CO 480 Lecture 15

Leonhard Euler and Theorems Named After Euler

June 27th, 2017

This Week... Leonhard Euler

- "Read Euler, read Euler, he is the master of us all." (Laplace)
- "There is a famous formula - perhaps the most compact and famous of all formulas - developed by Euler from a discovery of De Moivre: $e^{i\pi} + 1 = 0$. Elegant, concise, and full of meaning ... It appeals equally to the mystic, the scientist, the philosopher, the mathematician." (Kasner - Mathematics and the Imagination, p. 103, Dover, 2001, [Originally published 1940])
- See `http://www.goodreads.com/quotes/tag/euler` for more.

(Wikimedia Commons)

Chronology

This week, I'm going to try to go through the life of Euler in chronological order for a change of pace.

Preliminaries

- Born April 15, 1707 in Basel Switzerland, raised in Riehen (close to Basel)
- Named after a godparent (at the time, city privy councilors), Leonhard Respinger. [Cal16, p. 4]
- Father Paul Euler studied theology at University of Basel. Became a Protestant minister and married Margaret Brucker.
- Paul attended lectures of Jacob Bernoulli there and both he and Johann Bernoulli lived in Jacob's house while they were undergrads.

Jacob Bernoulli
https://upload.wikimedia.org/wikipedia/commons/1/19/Jakob_Bernoulli.jpg

Incredible Work

http://eulerarchive.maa.org/

Russia in the 1720s [Rit79, p. 101-111]

- Age of European Enlightenment (expansion in all areas)
- Russia was being ruled by czar/emperor Peter I (Peter the Great, self-given title) from 1682 until 1725 (died form intestinal illness).
- He was 10 when he came into power (mother was regent until he became of age)
- Had power struggles with his brother Ivan and stepsister Sophia.
- Reformed military powers in Russia by allowing for not just nobles but for serfs and peasants to participate as well.
- This required Peter to recruit more senior officers from Europe to help with training of these new recruits.
- The military reform paid dividends down the road.

Peter the Great Continued

- Founded St. Petersburg as the new capital city.
- Russia was poor at first but Peter helped to reform economically.
- Russia needed water access so they expanded with force.
- Very hands on ruler. Would build ships with his people.
- Ordered men to shave beards (if they refused, Peter did it himself). Changed hygiene and clothing.
- See `https://www.youtube.com/watch?v=4M1-wLy4YH4`

Formal Title [Wik]

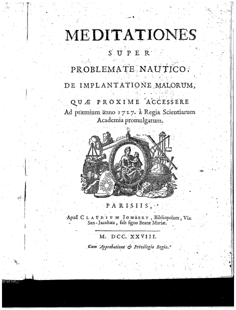
By the grace of God, the most excellent and great sovereign prince Pyotr Alekseevich the ruler all the Russias: of Moscow, of Kiev, of Vladimir, of Novgorod, Tsar of Kazan, Tsar of Astrakhan and Tsar of Siberia, sovereign of Pskov, great prince of Smolensk, Tversk, Yugorsk, Permsky, Vyatsky, Bulgarsky and others, sovereign and great prince of Novgorod Nizovsky lands, Chernigovsky, of Ryazan, of Rostov, Yaroslavl, Belozersky, Udorsky, Kondiisky and the sovereign of all the northern lands, and the sovereign of the Iverian lands, of the Kartlian and Georgian Kings, of the Kabardin lands, of the Circassian and Mountain princes and many other states and lands western and eastern here and there and the successor and sovereign and ruler.

In 1723 [OR]

- Euler graduates with a philosophy degree by comparing and contracting Descartes and Newton's ideas.
- Following his father's wishes, he began to pursue theology however his heart was never in it.
- At age 17 (1724), he completed his university studies but had no degree [Cal16, p. 27]
- Johann Bernoulli convinced Euler's father to let him study mathematics which he completed at the University of Basel in 1726.
- In the same year, Nicolaus II Bernoulli (yes they were everywhere...) died in St. Petersburg in July 1726.
- Euler applied for the position teaching mathematics applications and mechanics to physiology students [switched to physics upon arrival].

