, 296, 297]
- The key missing fact is that in a given arithmetic progression (under mild conditions), there exists infinitely many primes, a result first proved by Dirichlet in 1837. (Legendre needed there to be one prime in any AP not necessarily infinitely many).

Quadratic Reciprocity

Of the cases Legendre did prove, he used the following theorem which he derived from case work using infinite descent(!)

Legendre's Theorem

Let a, b, c be three integers not all of the same sign and such that abc is a squarefree integer. Then the equation

$$ax^2 + by^2 + cz^2 = 0$$

has a solution in integers x, y and z not all 0 if and only if $-bc$, $-ca$ and $-ab$ are all quadratic residues modulo $|a|$, $|b|$ and $|c|$ respectively.

Gauss

- The Prince of Mathematics.
- "Gauss is the world's greatest mathematician" (Laplace) [Hal70, p. 72].
- "It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment." (Bolyai in Sept. 2, 1808 letter to Gauss) [Pic08, p. 291].

(Wikimedia Commons)

Childhood (See also [Dun55, p. 130])

Birthplace of Gauss `http://gausschildren.org/genwiki/images/thumb/4/40/Fig2.JPG/180px-Fig2.JPG`

THE BIRTHPLACE OF CARL FRIEDRICH GAUSS IN BRUNSWICK

The birthplace of C. F. Gauss in Brunswick (picture taken 1884), which was destroyed in World War II

Childhood

- Most of what we know about Gauss of his childhood is from stories in his older age. These are suspect but others have claimed to have confirmed them [Hal70, p. 3].
- Born Johann Carl Friedrich Gauss on April 30th, 1777 in Brunswick (same Brunswick as Brunswick Manifesto, also known as Braunschweig).

Gauss' Easter Formula

- `https://en.wikipedia.org/wiki/Computus#Gauss_algorithm`
- In 1800, Gauss created a formula, valid from 1700-1899, that would compute exactly what day Easter falls on.
- Gauss claims he was motivated because he didn't know his birthday [Hal70, p. 60].
- His mother was illiterate and only knew that he was born on a Wednesday, eight days after Ascension Day (which occurs forty days after Easter) [Hal70, p. 60].

What a Mess! (Wikipedia)

Expression	*year* = 1777
$a = year \bmod 19$	$a = 10$
$b = year \bmod 4$	$b = 1$
$c = year \bmod 7$	$c = 6$
$k = \lfloor \frac{year}{100} \rfloor$	$k = 17$
$p = \lfloor \frac{13 + 8k}{25} \rfloor$	$p = 5$
$q = \lfloor \frac{k}{4} \rfloor$	$q = 4$
$M = (15 - p + k - q) \bmod 30$	$M = 23$
$N = (4 + k - q) \bmod 7$	$N = 3$
$d = (19a + M) \bmod 30$	$d = 3$
$e = (2b + 4c + 6d + N) \bmod 7$	$e = 5$
Gregorian Easter is 22 + *d* + *e* March or *d* + *e* – 9 April	30 March
if $d = 29$ and $e = 6$, replace 26 April with 19 April	
if $d = 28$, $e = 6$, and $(11M + 11) \bmod 30 < 19$, replace 25 April with 18 April	
For the Julian Easter in the Julian calendar $M = 15$ and $N = 6$ (*k*, *p*, and *q* are unnecessary)	

Childhood

- At the age of 7, he enrolled in St. Katharine's Volksschule (St Catherine's elementary school) in 1784 [Hal70, p. 3] [Dun55, p. 12].
- His schoolboy teacher, J. G. Büttner, who was a competent teacher who challenged Gauss [B81̈, p. 6].

Famous Story

- Note - the truth of this story is up to debate since it primarily comes from word of both from Gauss.
- Büttner gave his class an exercise to add the numbers from 1 to 100.
- Students would answer on tablets and put them in the middle of a centre table when they were done.
- Before the question was completely articulated, Gauss, then ten years old ([Kat93, p. 654] and [Bur91, p. 546] claims he was 9), had his number, 5050 already on the table with no other work shown saying "Ligget se!" (There it is) [Dun55, p. 12] [Hal70].
- Gauss then explained to the teacher how he did the problem.

Sum Slide

$$
\begin{array}{rcrcrcrcr}
1 & + & 2 & + & 3 & + & \ldots & + & 100 \\
+\ 100 & + & 99 & + & 98 & + & \ldots & + & 1 \\
\hline
101 & + & 101 & + & 101 & + & \ldots & + & 101
\end{array}
$$

This gives a total of 100 copies of 101. This is twice the sum we want so the total is 5050.

