CO 480 Lecture 11

Girolamo Cardano and Cardano's Formula

Special thanks to Steve Furino for most of these slides!

June 8th, 2017

This Day in History - June 8th, 1724

Euler received a masters degree in philosophy at age 17. Upon receipt of the degree, he gave a lecture comparing the philosophical ideas of Descartes and Newton. His bachelor's speech, in the summer of 1722, was "on temperance."

Announcements

- Assignment 3 has been posted. Due Tuesday June 27th.
- Your project's first edition (ie. a full version of your project) is due on Tuesday June 20th.
- Ensure that your project doesn't have large margins. (The default in LaTeX is too large. I believe at least one of my templates has a fix).
- Your quiz

Quiz 1

Find two [rational] square numbers and a [rational] cube such that when the squares are added together, another square is formed and when one of the squares is added to the cube, another cube is formed. Hint: Let x^3 and $9x^2$ be the cube and the square. Find a square to be added to the given square to give a second square and then consider the cubic condition.

Quiz 1

Divide 317 by 24 using the Egyptian arithmetic method taught in class.

This Week... Girolamo Cardano

- Born 24 September 1501. Died 21 September 1576.
- Illegitimate child of Fazio Cardano (was close friends to Da Vinci)
- Prolific mathematician and scientist
- Early adopter of binomial coefficients, Binomial Theorem.

Reference: School of Mathematics and Statistics, University of St Andrews, WIkimedia Commons

Cardano the Thief

Girolamo Cardano (1501 - 1576) was a turbulent man of genius, very unscrupulous, very indiscreet, but of commanding mathematical ability ... He was interested one day to find that Tartaglia held a solution of the cubic equation. Cardano begged to be told the details, and eventually under a pledge of secrecy obtained what he wanted. Then he calmly proceeded to publish it as his own unaided work in the Ars Magna which appeared in 1545... He seems to have been equally ungenerous in the treatment of his pupil Ferrari, who was the first to solve a quartic equation. Yet Cardano combined piracy with a measure of honest toil, and he had enough mathematical genius in him to profit by these spoils...

H W Turnbull, “The Great Mathematicians”

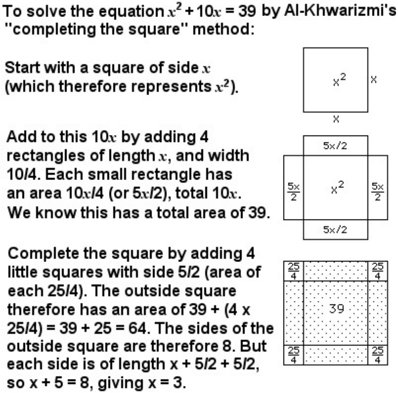
Revisit Muhammad ibn Musa al-Khwarizmi (c. 790 - 850)

- Father of modern algebra
- Wrote The Compendious Book on Calculation by Completion and Balancing, a book to solve quadratic equations.

Khwarizmi statute in Amir Kabir University, Tehran

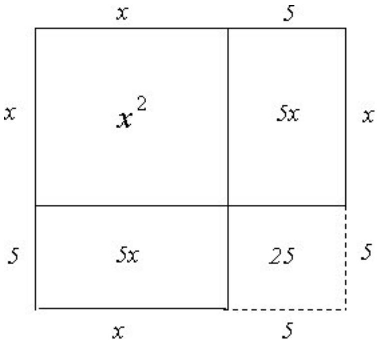
A Sample Problem

For instance, “one square and ten roots of the same amount to thirty-nine dirhems”, that is to say, what must be the square which when increased by ten of its own roots amounts to thirty-nine? [Kat07, p. 543]



http://www.storyofmathematics.com/islamic_alkhwarizmi.html

Alternate Approach

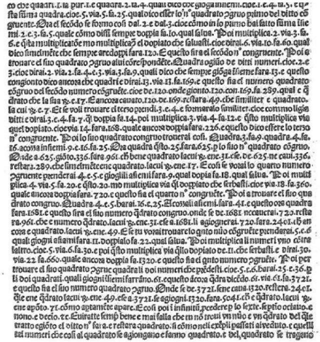


Abraham bar Hiyya Ha-Nasi (1070 - 1136)

- Spanish Jewish mathematician and astronomer
- Wrote "Hibbur ha-Meshihah ve-ha-Tishboret" (Treatise on Measurement and Calculation), which was translated into Latin by Plato of Tivoli as "Liber Embadorum" in 1145
- It contains a complete solution of the general quadratic and is the first European text to give such a solution.
- Strangely enough, al-Khwarizmi's "Algebra" was translated by Robert of Chester also in 1145

Luca Pacioli (1445 - 1517)

- Close friend of Leonardo da Vinci
- In 1494 the first edition of his "Summa de Arithmetica, Geometrica, Proportioni et Proportionalita" appeared.
- Does not discuss cubic equations but does discuss quartics



jeremycripps.com/docs/Summa.pdf
https://archive.org/details/161Pacioli

Scipione dal Ferro (1465 - 1526)

- Chair of Arithmetic and Geometry at the University of Bologna
- Presumably met Pacioli when Pacioli was at Bologna in 1501 -1502
- Was able to solve cubics of the form $x^3 + px = q, \quad p, q > 0$
- Kept this knowledge secret. Why?
- Shortly before his death in 1526, he passed on the secret to one of his students, Antonio Maria Fiore

Antonio Maria del Fiore (dates c. 1500)

- Also known as Antonio Maria Fior
- Couldn't keep a secret
- Rumours began to spread that some form of the cubic equation had been solved in Bologna

Niccolò Fontana (Tartaglia)

- Niccolò (or Nicolo)Fontana (Tartaglia) (1500 - 1557), originally of Brescia, moved to Venice in 1534
- Family name was Fontana.
- From a very poor family.
- Father was a despatch rider (mailman). Killed in 1506 by robbers further impoverishing the Fontana family. [Str13, p. 188-193]
- This wasn't the end of the Fontana misfortune.
- February 18, 1512: The Sack of Brescia.

1512 - War of the League of Cambrai [Str13, p. 188-193]

- Part of War of the League of Cambrai a.k.a. War of the Holy League (1508-1516)
- Main parties: France, Papal States, Venitians
- A mess of alliances (First Papal States allied with King Louis XII which led to friction)
- Then in 1510, formed a Venetial-papal aliance against France
- They managed to defend northern Italy from France but then disagreements with the Venetians about the spoils caused another broken alliance.