Prized Essay

- In 1727 at age 19, Euler submitted an essay on the best arrangements of ship masts to the Paris Academy [OR].
- He finished second place, impressive given that he had never seen a ship at sea in his life [Wei84, p. 163].
- First place went to Pierre Bouguer, a French surveyor and an expert on mathematics regarding ships. He was almost 30 at the time.
- First price was 2000 *livres* - which at the time were roughly an academic's annual salary at the St. Petersburgh Academy [Cal16, p.30]



http://eulerarchive.maa.org/ E004 in the Euler Archive

Euler's Voyage [Cal16, p. 38][Wei84, p. 163]

- Euler didn't arrive until 1727 despite getting the job in November of 1726 (one part to learn about the university and one part because a professor had also passed away at Basel opening up a spot)
- He started from Basel and sailed up the Rhine to Mainz. (His first time on a ship)
- From there he travelled by foot to Lübeck mostly on foot to visit Christian Wolff, philosopher and follower of Leibniz.
- From there he crossed the Baltic sea on ship to St. Petersburg arriving on May 24th, 1727.

Euler's Voyage [Cal16, p. 39]

Figure 2.1. Euler's trip from Basel to Saint Petersburg, in a drawing based on a map of Europe from about 1740.

Russia in 1727

- Euler never returned to basel but always kept his citizenship [Cal16, p. 36]
- At the time, Russia was ruled by Catherine I, the wife of the former leader Peter the Great.
- Russia was embarking on a time period of political unrest.
- Euler began his post at St. Petersburg Academy of Sciences two years after it's founding by Catherine I.

St. Petersburg Academy (Russian Academy of Sciences)

- Creation was from the work on Peter the Great and Gottfried Wilhelm von Leibniz.
- Leibniz believes that there should be more education and science research in Russia.
- They built a school system modelled after the European tiered system: schools, universities and academies.
- The academy, like the one in Paris, would consist of a small group of scientist. (Originally Daniel Bernoulli, Nicolaus II Bernoulli, Christian Goldbach, Johann Duvernoy, Christian Gross, and Gerhard Müller)
- Peter didn't get to see his dream come to fruition but his wife and next ruler Catherine I continued to support the establishment. (December 7th, 1725)

Issues with the Academy [Cal16, p. 64]

- Consisted of the gymnasium (pre-university school) and university.
- Gymnasium enrollment: 112 in 1727; 74 in 1729; 19 in 1737
- Russians wanted faculty who could speak Russian (Latin was required for university entry)
- Disliked discrimination of applicants based on social status
- Disliked emphasis on sciences versus humanities
- In retaliation, Russian aristocrats established "estate schools" exclusively for nobles.

Issues with the Academy [Cal16, p. 64]

- Classes at the university had to be in Latin and no instruction in Latin was given.
- Classes required a minimum number of students. (Academics would attend each other classes to boost enrollment)
- Students given training for careers in strategy and military.
- Students were academically weak.
- Eventually teaching was done away with and academics could focus more on research.

(Russian Academy of Sciences - Wikimedia Commons)

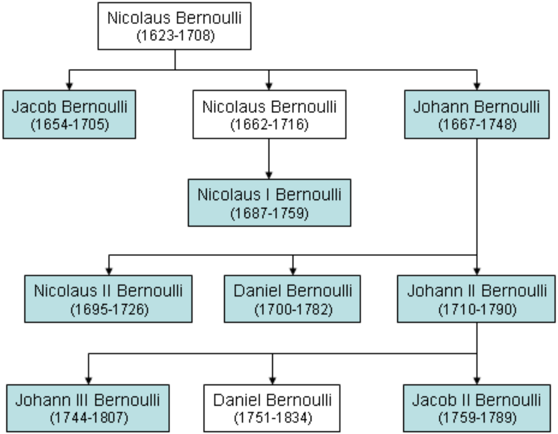
The Early Years

- Euler arrived at the academy in May 1727.
- Initial salary was 300 rubles (lodging, wood and candles included). Increase to 600 in 1733 and 1200 in 1740.
- The first few years, Euler lived with Daniel Bernoulli (Yes another Bernoulli).
- He worked for the navy for three years until he was able to become a full professor in physics, allowing him to leave the navy (this was based on his prized essay on ships!)
- Academic positions at St. Petersburg were under suppression which is why he began in the navy.

December 1728 [Cal16, p. 73]

- Euler (re)visits the topic of logarithms of negative numbers.
- Leibniz argued that $\log(-1)$ doesn't exist.
- Bernoulli argued that $\log(-x) = \log(x)$
- Euler's calculations led him to believe that $\log(-1) = \pi i$ (Euler was the first to use $\sqrt{-1} = i$ though he didn't use it yet).
- Later, d'Alembert also agreed with Bernoulli
- We know today that Euler was closest to being right (there are modulo $2\pi i$ which if you're interested should consider a course in Complex Analysis)

Bernoulli Family Tree (Wikimedia Commons)



More on the Political Climate in Russia

- During this time, from 1727-1730, Peter II was the Tsar of Russia.
- Began his rule at the age of 11. Was easily manipulated throughout his reign.
- Died of small pox after 3 years. He never named a successor and so Alexis Dolgoruky, one of Peter II manipulators, and his allies named Anna Ivanonva, the daughter of Peter I's half brother, as empress (from 1730-1740).
- Anna acceptance of the throne was conditional on her inability ... [Evt04]
  - of the empress to marry
  - to designate a successor
  - to declare war or peace
  - to raise taxes
  - to spend state revenue without the consent of the Council

Of course...