More of Gauss' Schooling (Picture of Bartels)

- Assistant to Büttner was Johann Christian Martin Bartels, an 18 year old who eventually became a mathematics professor and was able to mentor Gauss [Hal70, p. 5]
- The two educators approached his father, Gebhard, about the boy's education.
- Gebhard provided for his family by working as a stonemason, canal worker and gardener (they were not wealthy) - providing for education would be tough.
- The two educators assured Gebhard that they would be able to find funding for the boy and eventually Gauss' nightly chores turned into into late night reading sessions with Bartels.

(Wikimedia Commons)

Secondary School [Hal70, p. 6]

- Gauss, with the aid of Bartels and Büttner managed to get Gauss into Gymnasium Catharineum in Braunschweig in 1788 [OR].
- Professor Hellwig, a professor there, after seeing Gauss first assignment, claimed that Gauss no longer needed to attend his lectures.
- At the gymnasium, Gauss learnt about other languages, including Latin and High German (different than local dialect) [B̈81, p. 7].

Charles William Ferdinand, Duke of Brunswick-Wolfenbüttel

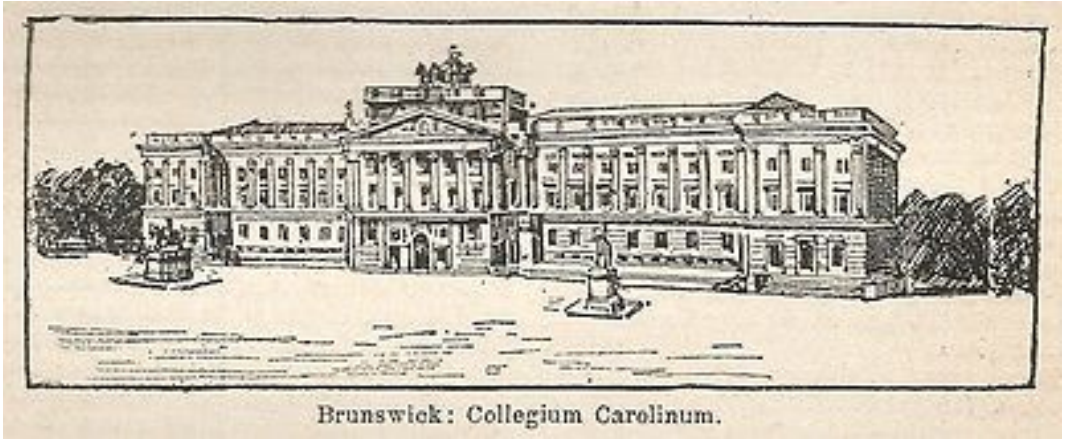
- In 1791, Gauss was introduced to the prince, the Duke of Brunswick-Wolfenbüttel [B̈81, p. 8]
- Impressed by the boy's talents, he awarded Gauss an initial yearly stipend of 10 talers. [1]
- With this money, he was able to attend the Collegium Carolinium, signing the register there in February 18th, 1792 as Johann Friderich Carl Gauss of Brunswick (The last time he used Johann).

[1] The best I can find for what this is worth is that a sow costs 8 talers in 1808 (see page 202 in Ordinary Prussians: Brandenburg Junkers and Villagers, 1500-1840 By William W. Hagen)

Duke of Brunswick (Wikimedia Commons)