Feb. 18th, 1512: The Sack of Brescia [Str13, p. 188-193]

- While the Italians managed to defend northern Italy from France, it wasn't without damage.
- The Sack of Brescia was a slaughtering of Italian citizens in Brescia
- Brescia revolted against French control - reinforcing itself with Venetian troops.
- Brescia was sacked and pillaged - more than 45000 citizens killed. (For comparison, in 1561, Brescia had 41,168).[1]

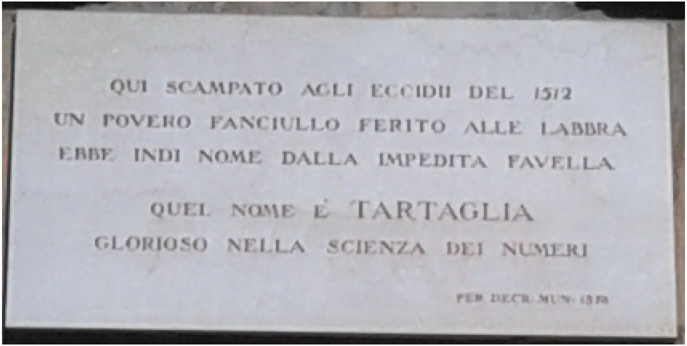
[1] Some estimate killed citizens 17,000-40,000 - see [HHH, p.10, 13]

Poor Fontana [Str13, p. 188-193]

- Niccolò, his mother and brothers took sanctuary in the local cathedral.
- Alas, a French soldier slashed Niccolò across the head leaving him for dead.
- Mother nursed him back to health but his speech was never the same.
- This gave him the nickname "Tartaglia" meaning "the stammerer"

https://upload.wikimedia.org/wikipedia/commons/0/0b/Niccol%C3%B2_Tartaglia.jpg

Tartaglia Plaque Currently at Cathedral in Brescia



Here escaped the massacre of 1512, a poor injured child unable to speak due to an injury to his lips. His name is Tartaglia who became glorious in the science of numbers.

Duomo at Brescia (from Google Maps)

Niccolò Tartaglia

- Tartaglia was very poor. Made his money as a teacher, tutor, and significantly, debater.
- Injury forced him to stay at home; became prolific at mathematics however he became unbearably proud and arrogant offsetting his lowly origins and gruesome figure [Str13]
- In 1534, he moved to Venice and was beating local mathematicians at contests that were becoming in vogue at the time.
- Translated Euclid's Elements into Italian (formerly only Greek and Latin).
- Tartaglia solved equations of the following form but kept the technique secret $x^3 + ax^2 = b, \quad a, b > 0$

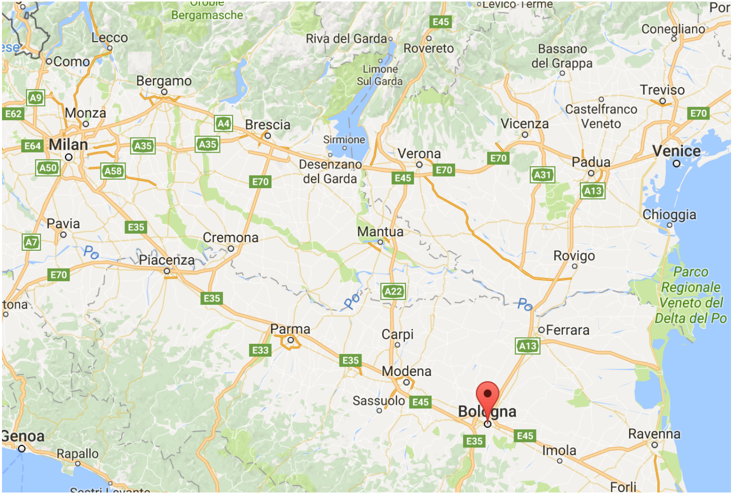
Tartaglia Statue in Brescia at Piazetta Santa Maria Calchera

(Near Universita Cattolica Del Sacro Cuore)

First ever Selfie?

Actual First Ever Selfie (Thanks J.P. Pretti)

Map of Italy


Bergamasche
Lecco
Como
Bergamo
Monza
Milan
Brescia
Riva del Garda
Rovereto
Limone
Sul Garda
Sirmione
Desenzano
del Garda
Verona
Vicenza
Bassano
del Grappa
Conegliano
Treviso
Castelfranco
Veneto
Padua
Venice
Chioggia
Pavia
Po
Cremona
Piacenza
Mantua
Rovigo
Parco
Regionale
Veneto del
Delta del Po
Parma
Carpi
Ferrara
Modena
Sassuolo
Genoa
Rapallo
Imola
Ravenna
Forlì
A9
E62
E64
A50
A35
A58
A4
E70
E35
E45
A7
A22
A31
A13
E33

Fiore vs Tartaglia

- Fiore challenged Tartaglia to a public contest: each gave the other 30 problems with 40 or 50 days in which to solve them, the winner being the one to solve most but a small prize was also offered for each problem.
- In what follows, I have two dates for the contest. On [ORb], it quotes “Glory to God, 1534, and the 22nd day of February, in Venice.” however, most other references claim that the contest took place on February 20th, 1535
- I've learnt that “In the Republic of Venice until its end in 1797 the year did not begin on January 1 as today but officially on March 1. Therefore, February 12 and 13 of our year 1535 were still in the Venetian year 1534.” [Kat]

Contest

The challenge begins as follows:
Glory to God, 1534, and the 22nd day of February, in Venice.
These are the thirty problems proposed by me Antonio Maria Fiore to you Master Niccolò Tartaglia.

1. Find me a number that when its cube root is added to it, the result is six, that is 6.
2. Find me two numbers in double proportion such that when the square of the larger number is multiplied by the smaller, and this product is added to the two original numbers, the result is forty, that is 40.

Oops!

- However, only 8 days before the problems were to be collected, Tartaglia had found the general method for all types of cubics.
- Tartaglia solved all Fiore's problems in the space of 2 hours
- Fiore couldn't solve most of Tartaglia's non-cubic related problems [Kat93, p. 359]
- Fiore was humiliated. Tartaglia becomes a celebrity
- The winners prize: 30 banquets prepared by the loser for the winner and his friends (Tartaglia declined) [Kat93, p. 359]

Contest Simulation

- Divide down the middle two teams (so yes you're going to have to move a bit).
- Nominate two captains
- We will have Team Tartaglia and Team del Fiore
- What follows is 10 polynomial equations. Each side will solve the questions and hand their solution to a team captain who will be responsible for official submissions. The captain can also work on them.
- You must find all real roots to each problem.

Cardano to Tartaglia, February 12, 1539 [ORb]

I wonder much, dear Master Niccolò, at the unhandsome reply you have made to one Zuan Antonio de Bassano, bookseller ... I would pluck you out of this conceit, as I plucked out lately Master Zuanne da Coi, that is to say, the conceit of being the first man in the world, wherefore he left Milan in despair; I would write to you lovingly and drag you out of the conceit of thinking you are so great - would cause you to understand from kindly admonition, out of your own words, that you are nearer to the valley than the mountain-top. In other things you may be more skilled and clever than you have shown yourself to be in your reply; and so I must in the first place state that I have held you in good esteem, and as soon as your book on Artillery appeared, I bought two copies, the only ones Zuan Antonio brought, of which I gave one to Signore the Marquis ...