- Anna accepted and then immediately removed from council those supporting these condition.
- Anna supported the Russian Academy of Science
- However, Anna often gave crucial positions to foreigners, largely Baltic Germans [Eula]
- This was a pro for St. Petersburg Academy in the short term however, xenophobic backlash soon ensued after her death in 1740.

https://commons.wikimedia.org/wiki/File:Louis_Caravaque,_Portrait_of_Empress_Anna_Ioannovna_(1730).jpg

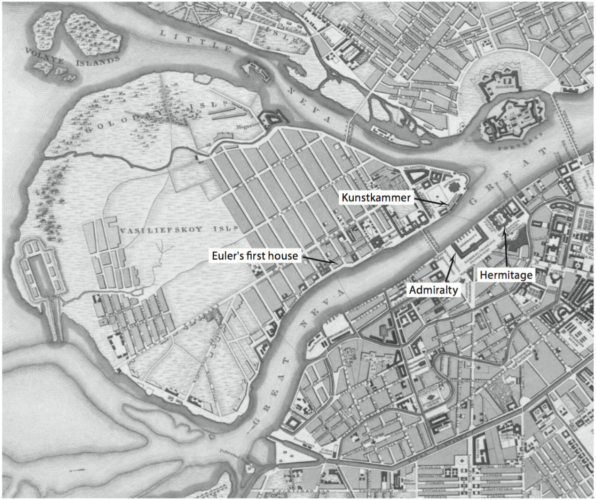
Benefits for Euler

- Euler managed to secure a position in physics and the Academy in 1730 allowing him to be relieved of his duties.
- During this time, others came to the Academy and this upset Euler to see that new faculty salaries were equal to his.
- As his reputation grew, his salary negotiating skills increased leading to the two aforementioned raises in 1735 and 1741.
- Daniel Bernoulli,senior chair of mathematics, left St. Petersburg in 1733 and in his stead, Euler took his place.

Marriage [Cal16, p. 88]

- Once the Bernoulli's left, Euler had more time for dating.
- With the salary increases, Euler also had enough money to get married.
- 1733 began courting Katharina Gsell (probably called her Katya).
- January 7th, 1734, Euler married Katharina.
- The St. Petersburg Academy, to celebrate, had fireworks and illuminations.
- They had 13 children though only 5 survived infancy.

Euler's House[Cal16, p. 91]



Basel Problem (1735)

- First formulated by Pietro Mengoli in 1650 [Wei84, p. 184]
- The question asks to find a closed form expression for $\zeta(2k)$ when $k = 1$ where

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

  or if we substitute $k = 1$, we seek a closed form expression for

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \ldots$$

- Euler 'solved' this in 1735 (and managed to extend at first to other small values for k and eventually for all such k).
- Interesting, the question for k half integers greater than 1 is still an open problem.

How Euler Approached Basel's Problem

(See E41 *De summis serierum reciprocarum* or E63 among others)
Euler began with the Taylor series of $\sin(x)$, namely,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

He then divided by x giving

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

Genius

- For finite polynomials, we can factor the polynomial in terms of its roots to get a factorization.
- Euler reasoned that surely he could do the same for infinite polynomials.
- The factors must be of the form $(1 - x\alpha)$ since the constant terms must align.
- The roots of $\frac{\sin(x)}{x}$ are $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$
- Thus...

A Product

$$\frac{\sin(x)}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\dots$$
$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\dots$$

This isn't formal - it requires the Weierstrass Factorization Theorem in order to rigorously prove (which was derived 100 years later!) We will not do this here.

Compare

Take the Taylor series expansion and the factorization based on the roots and compare coefficients. From the Taylor series, the coefficient of x^2 is $-1/6$. From the infinite product, the coefficient of x^2 is:

$$\frac{-1}{\pi^2} + \frac{-1}{4\pi^2} + \frac{-1}{9\pi^2} + \dots$$

Simplifying gives $\zeta(2) = \frac{\pi^2}{6}$.

Euler Not Done with Basel

In E20, Euler continued his work on the Basel problem by using a functional equation to determine that

$$\zeta(2) = (\log 2)^2 + \sum_{n=1}^{\infty} \frac{1}{n^2 2^{n-1}}$$

and from this he correctly computes $\zeta(2) = 1.644924$, correct to six decimal places (he would need to have summed around the first $30,000$ terms in the usual expansion to get this close of an estimate [Cal16, p. 95]

Also in 1735...