His Time in Carolinium

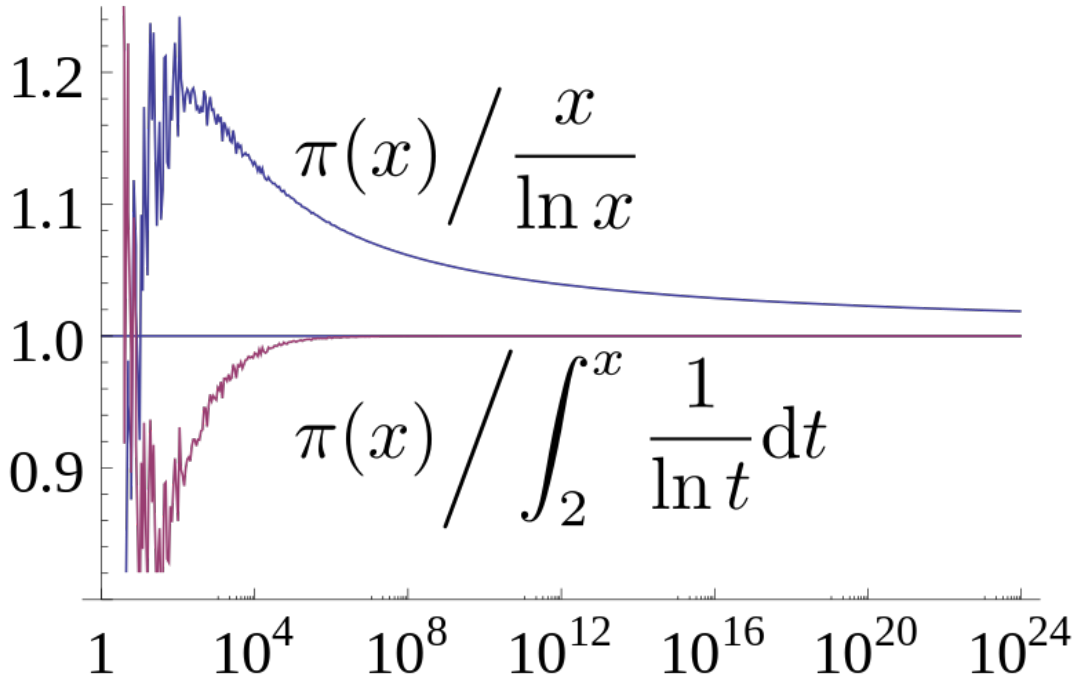
- While there, he met with the councillor, professor von Zimmerman.
- This relationship was crucial for Gauss as Zimmerman knew the Duke and would keep the flow of money coming to the young boy.
- The Duke insistent on giving the young boy what he needed "for the continued training of such a gifted person" [Hal70, p. 6]


Brunswick: Collegium Carolinum.

(Stamp)

Research at a Young Age

- Much like what we have done in the last few lectures, Gauss experimented with numbers and played with them trying to find patterns.
- It was here that Gauss made the observation about $\pi(x)$ and $\int_2^x \frac{dt}{\ln t}$
- Here he also discovered the method of least squares (first published by Legendre some 10 years later).



University in Göttingen

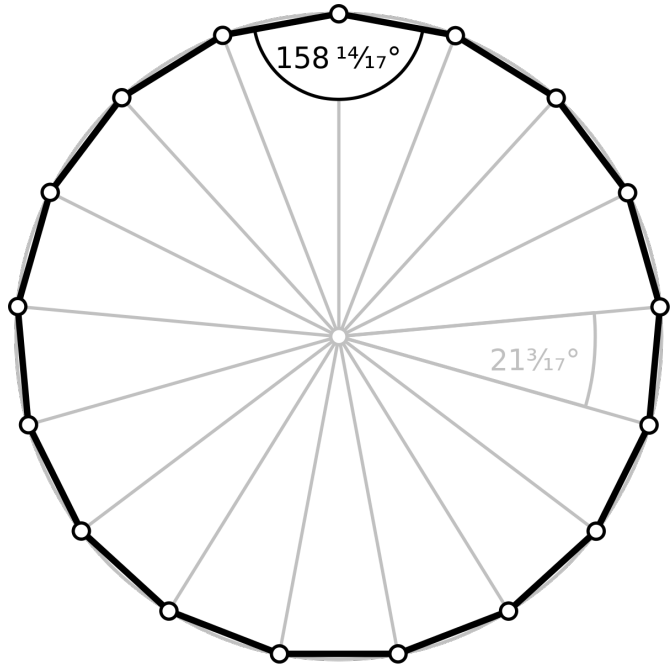
- August 21st, 1795 the order from the ducal office (Duke's office) "that 158 talers yearly shall be paid to the student named Gauss going to Göttingen, for assistance and that he be informed of this, as well as that the 'free table' is open to him in Göttingen" [Dun55, p. 21]
- This number increased to 400 talers in 1801 and 600 in 1803 (and free lodging!) [Dun55, p. 21]
- On October 11th, 1795, Gauss left Brunswick to study at Göttingen University, largely due the wealth of mathematical knowledge the university possessed in the library [B$\ddot{\text{o}}$81, p. 15].

