SAT Solving with Computer Algebra

for Fast, Verified Mathematical Search

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SAT:

Boolean satisfiability problem

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SAT solvers: Clever brute force

Effectiveness of SAT solvers

Surprisingly, many problems that have nothing to do with logic can be effectively solved by translating them into Boolean logic and using a SAT solver.

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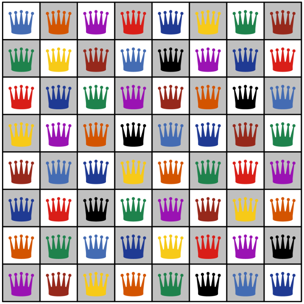
Examples

- Discrete optimization
- Hardware and software verification
- Proving/disproving conjectures (my specialty)



Limitations of SAT solvers

SAT solvers lack mathematical understanding beyond the most basic logical inferences and will fail on some trivial tiny problems.

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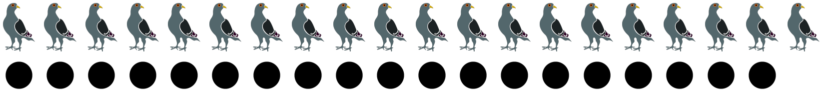
Example

Have a SAT solver try to find a way to put 20 pigeons into 19 holes such that no hole contains more than one pigeon...



CAS:

Computer algebra system

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Algorithmic mathematical computing

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Computer algebra systems can perform calculations and manipulate expressions from many branches of mathematics.

Example

What is the value of

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Example

What is the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\,?$$

Maple returns $\pi^2/6$ ... *not* 1.64493406685.

Limitations

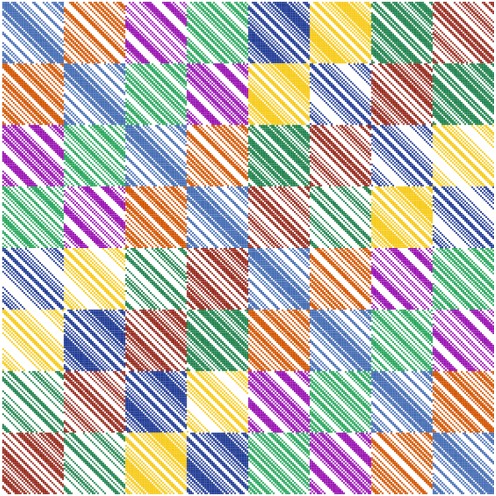
CASs are not optimized to do large searches (in an exponential-sized space).

SAT + CAS

Search + Math

MathCheck: A SAT+CAS system

MathCheck has found over 100,000 combinatorial matrices like this $\{\pm 1\}$-matrix of order 280 with pairwise orthogonal rows:



MathCheck results (see uwaterloo.ca/mathcheck)

Discrete Geometry:
Fastest verification of cases in Lam's problem (1800s).

Graph Theory:
Current best result in the Ruskey–Savage conjecture (1993).
Current best result in the Norin conjecture (2008).

Combinatorics:
Found the smallest counterexample of the Williamson conjecture (1944).
Found three new counterexamples of the good matrix conjecture (1971).
Current best result in the best matrix conjecture (2001).

Number Theory:
Verified a conjecture of Craigen, Holzmann, and Kharaghani (2002).

Lam's Problem

History

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The existence of projective planes (1800s) shows this is impossible!

Projective planes

A *projective plane* is a collection of points and lines and a relation between points and lines such that:

1. There is a unique line through any two points.
2. Any two lines intersect at a unique point.

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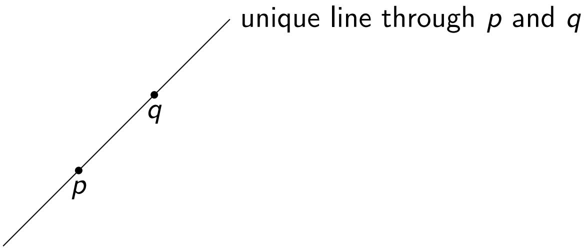