Euler gave in E43, the first known use of the constant

$$\gamma = \lim_{n\to\infty} - \ln(n) + \sum_{k=1}^{n} \frac{1}{k}$$

which is now known as Euler's constant (or the Euler-Mascheroni constant). Neither he nor Mascheroni used γ as the name (it is used now because of its connection with the Gamma function).

It is not even known today if γ is irrational.

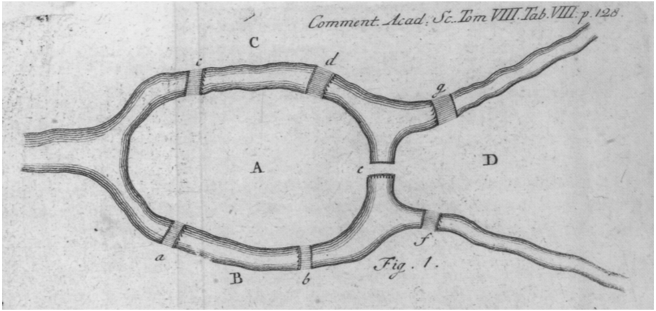
Deteriorating Vision

- In 1735, Euler suffered from a fever and nearly died.
- In 1738, Euler's eyesight deteriorated (Euler claimed from overwork however recent scholars believe that he had a massive infection in his right eye) [Dun99, p. xxiii].
- This didn't even phase Euler as he continued his work without pause.

Other exploits

- 1735, Euler (in three days of work!) managed to give a formula to compute the local time of day given their latitude and longitude. (The people requesting this of Euler thought it would take three months) [Cal16, p.114].
- Euler did a lot of work in cartography as well (he attributed his blindness to this).
- Also in 1735 he proposed his solution to the Bridges of Königsberg problem. (See E53)
- To this day, Eulerian circuits are named after him for his work in Graph Theory (Definition: A circuit that uses every edge of a graph).

Königsberg



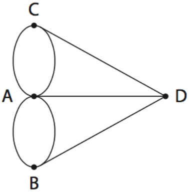


Figure 4.3. A drawing from the *Commentarii*, volume 8 (top), representing the seven bridges of Königsberg on the Pregel River, and a graph of the Königsberg bridges (bottom).

One More Cool Result 1937

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \dots}}}}$$

(Also, Euler's Formula $e^{ix} = \cos(x) + i\sin(x)$ was founded in this year).

The situation in 1740-1741

- In 1736-1738, soldiers were often billeted in or near Euler's house [Cal16, p. 164]
- Euler grew weary of this (having complained frequently to no avail) and wanted change.
- Czarina Anna Ioannovna fell ill in 1740
- Wanting to secure the throne, she named her grandnephew Ivan VI as the next ruler.
- She died on October 17th, 1740 (kidney stone) at age 47 leaving the throne to Ivan VI.

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- She died on October 17th, 1740 (kidney stone) at age 47 leaving the throne to Ivan VI.
- Ivan, being *two months old* at the time, could not rule and so his mother, Anna Leopoldovna was named regent.
- She was hated for her German (Prussian) counsellors and relations with Germany.

This Day in History - June 29th, 1904

Topologist Witold Hurewicz was born in Lodz, Poland (Cech was born on the same day!). He died in 1956 when he fell off a pyramid while attending a conference in Mexico. He and Cech invented the higher homotopy groups.

Announcements

- Typo in Assignment 3 Question 6b (See online). The base case is not correct in the induction problem. The definition of S_n we want to be using in the assignment is

$$S_n = \begin{cases} 4 & \text{if } n = 0 \\ S_{n-1}^2 - 2 & \text{if } n \geq 0 \end{cases}$$

  Further, we should have S_{n-1} in the question.
- Assignment 3 due tomorrow at 2:30pm. Please double check your submission on a computer!
- Editorial Review due in one week at 2:30pm Thursday July 6th - make sure to check your inbox for your paper to review and make sure to submit both to Crowdmark and to LEARN.

1740

- Prussia's Frederick the Great was seeking to ignite the Berlin Academy (Royal Prussian Society of Sciences).
- Offered an invitation to Euler to work at Berlin Academy.
- Euler insisted that a lack of billeting was to be part of the terms [Cal16, p. 164].
- Euler was originally offered $1,200$ Reichsthaler as an annual salary (Maupertuis was given 1600 later 3000, d'Alembert was given 12,000 and Voltaire was given 20,000!) [Cal16, p. 170] which he declined.
- Euler was making 1600 Reichsthaler at the time in Russia.