University in Göttingen in Hanover

- The Duke preferred that Gauss stay in Brunswick and study at the local university at Helmstädt [B̈81, p. 15].
- Gauss was not the first to disobey the wishes of the duke and the duke still kept Gauss' stipend.
- Gauss' teacher at Göttingen was Kästner whom Gauss often ridiculed [OR] [B̈81, p. 16].

Uncertainty

- During his time at Göttingen, Gauss was still undecided about whether to study languages or mathematics.
- The decision was made on one faithful day, March 30th, 1796, when Gauss figured out that a regular 17-gon can be constructed using only a straight edge and compass [Bur91, p. 547].
- Further, he proved that the n-gon with n a Fermat prime was always constructible.
- In fact, he was so proud of this result, he asked a stonemason if he could have this etched in his tombstone. The stonemason refused citing that it would be indistinguishable from a circle [Bur91, p. 547] [Hal70, p. 23] [Dun55, p. 28].



Gauss Monument in Brunswick (Check out near right foot!)

Thesis

- Gauss left the university in 1798 without a degree but had made several important discoveries.
- Part of his discoveries is rumoured to be a full proof of the Quadratic Reciprocity Theorem [Hal70, p. 56].
- This theorem he called "the gem of arithmetic" or "the golden theorem" [Hal70, p. 19]
- The Duke of Brunswick, despite Gauss leaving, continued his stipend but insisted that Gauss publish a thesis with the University of Helmstädt with no examination (somewhat) under Johann Fredrich Pfaff [OR] [Bur91, p. 548].
- Gauss did just this: *Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse.* (New Proof of the Theorem That Every Integral Rational Function of One Variable Can Be Decomposed into Real Factors of the First or Second Degree) [Hal70, p. 43] [Bur91, p. 548]

Notes about Thesis

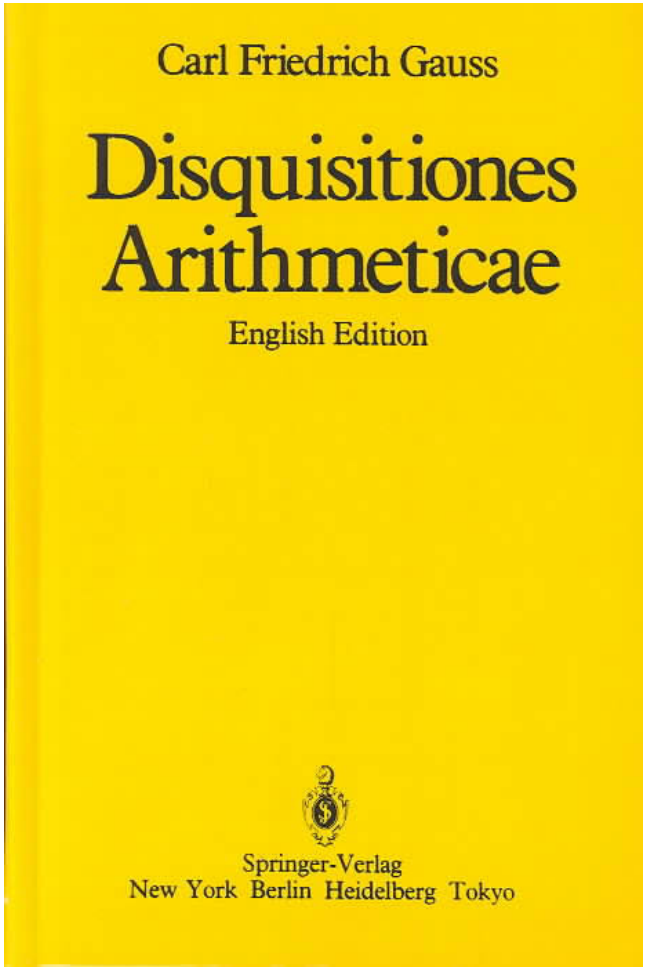
- The aforementioned paper contained the first 'proof' of the Fundamental Theorem of Algebra (Specifically a new proof that real polynomials can be reduced to linear and quadratic factors)
- Gauss was never really satisfied with his thesis, spending the next 50 years writing 4 clarified proofs to this statement.
- Part of his gripe with his thesis was that he assumed a lot of concepts about continuous functions that were not created until Bernard Bolzano did so in his work. [Hal70, p. 45]
- FTA was surely known before Gauss, but others, d'Alembert (1746), Euler (1749) and Lagrange (1772) all failed to give sufficient proofs [Bur91, p. 548]

Financial Times of Gauss in Brunswick

- Gauss managed to earn a partial living by doing some private tutoring while at Brunswick.
- There were occasions where Gauss was unsuccessful at obtaining new students and in those times, the Duke of Brunswick granted Gauss a fixed pension so that he could devote more time to research [Bur91, p. 548].