Cardano to Tartaglia, February 12, 1539 [ORb]

The third point is, that you told the said bookseller that if one of my questions were solved all would be solved, which is most false, and it is a covert insult to say that while thinking to send you six problems, I had sent but one, which would argue in me a great confusion of understanding; and certainly, if I were cunning, I would wager a hundred scudi upon that matter; that is to say, that they could not be reduced either into one, or into two, or into three questions. And, indeed, if you will bet them, I will not refuse you, and will come at an appointed time to Venice, and will give bank security here if you will come here, or will give it to you there in Venice if I go thither. This is not mere profession, for you have to do with people who will keep their word ...

Cardano to Tartaglia, February 12, 1539 [ORb]

I send you two questions with their solutions, but the solutions shall be separate from the questions, and the messenger will take them with him; and if you cannot solve the questions he will place the solutions in your hand. You shall have them each to each, that you may not suppose I have sent rather to get than to give them; but return first your own, that you may not lead me to believe that you have solved the questions, when you have not. In addition to this, be pleased to send me the propositions offered by you to Master Antonio Maria Del Fiore, and if you will not send me the solutions, keep them by you, they are not so very precious. And if it should please you, in receiving the solutions of my said questions - should you yourself be unable to solve them, after you have satisfied yourself that my first six questions are different in kind - to send me the solution of any one of them, rather for friendship's sake, as for a test of your great skill, than for any other purpose, you will do me a very singular pleasure.

Cardano to Tartaglia, February 12, 1539 [ORb]

1. Make me of ten four quantities in continued proportion whose squares added shall make sixty.
2. Two persons were in company, and possessed I know not how many ducats. They gained the cube of the tenth part of their capital, and if they had gained three less than they did gain, they would have gained an amount equal to their capital. How many ducats had they?

Cardano Persuades Tartaglia To Tell His Secret

- Cardano invites Tartaglia to visit him in Milan
- Cardano was just about to publish Practica Arithmeticae (1539)
- The following conversation between the two men on 25 March 1539 in Cardano's home is reported by Tartaglia later

Continuing the Conversation [ORb]

Cardano: And I also wrote to you that if you were not content that I should publish them, I would keep them secret.
Tartaglia: Enough that on that head I was not willing to believe you.
Cardano: I swear to you by the sacred Gospel, and on the faith of a gentleman, not only never to publish your discoveries, if you will tell them to me, but also I promise and pledge my faith as a true Christian to put them down in cipher, so that after my death nobody shall be able to understand them. If you will believe me, do; if not, let us have done.

Continuing the Conversation [ORb]

Tartaglia: If I could not put faith in so many oaths I should certainly deserve to be regarded as a man with no faith in him; but since I have made up my mind now to ride to Vigevano to the lord marquis, because I have been here already three days, and am tired of awaiting him so long, when I am returned I promise to show you the whole.
Cardano: Since you have made up your mind at any rate to ride at once to Vigevano to the lord marquis, I will give you a letter to take to his excellency, in order that he may know who you are; but before you go I should wish you to show me the rule for those cases of your, as you have promised.
Tartaglia: I am willing ...
(Warning: Poem Incoming)

Poem [ORb]

When the cube and things together
Are equal to some discreet number,
Find two other numbers differing in this one.
Then you will keep this as a habit
That their product should always be equal
Exactly to the cube of a third of the things.
The remainder then as a general rule
Of their cube roots subtracted
Will be equal to your principal thing
In the second of these acts,
When the cube remains alone,
You will observe these other agreements:

Poem [ORb]

You will at once divide the number into two parts
So that the one times the other produces clearly
The cube of the third of the things exactly.
Then of these two parts, as a habitual rule,
You will take the cube roots added together,
And this sum will be your thought.
The third of these calculations of ours
Is solved with the second if you take good care,
As in their nature they are almost matched.
These things I found, and not with sluggish steps,
In the year one thousand five hundred, four and thirty.
With foundations strong and sturdy
In the city girdled by the sea.

Finishing the Conversation [ORb]

Tartaglia: This verse speaks so clearly that, without any other example, I believe that your Excellency will understand everything.
Cardano: How well I understand it, and I have almost understood it at the present. Go if you wish, and when you have returned, I will show you then if I have understood it.

June 13th... 1676

Newton corresponded with Leibniz, sending Henry Oldenburg the "Epistola prior" for transmission to Leibniz. It contained the first statement of the binomial theorem for negative and fractional exponents, among other things.

Announcements

Rubric for your project:

- Name the file

  Group###.pdf

  where the number signs are numbers (eg. Group001.pdf or Group032.pdf)
- Aim for the quality of a history textbook
- Make connections between your person, place and problem (like was done for Tutte)
- Try to make this interesting.
- You won't receive feedback from your TAs (this is what the editorial review will be for) It simply will take too much time to get everything back to you before you need to submit your final project.

Rubric

1. Needs LOTS of work.
2. Needs some work.
3. At the level of Wikipedia.
4. Well written but lacking in content beyond that of Wikipedia.
5. Content is good but the style does not do justice to the content.
6. Competent.
7. More than competent but less than comparable to the text.
8. Comparable to the text.
9. Insightful commentary supported by strong presentation.
10. This is an undergraduate thesis, not a course project. This category is used for extensive projects, innovative presentation (animation, project specific software, ...) or deeply insightful work.
11. Contains information or analysis new to the scholarly world. Worthy of publication.

Cardano Asks For Help [ORb]

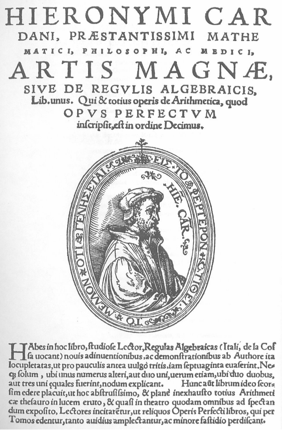
Tartaglia to Cardano (August 1539): Master Girolamo, I have received a letter of yours, in which you write that you understand the rule; but that when the cube of one-third of the coefficient of the unknown is greater in value than the square of one-half of the number you cannot resolve the equation by following the rule, and therefore you request me to give you the solution of this equation 'One cube equal to nine unknowns plus ten'. To which I reply, and say, that you have not used the good method for resolving such a case; also I say that such your proceeding is entirely false. And as to resolving you the equation you have sent, I must say that I am very sorry that I have given you already so much as I have done, for I have been informed, by person worthy of faith, that you are about to publish another algebraic work, and that you have gone boasting through Milan of having discovered some new rules in Algebra. But take notice, that if you break your faith with me, I certainly shall not break promise with you (for it is my custom); nay, even undertake to visit you with more than I have promised. ...

Note to Self [ORb]

Tartaglia (note to himself): I propose to see whether I can perhaps alter the data he possesses, that is, turn him away from the right track and make him take some other ...

Ars Magna

- The Great Art or Rules of Algebra by Girolamo Cardano
- Outstanding Mathematician, Philosopher and Physician In One Book, Being the Tenth in the Order of the Whole, Work on Arithmetic Which is called the Perfect Work.



HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,

ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inſcripſit, eſt in ordine Decimus.

HAbes in hoc libro, ſtudioſe Lector, Regulas Algebraicas (Itali, de la Coſſa uocant) nouis adinuentionibus, ac demonſtrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam ſeptuaginta euaſerint. Neq; ſolum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni ęquales fuerint, nodum explicant. Hunc aūt librum ideo ſeorſim edere placuit, ut hoc abſtruſiſſimo, & planè inexhauſto totius Arithmeticæ theſauro in lucem eruto, & quaſi in theatro quodam omnibus ad ſpectandum expoſito, Lectores incitarētur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore faſtidio perdiſcant.

Inside Cover

In this book, learned reader, you have the rules of algebra. It is so replete with new discoveries and demonstrations by the author - more than seventy of them - that its forerunners [are] of little account, or, in the vernacular, are washed out. It unties the knot not only where one term is equal to another or two to one but also where two are equal to two or three to one. It is a pleasure, therefore, to publish this book separately so that, this most abstruse and unsurpassed treasury of the entire [subject of] arithmetic, being brought to light and, as in a theatre, exposed to the sight of all, its readers may be encouraged and will all the more readily embrace and with the less aversion study thoroughly the remaining books of the Perfect Work which will be published volume by volume.

Chapter 1

This art originated with Mahomet the son of Moses the Arab. Leonardo of Pisa is a trustworthy source for this statement. There remain, moreover, four propositions of his with their demonstrations, which we will ascribe to him in their proper places. After a long time, three derivative propositions were added to these. They are of uncertain authorship, though they were placed with the principal ones by Luca Paccioli.

Trouble

In our own days Scipione del Ferro of Bologna has solved the case of the cube and first power equal to a constant, a very elegant and admirable accomplishment. Since this art surpasses all human subtlety and the perspicuity of moral talent and is a celestial gift and a very clear test of the capacity of men's minds, whoever applies himself to it will believe that there is nothing he cannot understand. In emulation of him, my friend Niccolò Tartaglia of Brescia, wanting not to be outdone, solved the same case when he got into a contest with [Scipione's] pupil, Antonio Maria Fior, and, moved by my many entreaties, gave it to me. Note: Cardano and Ferrari found Scipione's own notes about the discovery of solutions to one form of the cubic equation and hence thought they need not be bound by Cardano's vow of silence.

Cardano Continues

For I had been deceived by the words of Luca Paccioli, who denied that any more general rule could be discovered than his own. Notwithstanding the many things which I had already discovered, as is well known, I had despaired and had not attempted to look any further. Then, however, having received Tartaglia's solution and seeking the proof for it, I came to understand that there were a great many other things that could also be had. Pursuing this thought and with increased confidence, I discovered these others, partly by myself and partly through Lodovici Ferrari, formerly my pupil. Hereinafter those things that have been discovered by others have their names attached to them; those to which no name is attached are mine. The demonstrations, except for the three by Mahomet and the two by Lodovico, are all mine.

Tartaglia Retaliates

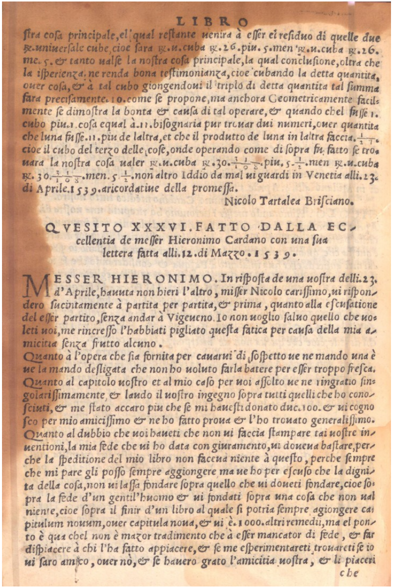
Tartaglia publishes a book to defend his honour

QVESITI, ET INVENTIONI DIVERSE DE NICOLO TARTALEA BRISCIANO.

Con gratia, & priuilegio dal Illuſtriſſimo Senato Veneto, che niun[illegible]rdiſca ne preſuma, di ſtampare la preſente opera, ne ſtampate altroue uendere ne far uendere in Venetia, ne in alcuno altro luoco, o terra del Dominio Veneto, per anni diece ſotto pena de ducati trecento, & perdere le opere, el terzo della qual pena immediate che ſia denontiata, ſi applica al Arſenale, & un terzo ſia del magiſtrato, ouer rettore del luoco doue ſe ſara la aſſecutione, & laltro terzo ſara del denuntiante, ouer accuſato-

Tartaglia Retaliates

Records the villainy of Cardano



ſtra coſa principale, el qual reſtante uenira à eſſer el reſiduo di quelle due ℞. uniuerſale cube, cioe ſara ℞. u. cuba ℞. 26. piu. 5. men ℞. u. cuba ℞. 26. me. 5. & tanto ualſe la noſtra coſa principale, la qual concluſione, oltra che la iſperienza, ne renda bona teſtimonianza, cioe cubando la detta quantita, ouer coſa, & à tal cubo giongendoui il triplo di detta quantita tal ſumma ſara preciſamente. 10. come ſe propone, ma anchora Geometricamente facilmente ſe dimoſtra la bonta & cauſa di tal operare, & quando chel fuſſe 1. cubo piu. 1. coſa equal à. 11. biſognaria pur trouar dui numeri, ouer quantita che luna fuſſe. 11. piu de laltra, et che il produtto de luna in laltra faccia. $\frac{1}{27}$. cioe il cubo del terzo delle coſe, onde operando come di ſopra fu fatto ſe trouara la noſtra coſa ualer ℞. u. cuba ℞. $30.\frac{31}{108}$. piu. $5.\frac{1}{2}$. men ℞. u. cuba ℞. $30.\frac{31}{108}$. men. $5\frac{1}{2}$. non altro Iddio da mal ui guardi in Venetia alli. 23. di Aprile. 1539. aricordatiue della promeſſa.

Nicolo Tartalea Briſciano.

QVESITO XXXVI. FATTO DALLA ECcellentia de meſſer Hieronimo Cardano con una ſua lettera fatta alli. 12. di Mazzo. 1539.

MESSER HIERONIMO. In riſpoſta de una uoſtra delli. 23. d'Aprile, hauuta non hieri l'altro, miſſer Nicolo cariſſimo, ui riſpondero ſucintamente à partita per partita, & prima, quanto alla eſcuſatione del eſſer partito, ſenza andar à Vigeueno. Io non uoglio ſaluo quello che uoleti uoi, me rincreſſo l'habbiati pigliato queſta fatica per cauſa della mia amicitia ſenza frutto alcuno.

Quanto à l'opera che ſia fornita per cauarui di ſoſpetto ue ne mando una è ue la mando deſligata che non ho uoluto farla batere per eſſer troppo freſca.