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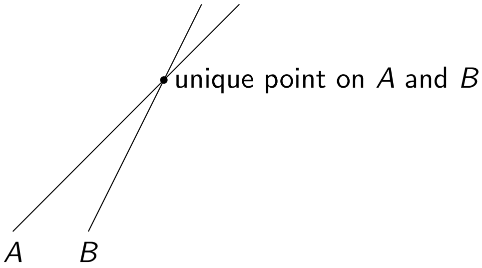

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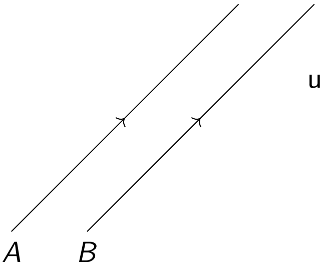

unique point on A and B?

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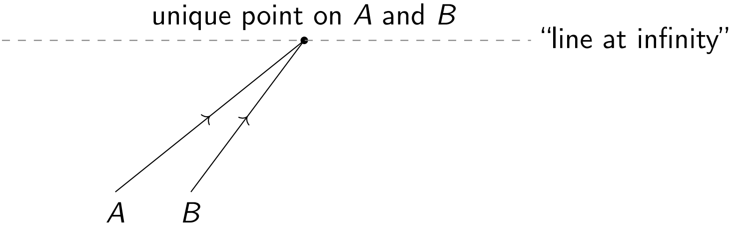


Finite projective planes

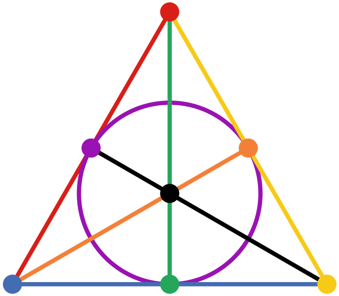
Must a projective plane have an infinite number of points?

Finite projective planes

Must a projective plane have an infinite number of points?

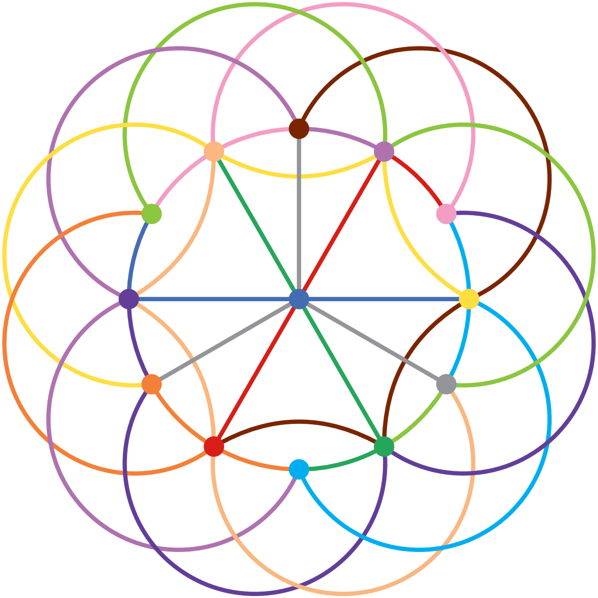
If not, it must have exactly $n^2 + n + 1$ points for some integer n (called the *order* of the plane).

Projective plane of order 2



point 1	line 1
point 2	line 2
point 3	line 3
point 4	line 4
point 5	line 5
point 6	line 6
point 7	line 7

Projective plane of order 3



Projective planes of small orders

2	3	4	5	6	7	8	9	10
✓	✓	✓	✓	✗	✓	✓	✓	**?**

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Lam's Problem

Projective planes of small orders

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✓	✓	✓	✓	✗	✓	✓	✓	✗

Supercomputer Search (1973–1989)

Projective plane of order 2: Incidence matrix

1	1	0	1	0	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	0
0	0	0	1	1	0	1
1	0	0	0	1	1	0
0	1	0	0	0	1	1
1	0	1	0	0	0	1

Boolean matrix of size 7×7 where (i, j)th entry is 1 exactly when the ith line is incident with the jth point.