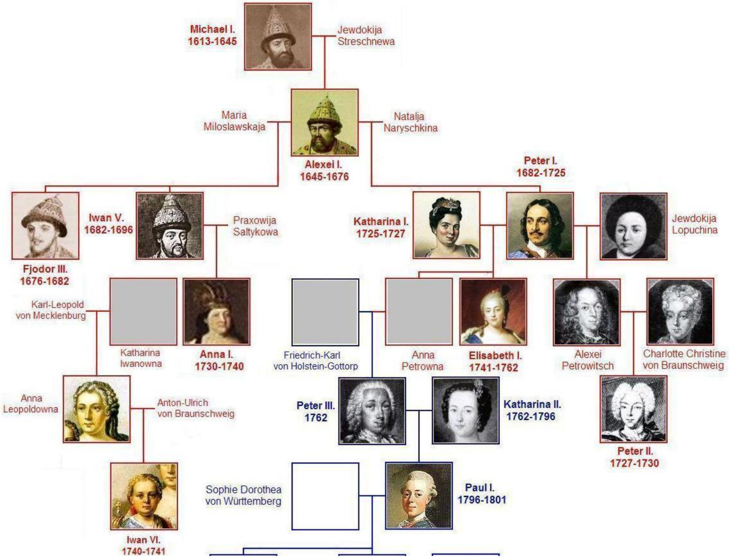
Unfortunate Circumstances

- Euler send a counter offer to Prussia (1600 Reichsthaler and 500 Reichsthaler for moving expenses) but the messenger (Suhm) died delivering it [Cal16, p. 170]
- The death of Anna was causing political unrest which persuaded Euler to leave Russia.
- In 1741, Euler reached out to a Prussian ambassador with the offer which Prussia accepted.

November 25th, 1741 [Vol11, Chp. 2]

- A squad of 300 imperial guards led by Peter's daughter Elizabeth (age 32) began a coup d'état.
- Was successful largely because of Anna's pro-German regimes and Elizabeth's pro-Russia narrative.
- The establishment surrendered to her and she imprisoned Ivan VI and his mother in various prisons.
- Ivan VI remained in solitary confinement for the next 20 years. Vasily Yakovlevich Mirovich (a lieutenant), tried in 1764 to free Ivan seeking a coup of Catherine II the Great (ruler from 1762-1796). In the course of Mirovich's mutiny, however, Ivan was assassinated by his jailers. [oEB]

Peter the Great Family Tree



https://www.pinterest.com/pin/185140234651014924/

Thus in 1741...

- Euler seeing a vastly deteriorating Russia decided to move with Katharina and family to Berlin [Dun99, p. xxiv].
- Despite this, he was able to keep his involvement with the Petersburg Academy and his pension with the Academy was continued.

Berlin 1741

- Euler makes his way with his family to Berlin after a 3 week voyage on the Baltic sea in July 1741 [Wei84, p. 165].
- In Berlin, to the Queen Mother of Prussia, on his lack of conversation in his meeting with her, on his arrival from Russia:

Berlin 1741

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- In Berlin, to the Queen Mother of Prussia, on his lack of conversation in his meeting with her, on his arrival from Russia:

  > *Madam, I have come from a country where every person who speaks is hanged.*

  See [Dun99, p. xxiv] or from
  `https://en.wikiquote.org/wiki/Leonhard_Euler#Quotes`

Silesian Wars

- A series of three wars for control of Silesia with Prussia versus Austria.
- Silesia was a country rich of resources and a booming economy.
- All three wars (1740-1742, 1744-1745, 1756-1763) were won by Prussia.
- Euler worried that the Academy in Berlin would not come to fruition because Frederick the Great was preoccupied with the wars (he was right but only in the sense that the Academy was delayed).

Silesia

North Sea
Baltic Sea
GREAT BRITAIN
DUTCH REPUBLIC
PRUSSIA
PRUSSIA
N
S
E
W
Elbe R.
POLAND-LITHUANIA
Brussels
Aix-la-Chapelle
Silesia
Frankfurt
Bohemia
AUSTRIAN NETHERLANDS
Moravia
Strasbourg
Rhine R.
Loire R.
Bavaria
Vienna
FRANCE
SWISS CANTONS
AUSTRIA
HUNGARY
Savoy
Turin
Parma
REPUBLIC OF VENICE
Danube R.
PAPAL STATES

Royal Academy of Sciences

- The Royal Academy of Sciences in Berlin was founded in 1746 and its first president was Pierre Louis Maupertuis with Euler as the director of mathematics
- However, Euler ended up doing most of the deputizing: [OR]