Quanto al capitolo uoſtro et al mio caſo per uoi aſſolto ue ne ringratio ſingolariſſimamente, & laudo il uoſtro ingegno ſopra tutti quelli che ho conoſciuti, & me ſtato accaro piu che ſe mi haueſti donato duc. 100. & ui cognoſco per mio amiciſſimo & ne ho fatto proua & l'ho trouato generaliſſimo.

Quanto al dubbio che uoi haueti che non ui faccia ſtampare tai uoſtre inuentioni, la mia fede che ui ho data con giuramento, ui doueua baſtare, perche la ſpeditione del mio libro non faceua niente à queſto, perche ſempre che mi pare gli poſſo ſempre aggiongere ma ue ho per eſcuſo che la dignita della coſa, non ui laſſa fondare ſopra quello che ui doueti fondare, cioe ſopra la fede d'un gentil'huomo & ui fondati ſopra una coſa che non ual niente, cioe ſopra il finir d'un libro al quale ſi potria ſempre agiongere capitulum nouum, ouer capitula noua, & ui è. 1000. altri remedii, ma el ponto è qua chel non è mazor tradimento che à eſſer mancator di fede, & far diſpiacere à chi l'ha fatto appiacere, & ſe me eſperimentareti trouareti ſe io ui ſaro amico, ouer nò, & ſe hauero grato l'amicitia uoſtra, & li piaceri

che

Ferrarri Enters The Dispute [ORb]

Ferrari to Tartaglia: You have the infamy to say that Cardano is ignorant in mathematics, and you call him uncultured and simple-minded, a man of low standing and coarse talk and other similar offending words too tedious to repeat. Since his excellency is prevented by the rank he holds, and because this matter concerns me personally since I am his creature, I have taken it upon myself to make known publicly your deceit and malice.

Ferrari to Tartaglia (again): As for the twenty-second problem in the disputation. You at first say that it is not a question for a mathematician. To which I reply, that, if by a mathematician you mean someone like you, that is, someone who spends the whole time on roots, fifth powers, cubes and other trifles, then you are quite right. I promise you that if it were up to me to reward you, taking example from the custom of Alexander, I would load you up so much with roots and radishes that you would never eat anything else in your life.

Back to Cardano [Car02][Bur91, p.319-323]

- English: Jerome Cardan (Girolamo Cardano)
- 1501, born illegitimately in Pavia
- Father was a lawyer in Milan but mathematically skilled. Consulted by Leonardo da Vinci on questions of geometry.
- His mother had left her residence in Milan because of the plague, and gave birth to Cardano in Pavia
- Worked as his father's assistant but was so sickly that his father hired others
- 1520, entered the University of Pavia to study medicine. His father wanted him to study law. Schooling interrupted in 1523 because of war.
- 1524, became rector of the University of Padua. His term is not recorded in the official records but he repeatedly attests to it.
- 1525, graduates.

Early Life As a Doctor [Car02][Bur91, p.319-323]

- 1525, applied (repeatedly) to join the College of Physicians in Milan. "The College did not wish to admit him for, despite the respect he had gained as an exceptional student, he had a reputation as a difficult man, whose unconventional, uncompromising opinions were aggressively put forward with little tact or thought for the consequences. The discovery of Cardan's illegitimate birth gave the College a reason to reject his application."[ORb]
- 1525, practiced in Sacco, a village near Padua. Plague, government instability, crop failures, war and high taxes made it very difficult to survive.
- 1529, returns to Milan. Again rejected by the College.
- 1536, published a book attacking the College.
- 1537, again rejected by the College.

Success As a Doctor

- Practiced medicine very successfully without a license
- 1539, accepted to the College because of the influence of friends
- 1543, begins an extended period of lecturing in medicine at universities. Frequently interrupted because of the universities' failure or inability to pay.
- Tremendous reputation as a doctor, perhaps the best known doctor in Europe
- Rejects offers from the Danish and French kings.
- 1553, successfully treats the Archbishop of St. Andrews. Paid 1400 gold crowns. [Fie83, p. 18] [Ber96]

Mathematical Work

- 1530 (?), takes up a position in mathematics that his father held
- 1537, first two mathematical books published
- 1539, first contact with Tartaglia
- 1545, publishes Ars Magna
- 1663, Liber de Ludo Aleae (On Casting the Die) published posthumously

Personal Disasters

- 1531, married to Lucia
- 1533, gambled away all of his family's assets. Entered the poorhouse.
- 1560, the execution of his son Giambatista who poisoned his wife
- 1569, second son Aldo gambled and lost a large fraction of Cardano's wealth
- 1570, arrested and jailed for heresy for publishing a horoscope of Jesus. Tartaglia may have bribed Cardano's son to contribute to the prosecution.

Cardano In His Own Words

- In 1575 wrote an autobiography, De Vita Propria Liber or The Book of My Life
- Translated from the Latin by Jean Stoner
- Arguably one of the archetypes of the autobiographical form (see Anna Robeson Burr)[2]
  - Caesar: The Gallic Wars and The Civil War, historical
  - St. Augustine: The Confessions, religious - emotional
  - Cardano: The Book of My Life, psychological or scientific self-study
- Remarkably candid look at himself, but omits details about the events of his life, particularly his accomplishments and the fame that attended them
- What follows is from [Car02].

[2] https://archive.org/details/autobiographycri00burr

Prologue

This Book of My Life I am undertaking to write after the example of Antoninus the Philosopher, acclaimed the wisest and best of men, knowing well that no accomplishment of mortal man is perfect, much less safe from calumnny; yet aware that none, of all ends which man may attain, seems more pleasing, none more worthy than recognition of the truth.

My Nativity

Although various abortive medicines - as I have heard - were tried in vain, I was normally born on the 24th day of September in the year 1500 ...
The most significant positions of the horoscope were as I have indicated in the eighth nativity of the supplement to the four sections of my Commentaries on Ptolemy
...I am a man bereft of bodily strength, with few friends, small means, many enemies - a very large part of whom I recognize neither by name or face - a man without ordinary human wisdom, inclined to be faulty of memory, though rather better in the matter of foreseeing events

A Brief Narrative of My Life

Having been born at Pavia, I lost, in the very first month of my life, my wet-nurse on the day when she fell ill, so they tell me, of the plague; and my mother returned to me. Five carbuncles came out on my face ...
In my fourth year I was taken to Milan. Here, although I received kindly treatment (except on occasions when I was unjustly whipped by both my father and my mother)...
I was many times ill to the very verge of death. At length when I was just seven years old - my father and my mother were not living together then - and really justly deserved an occasional whipping, they decided to refrain from this punishment

A Brief Narrative of My Life

My bad luck had by no means deserted me, but had simply changed my misfortunes without removing them. Having rented a house, my father took me, my mother, and my aunt home with him; and, thereupon I was ordered to accompany him daily in spite of my tender age and frail little body ... Consequently at the beginning of my eighth year I fell ill of a dysentry and fever

[after his marriage] And I ceased to be poor because I had nothing left.

[much later in life] Shortly after my return to Pavia in the year 1559, the occasion of my son's death came about. Thereafter, I endured my existence ...