1763 taler

Disquisitiones Arithmeticae



Disquisitiones Arithmeticae

- In 1801, Gauss published Disquisitiones Arithmeticae.
- He dedicated this book to the Duke of Brunswick.
- The duke continued his support of Gauss until he was killed in battle in November of 1806 from France [Kat93].
- However, in this, the French general was told to see to the welfare of Gauss.
- This was a part of the Napoleonic rise, specifically at the Battle of Jena-Auerstädt on October 14th, 1806.
- The Prussian army was outdated and France quickly overtook them under the lead of Napolean.

Contents [Gau86]

- First three chapters: Review of known results including Fermat's Little Theorem, Wilson's Theorem and the existence of primitive roots modulo p.
- These results were known but these present the first modern treatment of these subjects, including the modular notation.
- Chapter 4 contains the Gem of Arithmetic (a proof of Quadratic Reciprocity)
- Chapter 5 is about half the book and consists of binary and ternary quadratic forms.
- Chapter 6 contains a few primaility tests.
- Last chapter was the result about constructibility of polygons corresponding to Fermat primes.
- Written in Latin, first translation into English was in 1965.

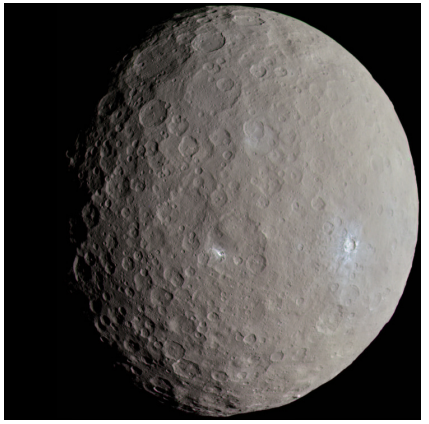
Wilson's Theorem

Wilson's Theorem

We have $(p-1)! \equiv -1 \mod p$ if and only if p is a prime.

Ceres [Bur91, p. 549]

- One of Gauss' greatest feats was computing the position of the dwarf planet Ceres between Mars and Jupiter
- Italian Piazzi discovered this on January 1st, 1801 but no astronomer could find it again.
- Gauss, using poor data, was able to, with remarkable accuracy, rediscover the planet by predicting it's orbit around the sun.
- This earned him an offer from St. Petersburg Academy which he declined.



Marriage and Göttingen

- In 1803, Gauss met Johanna Osthoff, daughter of a tannery owner in Brunswick [Hal70, p. 68].
- Johanna and Gauss' families appear to have known each other and there is some evidence that they knew each other as children but we know little of their private life [Bö81, p. 48]
- In 1804, the University of Göttingen was establishing an observatory and they wanted gauss at the helm [Hal70, p. 67].
- Johanna and Gauss were engaged at the end of 1804 and seemingly immediately, he wrote a letter to his good friend Boylai [Hal70, p. 68]:

  > *For three days now this angel, almost too heavenly for our earth, has been my fiancée... Life lies before me like an eternal spring with new radiant colours.*

- On October 9th, 1805, the couple were married.
- Recall: During this time, Gauss was in communication with one M.LeBlanc (Sophie Germain) [Bö81, p. 52-53]