  > *... he supervised the observatory and the botanical gardens; selected the personnel; oversaw various financial matters; and, in particular, managed the publication of various calendars and geographical maps, the sale of which was a source of income for the Academy. The king also charged Euler with practical problems, such as the project in 1749 of correcting the level of the Finow Canal... At that time he also supervised the work on pumps and pipes of the hydraulic system at Sans Souci, the royal summer residence.*

Euler's Criterion

Euler's Criterion

Let a be an integer not divisible by a prime p. If there is a solution to $x^2 \equiv a \mod p$ then $a^{(p-1)/2} \equiv 1 \mod p$. Also, if $x^2 \equiv a \mod p$ does not have a solution then $a^{(p-1)/2} \equiv -1 \mod p$.

Euler's Criterion

In E134,

Theorem 11

If $a = f^2 \pm (2m+1)\alpha$ and $2m+1$ is a prime number, then the expression $a^m - 1$ is divisible by $2m+1$

Euler's Criterion

In E134,

Theorem 11 Discussion

[The number of values for which] $a^m - 1$ are not divisible by $2m + 1$ [is the same for] so many values of $a^m + 1$ that are able to be divisible by $2m + 1$.

Euler's Criterion

In E262,

Theorem 17 Corollary 2

Therefore, if it were $a \equiv c^2 \mod p$, then $a^{(p-1)/2}$ would leave unity when divided by p.

Euler's Criterion

In E262c(1755),

Corollary to Theorem 6

If a^{2n} whose exponent is an even number divided by a prime number p leaves a residue of 1, then the power a^n divided by the same number p will give a residue of either 1 or -1.

Actual quotes

Scholion.

51. Ut usus huius theorematis clarius appareat, atque per exempla numerica illustrari possit, sequentia problemata adiicere visum est, ex quibus non solum veritas theorematis lucelentius perspicietur, sed etiam vicissim patebit, quoties a non habuerit valorem hic assignatum, toties formulam $a^m - 1$ non esse divisibilem per $2m+1$. Cum igitur haec formula $a^{2^m} - 1$ semper sit divisibilis per $2m+1$, quoties $a^m - 1$ divisionem per $2m+1$ non admittit, toties $a^m + 1$ per $2m+1$ divisibile esse oportebit.

Theorema 6.

24. Si poteſtas a^{2n}, cuius exponens eſt numerus par, per numerum primum p diuiſa, reſiduum $= 1$ relinquit, tum poteſtas a^n per eundem numerum p diuiſa, dabit reſiduum $= +1$, vel $= -1$.

Reminder

Euler's Criterion

Let a be an integer not divisible by a prime p. If there is a solution to $x^2 \equiv a \mod p$ then $a^{(p-1)/2} \equiv 1 \mod p$. Also, if $x^2 \equiv a \mod p$ does not have a solution then $a^{(p-1)/2} \equiv -1 \mod p$.

Proof

If $x^2 \equiv a \mod p$, then

$$a^{(p-1)/2} \equiv (x^2)^{(p-1)/2} \equiv x^{p-1} \equiv 1 \mod p$$

For the next part recall Euclid's Lemma that if $ab \equiv 0 \mod p$ either $a \equiv 0 \mod p$ or $b \equiv 0 \mod p$ (it is important here that p is prime!)

Proof Continued

If instead $x^2 \equiv a \mod p$ doesn't have a solution, then since by Fermat's Little Theorem, $a^{p-1} \equiv 1 \mod p$, we see that $(a^{(p-1)/2} - 1)(a^{(p-1)/2} + 1) \equiv 0 \mod p$. The polynomial $x^{(p-1)/2} \equiv 1 \mod p$ as we've seen has all the numbers b^2 as a solution where $1 \leq b \leq p - 1$. This constitutes half the total values. The other half of the values must be the solutions to $x^{(p-1)/2} \equiv -1 \mod p$

Visualizing the Above Argument with an Example

When $p = 7$,

x	1	2	3	4	5	6
x^2 reduced mod 7	1	4	2	2	4	1

As the above shows, there are only half the values which are squares modulo p.

Fermat Primes

- Also in the aforementioned paper E134 was work on Fermat Primes. (First reference is E26)
- In 1729, in a correspondence with Goldbach (probably Euler's biggest supporter of his studies in number theory), they discussed that neither knew of a proof of Fermat's Conjecture on Fermat Primes, that is numbers of the form $2^{2^n} + 1$ that are prime (Fermat claimed all such numbers were prime for $n \geq 0$). [Cal16, p. 80]
- In E34, he manages to show that if $2^{2^5} + 1$ is divisible by a prime, it must be of the form $64n + 1$.
- In general, he shows that if $2^{2^n} + 1$ has a factor, it has one of the form $2^{n+1}k + 1$ for some k.
- After trying some values, he identifies correctly that 641 is a prime factor of this number (see Assignment 4).