Customs, Vices and Errors

An account of the nature is by its own character a most difficult thing to write, and so much more for me as I reflect that those who have been wont to read the autobiographical works of writers are not used to seeing such a straightforward narrative ...

This I recognize as unique and outstanding among my faults - the habit, which I persist in, of preferring to say above all things what I know to be displeasing to the ears of my hearers ...

I used to be just as immoderate in living when I well knew what course was most expedient to follow, and what I ought to do; scarcely another man could be found so obstinate in a fault of the sort

From my youth, I was immoderately given to gambling ... The dice turned out to be far worse, and once my sons were instructed in the attractions of the games of chance, our home was too frequently thrown open to gamblers

Perils, Accidents and Persistent Treacheries

Once when I was in Venice on the birthday of the Blessed Virgin, I lost some money while gambling; on the following day I lost the rest, for I was in the house of a professional cheat. When I observed that the cards were marked, I impetuously slashed his face with my poniard ...

(The picture that follows is from 1594 gives an account of what Cardano had to deal with)

The Cardsharps (Caravaggio, 1594 Wikimedia Commons)

Marriage and Children

For this unfortunate union was the cause of all the calamities which befell me throughout my whole life.

The Disasters of My Sons

My son, between the day of his marriage and the day of his doom, had been accused of attempting to poison his wife while she was still in weakness attendant upon childbirth. On the 17th day of February he was apprehended and ... on April 13th, he was beheaded in prison.

Hard upon this succeeded the folly, the ignominious conduct, and violent actions of my younger son until nothing could have contributed to his further disgrace. I was obliged to have him imprisoned more than once, to condemn him to exile, and to cut him off from his paternal inheritance

Perils, Accidents and Persistent Treacheries (2)

I was the victim of [four] great discouragements and obstacles in my life. The first was my marriage; the second, the bitter death of my son; the third, imprisonment; the fourth, the base character of my youngest son.

Ars Magna - On The Types of Equations

...we conclude our detailed consideration with the cubic, others being merely mentioned, even if generally, in passing. For as positio [the first power] refers to a line, quadratum [the square] to a surface, and cubum [the cube] to a solid body, it would be very foolish for us to go beyond this point. Nature does not permit it.

Thus, it will be seen, all those matters up to and including the cubic are fully demonstrated, but the others which we will add, either by necessity or curiosity, we do not go beyond barely setting out.

On Negative Numbers

[It will be remembered also that] 9 is derivable equally from 3 and -3, since a minus times a minus produces a plus. But in the case of the odd powers, each keeps its own nature: it is not a plus unless it derives from a true number, and a cube whose value is minus, or what we call debitum, cannot be produced by any expansion of a true number. It behooves us to remember this very clearly.

nes. At vero quòd tam ex 3. quàm ex m. 3.
fit 9. quoniam minus in minus ductum pro-
ducit plus. At in imparibus denominationi-
bus eadem ſeruatur natura : ſeu quòd dici-
mus debitum, expoſitione ulla numeri veri
produci poteſt, iam meminiſſe oportet di-
lucidius explicatum.

Solving Cubic Equations

- Cardano argued geometrically
- Our presentation will be mainly algebraic.
- We first solve the simplified equation given by

$$x^3 + px = q$$

where $p, q > 0$.

Brilliant Idea

Notice that for a and b numbers with $a > b > 0$,

$$\begin{aligned}(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\ (a-b)^3 &= -3ab(a-b) + a^3 - b^3 \\ (a-b)^3 + 3ab(a-b) &= a^3 - b^3\end{aligned}$$

Pattern matching this to $x^3 + px = q$, we see that $x = a - b$ is a solution to

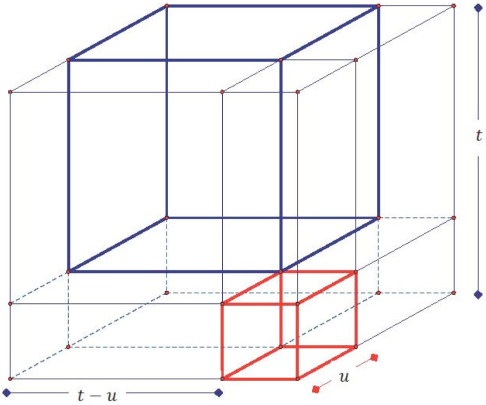
$$x^3 + px = q$$

where $p = 3ab$ and $q = a^3 - b^3$.

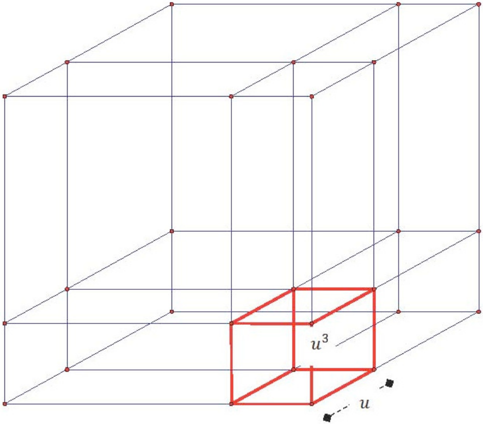
Dissecting the Cube

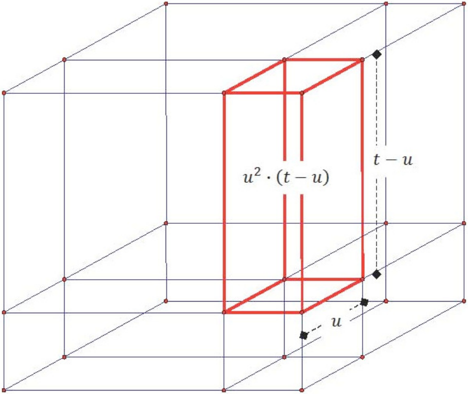
Tartaglia (and others) however thought of this dissection geometrically. The slides that follow are due to Marty Bonsangue at Cal State Fullerton.