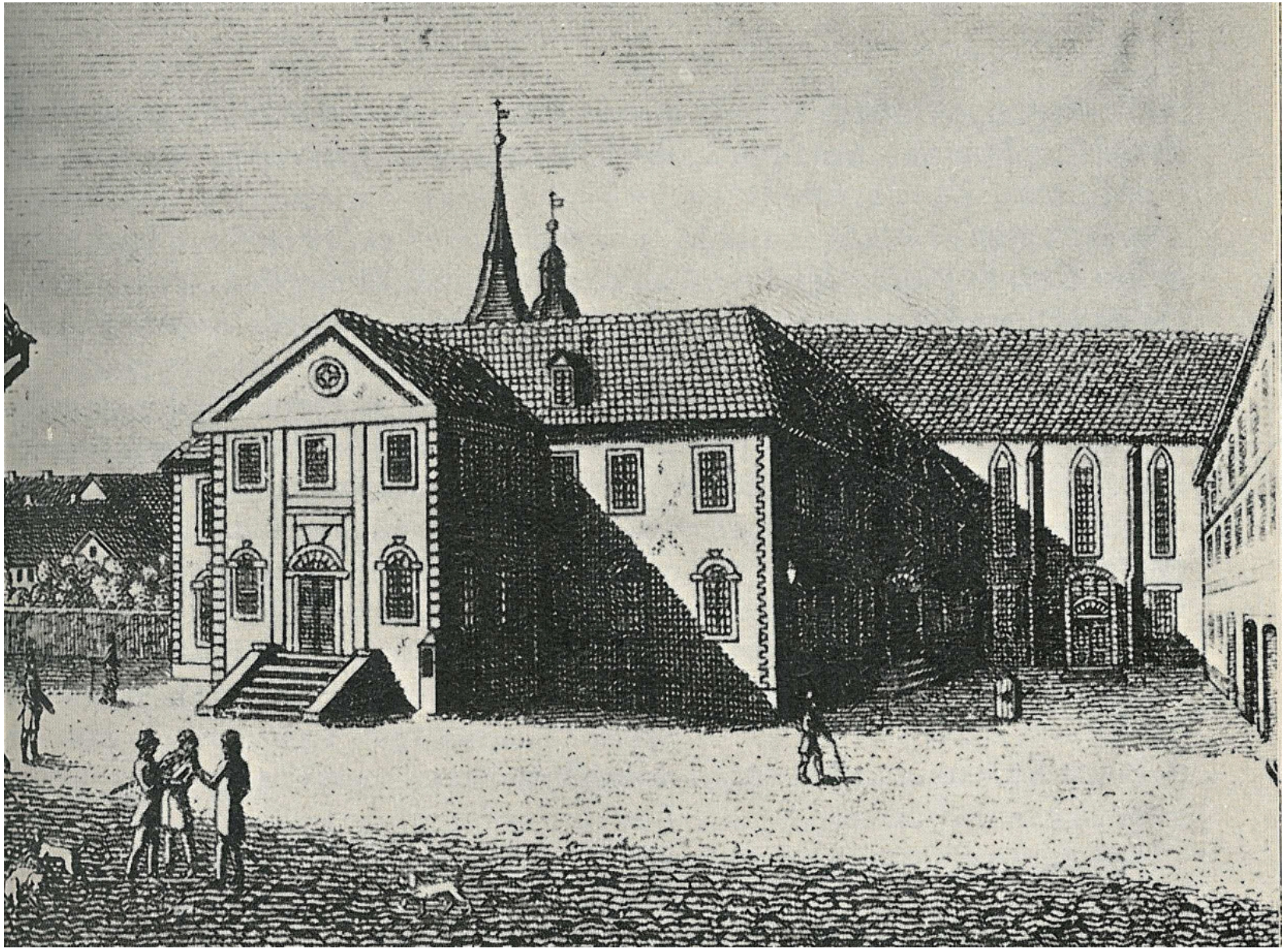
Move to Göttingen

- The Duke of Brunswick, in 1806, once again increased Gauss' stipend (probably influenced by the offer by the Saint Petersburg Academy) [Hal70, p. 69]
- In May of 1806, he went to see the duke to thank him - they were good friends by now.
- This was lastly the last time they met (remember the duke died in battle in October of 1806)
- Göttingen is about 115 kilometres south west of Brunswick in the northern/central part of modern day Germany
- On August 21st, 1806, Gauss' first child, Joseph (named after Piazzi) was born in Brunswick. (He worked in military and as a railroad worker) [Dun55, p. 362].

Other Children

- With Johanna, Gauss had a daughter Minna on February 29th, 1808.
- Gauss jokingly complained that her daughter would only have a birthday once every 4 years [Hal70, p. 70]
- This birth taxed Johanna and by time their third child, Ludwig (Louis) was born on September 10th, 1809, Johanna was taxed and soon died on October 11th, 1809. Louis died soon after on March 1st, 1810.

University of Göttingen (Public Domain)



July 18th...

1768 - Jean Robert Argand born in Geneva, Switzerland. His single original contribution to mathematics was the invention and elaboration of a geometric representation of complex numbers and operations on them. In this he was preceded by Wessel and followed by Gauss.