Euler's Theorem

Proved in E271 in 1758.

Euler's [Totient] Theorem

Let n be a positive integer and a an integer coprime to n (that is, $\gcd(n, a) = 1$). Then $a^{\phi(n)} \equiv 1 \mod n$

Notation

Euler's Totient Function

Define $\phi(n)$, *Euler's totient function*, for positive n to be the number of integers $1 \leq i \leq n$ such that $\gcd(n, i) = 1$

Examples: $\phi(4) = 2$, $\phi(10) = 4$, $\phi(12) = 4$.

Important Concept in Analytic Number Theory

Multiplicative Function

A function f from the positive integers to the positive integers is *multiplicative* if and only if whenever $\gcd(m, n) = 1$ for positive integers m and n, we have that $f(mn) = f(m)f(n)$.

Claim: ϕ is multiplicative.

Proof is Combinatorial

- Show number line idea.
- Take a look at the numbers from 1 to mn as $a + bm$ where $0 \leq a \leq m-1$ and $0 \leq b \leq n-1$. Now, if $d = \gcd(a, m) \neq 1$, then $d \mid (a + bm)$ and so $\gcd(a + bm, m) \neq 1$. Hence, we look only at the $\phi(m)$ values for a that are coprime to m.
- Fix an a with $\gcd(a, m) = 1$. Now, The values a, $a + m$, $a + 2m$, ..., $a + (n-1)m$ are n numbers and modulo n they are all distinct (for otherwise $a + km \equiv a + k'm \mod n$ and thus $k \equiv k' \mod n$ results since m is invertible modulo n as it is coprime with n).
- Thus, these values must be, modulo n, the numbers $\{0, 1, ..., n-1\}$ and so we know that exactly $\phi(n)$ of these values are coprime to n by definition of $\phi(n)$.
- Hence, there are a total of $\phi(m)\phi(n)$ numbers between 1 and mn coprime to mn. By definition, this number is equal to $\phi(mn)$ and we are done.

Recall Then Prove

Euler's [Totient] Theorem

Let n be a positive integer and a an integer coprime to n (that is, $\gcd(n, a) = 1$). Then $a^{\phi(n)} \equiv 1 \mod n$

Proof

- Works just like the proof in Math 135 of Fermat's Little Theorem.
- Take $R = \{x \in \mathbb{Z}_n : \gcd(x, n) = 1\}$.
- Notice that $aR = \{ax \in \mathbb{Z}_n : \gcd(x, n) = 1\} = R$, valid since $\gcd(a, n) = 1$.
- Note that

$$\prod_{aj \in aR} aj \equiv \prod_{j \in R} j \mod n$$

  and canceling $\prod_{j \in R} j \mod n$ on both sides gives $a^{\phi(n)} \equiv 1 \mod n$.

Fermat's Last Theorem

- The $n = 3$ case was first written down in 1760 (unpublished) and in 1770 completely published in Euler's *Elements of Algebra* (see Theorem 243 in E388 [Eulb, p. 450]).
- The original proof was incomplete but still sufficient to still be cited as the first person to solve this case.
- Now, the solution to this case has been done many times, the most elegant solution I know uses theory about elliptic curves (my thesis has more on this if you're interested).

Changing of the Guard

- In 1759, Maupertuis died and Euler assumed the leadership role of the Berlin Academy.
- However, Euler was not given the title of presidency.
- Frederick the Great was in charge of operations and Euler was now not on good terms with him despite being in favour earlier. [OR]
- Euler was dismayed when Frederick in 1763 offered Jean d'Alembert (French physicist who did work on differential equations) the title of presidency (he refused to move to Berlin however).
- Euler tried to convince Frederick to give him the presidency but alas this was not to be.
- Frederick's continual interference with the operations of the academy however signaled to Euler that it was time to leave.

Catherine the Great

- Euler missed Russia.
- Positive political climate was beginning - Catherine the Great (Empress of Russia from 1762-1796).
- First project: Restore Academy of St. Petersburg.
- Priority - Getting Euler back.
- After three years of negotiations, Catherine the Great instructed the Russian ambassador in Berlin to request that Euler "write his own contract" [Wei84, p. 167]!
- Frederick, tried to put obstacles in the way but in the end could not afford to upset the imperial lady.

https://commons.wikimedia.org/wiki/File:Profile_portrait_of_Catherine_II_by_Fedor_Rokotov_(1763,_Tretyakov_gallery).jpg

After 25 years in Berlin

- During the 25 years in Berlin, he published more than 100 memoirs that were sent to Petersburg and 127 published in Berlin [Wei84, p. 165]. Some bring the total to more than 380 articles during his tenure in Berlin [OR]
- Published in all things pure and applied (number theory, ship building, lunar theory, artillery, etc.)