Tartaglia’s Cube



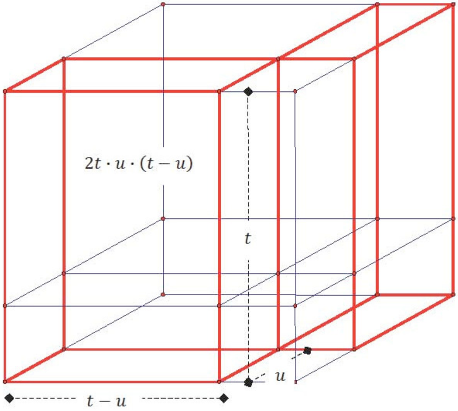
Small Cube



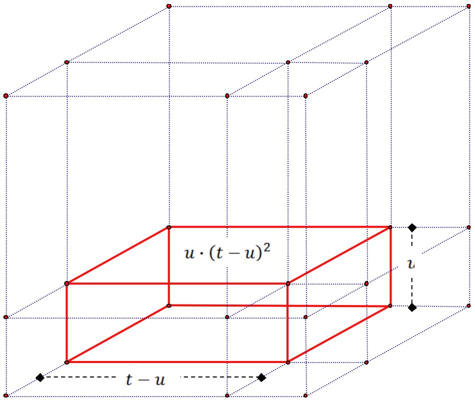


Tower

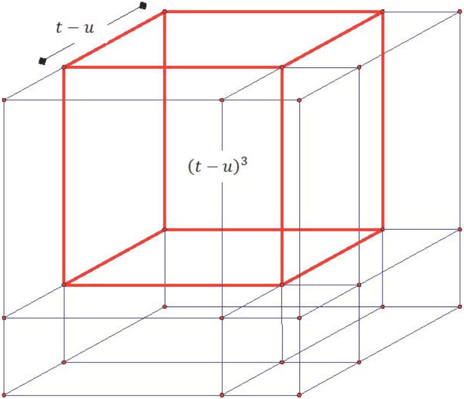
Two Slabs



Base



Big Cube



Summing

Big Cube	$(t-u)^3$
Small Cube	u^3
Tower	$u^2(t-u)$
Slabs	$2tu(t-u)$
Base	$u(t-u)^2$
Total	t^3

Simplifying the algebra

$$
\begin{aligned}
t^3 &= (t-u)^3 + u^3 + u^2(t-u) + 2tu(t-u) + u(t-u)^2 \\
t^3 - u^3 &= (t-u)^3 + 2tu(t-u) + u(t-u)(u + (t-u)) \\
t^3 - u^3 &= (t-u)^3 + 2tu(t-u) + tu(t-u) \\
t^3 - u^3 &= (t-u)^3 + 3tu(t-u)
\end{aligned}
$$

which is precisely the algebraic deconstruction we had from before.

Finding a and b

Hence, it suffices to find values for a and b satisfying

$$\begin{aligned} a^3 - b^3 &= q \\ ab &= \frac{p}{3} \end{aligned}$$

and then we can compute $x = a - b$ which will be a solution to our given equation.

Another idea

If we could find a value r satisfying $a^3 + b^3 = r$, then this with $a^3 - b^3 = q$, we can easily solve for a and b. Summing these two equations gives

$$2a^3 = r + q \quad \Rightarrow \quad a = \sqrt[3]{\tfrac{1}{2}(r+q)}$$

and subtracting these two equations gives

$$2b^3 = r - q \quad \Rightarrow \quad b = \sqrt[3]{\tfrac{1}{2}(r-q)}$$

Getting $a^3 + b^3$

Notice that

$$(a^3 + b^3)^2 = a^6 + 2(ab)^3 + b^6$$
$$(a^3 - b^3)^2 = a^6 - 2(ab)^3 + b^6$$

Taking the difference of these two equations gives

$$(a^3 + b^3)^2 - (a^3 - b^3)^2 = 4(ab)^3$$

However, we know that $a^3 - b^3 = q$ and $ab = p/3$ and thus,

$$(a^3 + b^3)^2 = q^2 + \frac{4p^3}{27}$$

Summary

Therefore, as

$$(a^3 + b^3)^2 = q^2 + \frac{4p^3}{27}$$

we see that

$$a^3 + b^3 = \sqrt{q^2 + \frac{4p^3}{27}}.$$

Notice above we can take the positive root since a and b are positive. It turns out even if these weren't positive, we could still take the positive root because the value of $a - b$ would eventually be the same (exercise).

Solving for a and b

Now, $a^3 - b^3 = q$ and $a^3 + b^3 = \sqrt{q^2 + \frac{4p^3}{27}}$ gives

$$a^3 = \frac{1}{2}\left(q + \sqrt{q^2 + \frac{4p^3}{27}}\right) = \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$b^3 = \frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right) = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Solving For x

This yields

$$a = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$b = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

and thus,

$$x = a - b = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Cardano's Example

Cardano used the example $x^3 + 6x = 20$. In this case, $p = 6$ and $q = 20$ and hence

$$\frac{p^3}{27} = 8 \qquad \frac{q^2}{4} = 100$$

and thus, the formula

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

yields

$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

So now...

... surely you can solve $x^3 + 6x = 20$ over the complex numbers right? Just factor out that root and solve the remaining quadratic over $\mathbb{C}$.

However...

... did you get the roots:

$$x = 2, -1 \pm 3i$$

?

However...

... did you get the roots:

$$x = 2, -1 \pm 3i$$

? How on earth does

$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

make any sense?

Magic!!!

Well if we start by believing that there must be a nice solution, it probably means that

$$10 + \sqrt{108} = 10 + 6\sqrt{3} = (a + b\sqrt{3})^3$$

for some values of a and b. Expanding:

$$10 + 6\sqrt{3} = (a + b\sqrt{3})^3 = a^3 + 9ab^2 + (3a^2b + 3b^3)\sqrt{3}$$

and comparing coefficients of the constant term and the $\sqrt{3}$ term gives us that

$$a^3 + 9ab^2 = 10 \qquad 3a^2b + 3b^3 = 6$$

Finishing the magic

Simplifying gives

$$a(a^2 + 9b^2) = 10 \qquad b(a^2 + b^2) = 2$$

We know the answer is positive so if this is solvable for a and b integers, a solution has to exist when $b \leq 2$ and sure enough, $b = 1$ and $a = 1$ gives a solution, namely that $10 + 6\sqrt{3} = (1 + \sqrt{3})^3$. Similarly, $-10 + \sqrt{108} = (-1 + \sqrt{3})^3$ and so

$$\begin{aligned} x &= \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}} \\ &= \sqrt[3]{(1 + \sqrt{3})^3} - \sqrt[3]{(-1 + \sqrt{3})^3} \\ &= 1 + \sqrt{3} - (-1 + \sqrt{3}) \\ &= 2 \end{aligned}$$

June 15, 1894

Nikolai Chebotaryov born in Ukraine. He proved a density theorem generalizing Dirichlet's Theorem on primes in an arithmetic progression in 1922 and translated or wrote many of the first advanced algebra texts in Russian.

The Chebotaryov Density Theorem (Special Case)

A special case that is easier to state says that if K is an algebraic number field which is a Galois extension of $\mathbb{Q}$ of degree n, then the prime numbers that completely split in K have density $1/n$ among all primes.

Note: This is why as we will see this week that 'half' the primes can be represented as the sum of two squares and half the primes can't!

Announcements

- Project due Tuesday. Dropboxes are live! Turnitin Note. Only keeps last submission.
- Note: The rubric I outlined doesn't mean "Thesis level = 80%".
- Assignment 3 due the Tuesday after. I just sent out Crowdmark emails - check for yours!
- Survey results

Brief Comments on Survey

- Using LaTeX
- BC
- Quiz was long (discuss what we should shorten)
- Connecting assignment to quiz Q4- A1Q2,3; Q5-A2Q3; Q6-A1Q6,7; Q7-A2Q8
- Make slides more detailed (I don't think I physically can)
- Learning objectives before class (Will do my best)
- Why I write on the board (also - it's nice to have slide and board simultaneously)
- *** Students talk to much in class
- Stop reading form notes (Agreed! Will not be repeating Cardano)

Key Steps in Cardano's Formula

What are they?