1872 - Weierstrass lecture. In a lecture to the Berlin Academy, Karl Weierstrass gave the classic example of a continuous nowhere differential function.

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- Final Quiz is on Tuesday July 25th in M3 at 2:30pm.
- Quiz will cover from post Quiz 1 material. Math will cover topics after Alain (history will include Alain's topics).
- If you miss the final quiz, you must document it and will be given an INC grade and at a later date, a make up quiz (Hint: I don't want students to miss the quiz!) You must write the final quiz. Note failure to write this may result in delaying your graduation.
- Failure to document absence will result in a failure (grade of 49% [or maybe 46% depending on policies]).

University of Göttingen

- Gauss began working in Göttingen
- Göttingen was part of the new French-dominated Kingdom of Westphalia as agreed to by the Treaties of Tilsit [B81, p. 54].
- Austerity measures form the Treaties of Tilsit were very taxing on Gauss.

Digression - Napoleonic Rise

- What follows is a far too brief interlude on the rise of Napoleon in France.
- Napoleon was born in Corsica off the coast of Italy (invaded by France when Napoleon was young).
- He grew up resenting France.
- Rose into power based on his merits.

Napoleon Bonaparte

- Father secured boy a scholarship in France.
- Seeing himself as a foreigner, he finds nothing in common with the French aristocrats and spends most of his time alone.
- Enrolled in the military academy at age 15.
- Siege of Toulon in August 29th - December 19th 1793; Napoleon impresses the generals with his understanding of war.

French Rebellion Part 2

- In 1795, some Frenchmen wanted to revive the Monarchy.
- Napoleon Bonaparte seized control; using cannons on angry mobs - focusing on one part quickly then moving onto another. (See Napoleonic Warfare)
- He becomes a general after this.
- He rose to power winning battle after battle with a ruthless attacking regime.
- Napoleon seeks to invade Africa which he does but Britain finds their fleets and destroys them.

Napoleon the Saviour

- Napoleon leaves his soldiers in France and sneaks his way back to France.
- He is seen as a hero for conquering the pyramids.
- In 1802, the Treaty of Amiens ends the Revolutionary war
- In 1804 he is named Emperor of France and proceeds to invade Europe.
- In 1806, he proceeds to invade Prussia.

Sophie Germain and Gauss [Dun55, p. 67]

- On November 27th, 1806, a French officer Chantel entered a room where Johanna and Gauss were
- Chantel claims that Demoiselle Sophie Germain in Paris send him to see how Gauss' health was and to offer his protection if needed.
- Gauss was confused - after all - he only knows Sophie Germain by M. Le Blanc and the only lady Gauss knows who is French is "Madame Lalande".
- The officer asks if Gauss wants to send a letter back but at this point Gauss is extremely confused and simply thanks the officer and his general for their kindness.

Treaties of Tilsit

- Two agreements between France and Prussia ending the Napoleonic Invasion (for the time being)
- The treaties forced Prussia to cede a vast amount of land to France.
- Professors at the University of Göttingen were also forced to pay tariffs amounting to 2000 francs, then a substantial sum of money that Gauss had no means of producing [Hal70, p. 71].
- Many people helped Gauss
  - Olbers, his good friend paid for one such instance (but Gauss repaid him)
  - Laplace also paid Gauss sum once claiming in a letter that is was an honour to help (Gauss repaid him as well with interest)
  - He once received an anonymous gift to help
- Gauss resented Napoleon and France who were ruining his country but held Laplace in high esteem [Hal70, p. 72]

Laplace on Gauss [Hal70, p. 72]

- Alexander von Humboldt, a key player in Gauss being nominated as the head of the observatory at Göttingen, knew of Gauss' reputation but wanted to hear what other mathematicians felt.
- When Laplace was asked who Germany's greatest mathematician was, he replied with "Pfaff" (Gauss' 'supervisor')
- von Humboldt was taken aback - he replied "But what do you think of Gauss?"
- Laplace replied "Gauss is the world's greatest mathematician".

Death of Johanna

- The death of Johanna hit Gauss hard. In a letter to Olbers shortly after Johanna's death [B$\ddot{8}$1, p. 66]:

  > *Yesterday evening at 8 o'clock, I closed her angelic eyes in which I have found heaven for the last five years. Heaven give me the strength to bear this blow. Grant me a few weeks dear Olbers to gather new strengths in the arms of your friendship.*

- He stayed with his wealthy friend Olbers until the end of Octobet (Johanna died October 11th, 1809)

New Marriage

- Gauss then proceeded to marry Minna Waldeck, the daughter of a professor at Göttingen. It is unclear how close as friends they were [B81, p. 67].
- They got engaged on March 27th 1810 and were married on August 4th, 1810. They shared two new sons and one new daughter together [Hal70, p. 71].