Euler's Method

- Euler's eyesight continued to deteriorate upon his return to St. Petersburg (1766) [Wei84, p.168]
- in 1768-1770, Euler produced and published a book Institutionum calculi integralis where he outlines a way to approximate a solution to a differential equation
- This is called Euler's Method

Euler's Method

Problem: Given a differential equation

$$\frac{dy}{dx} = f(y, x) = \text{function in variables x and y}$$

with initial condition (say $y_0 = y(x_0)$), find $y(c)$ for some value of c.

Euler's Idea

Write $\frac{dy}{dx} = \frac{y_{k+1} - y_k}{\Delta x} = f(y_k x_k)$, solve for y_k then choose Δx to be some step size that when applied to this iteratively will eventually reach $y(c)$ (say pick $\Delta x = \frac{c - x_0}{\#\text{ of steps}}$).

The more steps you use, the more accurate your approximation will be.

Concrete Example

Use Euler's method with a step size of $\Delta x = 0.5$ to estimate $y(1)$ given the initial value problem

$$y' = xy - x^2 \qquad y(0) = 1$$

Concrete Example

Use Euler's method with a step size of $\Delta x = 0.5$ to estimate $y(1)$ given the initial value problem

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Solution: From $\frac{y_{k+1} - y_k}{\Delta x} = x_k y_k - x_k^2$, we see that

$$y_{k+1} = y_k + (0.5)(x_k y_k - x_k^2)$$

with initial condition $y_0 = 1$ and $x_0 = 0$, we see that

$$y_1 = y_0 + 0.5(x_0 y_0 - x_0^2) = 1 + 0.5(0 - 0) = 1$$

Thus, $x_1 = 0.5$ and $y_1 = 1$. Applying again gives

$$y_2 = y_1 + 0.5(x_1 y_1 - x_1^2) = 1 + 0.5(0.5 - 0.25) = 1.125$$

and so $y(1) \approx 1.125$.

Surgery

- In 1771, Euler underwent surgery which was successful at first.
- However, Euler's eye eventually became infected and by 1771, Euler was completely blind [Wei84, p. 168], [Dun99, p. xxvi]
- While blind, he had assistants to help him produce, including his own son, people from Basel (found by Daniel Bernoulli) - Nicolas Fuss in particular we owe a lot to for preserving his work (he married a granddaughter of Euler)
- Blindness was no match for Euler as he produced a treatise on algebra, a 775 page book on the celestial movements of the moon and a three volume tome on integral calculus.
- In 1775, Euler averaged one paper *per week* (even while blind!) [Dun99, p. xxvi].

Euler's Conjecture (First August 4th, 1753 in letter go Goldbach for $k = 4$))

Euler conjectured the following

Euler's Conjecture

If $a_1^k + a_2^k + ... + a_n^k = b^k$ for $n > 1$ and a_i, b nonzero integers for $1 \leq i \leq n$, then $n \geq k$

When $n = 2$, this is Fermat's Last Theorem, namely that if $a_1^k + a_2^k = b^k$ has a nontrivial solution, then $k \leq 2$.

The claim is true for $k = 3$...

Euler Makes a 'Mistake'

However the claim is false for $k = 4$ and $k = 5$!

In 1966, Lander and Parkin showed that
$27^5 + 84^5 + 110^5 + 133^5 = 144^5$.

More impressively, in 1986, Noam Elkies managed to construct an infinite set of counter examples as parameterized by the identity:

$$(85v^2+484v-313)^4+(68v^2-586v+10)^4+(2u)^4 = (357v^2-204v+363)^4$$

where $u^2 = 22030 + 28849v - 56158v^2 + 36941v^3 - 31790v^4$. His smallest counter example (for some definition of smallest) was

$$(2,682,440)^4 + (15,365,639)^4 + (18,796,760)^4 = (20,615,673)^4$$

The result is open for $k \geq 6$

Final days

- Euler's wife died in 1773 and he proceeded to marry her sister Abigail in 1776 [Dun99, p. xxvii].
- They spent their last days together keeping each other company.
- Save his blindness, Euler enjoyed very good health and his mental acuity remained with him until his final day on September 18th, 1783 (died from a hemorrhage).

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