Ars Magna - On the Rule for Postulating a Negative Ch. XXXVII

The second species of negative assumption involves the square root of a negative. I will give an example: If it should be said, Divide 10 into two parts the product of which is 30 or 40, it is clear that this case is impossible. Nevertheless, we will work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth book, leaving a remainder of -15, the square root of which added to or subtracted from 5 gives parts the product of which is 40. These will be $5+\sqrt{-15}$ and $5-\sqrt{-15}$.

Mental Tortures

Putting aside the mental tortures involved, multiply $5+\sqrt{-15}$ by $5-\sqrt{-15}$, making $25-(-15)$. Hence this product is 40. Yet the nature of [the square] AD is not the same as that of 40 or of [the line segment] AB, since a surface is far from the nature of a number and that of a line, though somewhat closer to the latter. This truly is sophisticated ...

... So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless.

[From the 38th problem of Ars Magna Arithmeticae] Note that $\sqrt{9}$ is either 3 or -3, for a plus [times a plus] or a minus times a minus yields a plus. Therefore $\sqrt{-9}$ is neither 3 nor -3 but is some recondite third sort of thing.

Another type of cubic

Cardano (and previously Scipione and Tartaglia) solved

$$x^3 + px = q \qquad p, q > 0$$

Cardano also solved

$$x^3 = px + q \qquad p, q > 0$$

in much the same way as $x^3 + px = q$. His value for x was

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

Things are Getting Complex...

Contrasting the two solutions, for $x^3 + px = q$ with $p, q > 0$, Cardano's solution

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

gives only real solutions. However, Cardano's solution to $x^3 = px + q$ with $p, q > 0$:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

will give imaginary solutions whenever $\dfrac{q^2}{4} < \dfrac{p^3}{27}$.

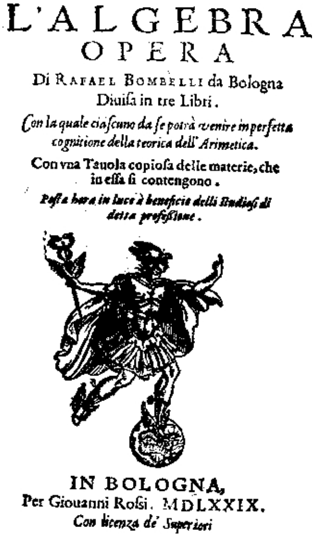
Bombelli's Approach [Kat93, p.365-366]

- Keep in mind that imaginary numbers are not known to have any real meaning yet.
- *Ars Magna* was a difficult text to absorb and read
- 15 years later, Bombelli decided to write down his version of the text, *Algebra*
- He created an algebraic way to handle numbers that were neither positive (*più*) or negative (*meno*)
- He called bi and $-bi$ (in modern notation) *più di meno* and *meno di meno*
- He also gave our modern day version of the operations with complex numbers.

Brief Biography of Bombelli [ORa]

- Baptized Jan 20, 1526 died 1572 in Bologna
- Son of Antonio Mazzoli, a powerful family name in Bologna forcing him to eventually change his name to Bombelli to avoid the association
- Received no formal college education; was taught by engineer-architect Pier Francesco Clementi

Bombelli's Algebra



L'ALGEBRA
OPERA

Di RAFAEL BOMBELLI da Bologna
Diuisa in tre Libri.

Con la quale ciascuno da se potrà venire in perfetta cognitione della teorica dell'Arimetica.

Con vna Tauola copiosa delle materie, che in essa si contengono.

Posta hora in luce à beneficio delli Studiosi di detta professione.

IN BOLOGNA,
Per Giouanni Rossi. MDLXXIX.
Con licenza de' Superiori

Bombelli's Approach

Consider $x^3 = 15x + 4$. Cardano's formula gives

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

and the three roots are

$$4, -2 \pm \sqrt{3}$$

Bombelli's Idea

Notice that

$$\sqrt[3]{2+\sqrt{-121}} \quad \text{and} \quad \sqrt[3]{2-\sqrt{-121}}$$

differ only by a sign. What if the imaginary numbers they represented also only differed by a sign? That is, do there exist positive real numbers a and b such that

$$\sqrt[3]{2+\sqrt{-121}} = a+bi \quad \text{and} \quad \sqrt[3]{2-\sqrt{-121}} = a-bi$$

Bombelli's Thoughts [Bur91, p. 325]

It was a wild thought in the judgement of many; and I too for a long time was of the same opinion. The whole matter seemed to rest on sophistry rather than on truth. Yet I sought so long, until I actually proved this to be the case.

Attempting to solve for a and b

Setting $\sqrt[3]{2+\sqrt{-121}} = a + bi$, we see that

$$\begin{aligned} 2+\sqrt{-121} &= (a+bi)^3 \\ &= a^3 + 3a^2bi - 3ab^2 - b^3i \\ &= (a^3 - 3ab^2) + (3a^2b - b^3)i \end{aligned}$$

and so $2 = a(a^2 - 3b^2)$ and $b(3a^2 - b^2) = 11$

So...

If integer solutions are going to work, a is one of 1 or 2 and $a^2 - 3b^2 > 0$ so $a = 2$ either works or there is no integer solution. Checking yields $a = 2$ and $b = 1$. Hence,

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + i \quad \text{and} \quad \sqrt[3]{2 - \sqrt{-121}} = 2 - i$$

Therefore,

$$\begin{aligned} x &= \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\ &= (2 + i) + (2 - i) \\ &= 4 \end{aligned}$$

The Final Step

- In much the same way, one can handle $x^3 + px + q = 0$ for and p and q with knowledge of complex numbers.
- What about handling equations of the form $x^3 + ax^2 + bx + c = 0$?
- Idea: Eliminate the ax^2 term.

Eliminating ax^2

Set $x = y - a/3$. This gives

$$\begin{aligned}
0 &= \left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c \\
&= \left(y^3 - 3y^2\frac{a}{3} + 3y\frac{a^2}{9} - \frac{a^3}{27}\right) + a\left(y^2 - 2y\frac{a}{3} - \frac{a^2}{9}\right) + by - \frac{ab}{3} + c \\
&= y^3 + \left(b - \frac{a^2}{3}\right)y + \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right)
\end{aligned}$$

The rest of the story

- As I alluded to earlier, the quartic was solved by Cardano's student Ferrari using similar types of tricks (eliminating the cubic term, doing some weird completing tricks etc.)
- For fifth degree and higher, despite efforts, no solutions were found.
- It wasn't until 1824 when Abel from work of Ruffini managed to show that if a polynomial had degree at least 5, then one could not in general find an algebraic solution to every equation.
- This coincided with the founding of Galois Theory which helped explain the proof in a different way.

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