MINNA WALDECK GAUSS
SECOND WIFE OF CARL FRIEDRICH GAUSS
(FROM A PASTEL PORTRAIT).

http://gausschildren.org/genwiki/images/2/29/Fig4.JPG

New Marriage

- It is clear from letters that Gauss wrote that the marriage was more of "a happy solution to Gauss's nonscientific problems" than that Gauss had the same infatuation as for Johanna [Hal70, p. 71]

Gauss in the Observatory

- Gauss stayed in the old observatory until the new one (depicted below - Wikimedia Commons) was built in 1816.
- The death of Johanna seemed to make Gauss more melancholy but he still managed to produce work in many different fields, including astronomy, differential geometry and magnetism with Weber (More later)

Gauss on Teaching [Hal70, p. 165]

Gauss was a decent lecture however he believed that truly gifted students could take care of themselves and needed little more than a few suggestions now and then.

> *I have a real aversion to teaching. For a professor of mathematics, it consists of eternal work to just teach the ABC's of his science; most of the few students who go on continue to gather a pile of information and become only half-educated, for the rare gifted students will not allow themselves to be educated through lectures but instead learn by themselves. And through this thankless work the professor loses his precious time.*

Gauss' Lectures

- Students sat at a table covered with books - Gauss himself in an armchair on one side along with a small blackboard easel.
- Gauss would instruct students not to write rather to listen carefully instead.

Maybe Gauss Was Onto Something...

- Gauss claimed that students were uninterested and ill-prepared [Kat93, p. 654]
- This being said, Gauss was willing to work privately with any interested student who approached him [B̈81, p. 71].
- Some of Gauss' students included Riemann, Dedekind, Eisenstein, Möbius [Hal70, p. 166].

Gauss in His Later Years [OR]

- Gauss invited Wilhelm Weber (1804-1891) to become the chair in physics at Göttingen in 1831.
- Made advanced in electricity magnetism.
- Work done on terresterial magenetism discovering Kirchoff's laws.
- Also constructed a telegraph system that linked Weber's laboratory with Gauss' Observatory to help communicate faster.
- Made advanced in developing sensitive instruments to more accurately measure effects of electricity and magnetism.

Gauss in his Later Days

- Gauss' Collected Works number about 150 total papers from 1799 until his death.
- This only spans 12 volumes, less than other mathematicians at the time like Euler and Cauchy [Kat93, p. 654]
- After his death, a journal of his thoughts was found in 1898 dating back from March 30th, 1796 until July 9th, 1814.

Notizenjournal

- The journal is a small book of nineteen octavo pages consisting of a total of 146 discoveries [Hal70, p. 37].
- The results here are mentioned very very briefly (Gauss was concerned about saving paper).
- One such result, July 10th, 1796 [B$\ddot{\text{o}}$1, p. 33]

$$\textit{EYPHKA!} \quad \textit{num} = \Delta + \Delta + \Delta$$

- Not all is known about the book, for example on October 21, 1796, he wrote "Vicimus Gegan" (we have defeated Gegan) which no one knows what Gauss meant [Hal70, p. 38]
- Discoveries of non-Euclidean geometries are also in here!

Gauss' Final Years

- Not much to report in Gauss' final years. He became slightly deaf in 1838 but otherwise had good health.
- Gauss died peacefully in his sleep at the age of 77 on February 23rd, 1855 in Göttingen, Germany.
- His brain, deep convolutions and all, if part of the anatomical collections at the University of Göttingen. [B̈81, p. 155]
- Gauss on Fermat's last theorem in a letter to Olbers dated March 1816 [Hal70, p. 151]:

> *... Fermat's theorem, considered as an isolated proposition interests me very little; I could very easily propose a whole string of such propositions which no one should be able to prove or use.*

The Gem of Arithmetic

Finally, we finish with a proof due to Eisenstein of quadratic reciprocity.

July 20th, 1886

Bernhard Riemann died in Selasca, Italy. Most famous for Riemannian space, the Riemann Hypothesis and the Riemann integral, he was a student of Eisenstein, Dirichlet, and Gauss.

References I

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