SAT encoding: false $\equiv 0$, true $\equiv 1$

Lam's problem: First case

The first case of Lam's problem was solved in 1973 and has been verified by at least four independent implementations on modern desktops:

Authors	**Year**	**Language**	**Time**
Roy	2005	C	78 min
Casiello, Indaco, and Nagy	2010	GAP	3.3 min
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Bright et al.	**2019**	**SAT+CAS**	**6.8 sec**

Lam's problem: Second case

The second case was initiated in 1974 and not entirely searched until 1986. I am only aware of a single verification on a modern desktop prior to our work:

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Authors	**Year**	**Language**	**Time**
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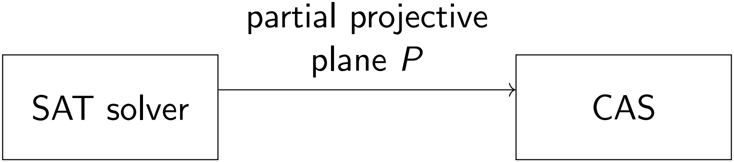
Bright et al.	**2020**	**SAT+CAS**	**30 hours**

Learning method

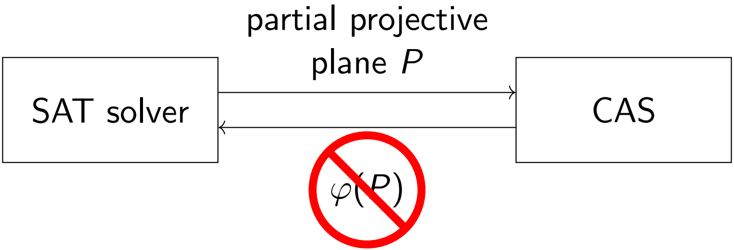


Learning method

$$\boxed{\text{SAT solver}} \xrightarrow{\text{partial projective plane } P} \boxed{\text{CAS}}$$

The CAS computes a nontrivial symmetry φ of the plane...

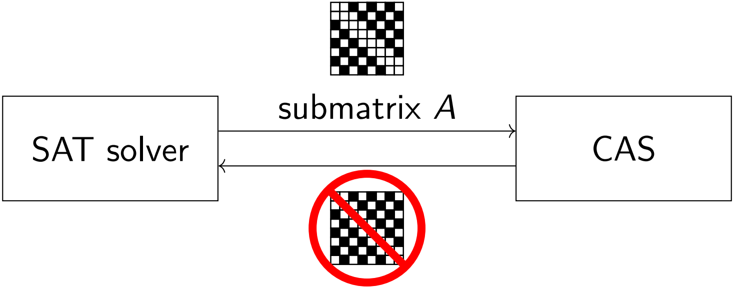
Learning method



The CAS computes a nontrivial symmetry φ of the plane...

...and a symmetry "blocking clause" is learned.

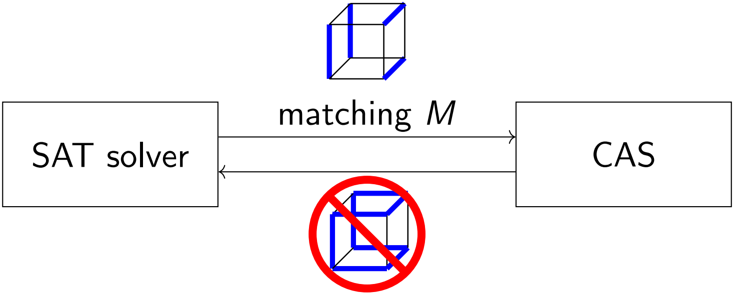
Learning method: Hadamard matrices



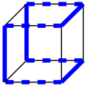
The CAS computes the largest magnitude in the discrete Fourier transform of A. If it is too large...

...a "conflict clause" is learned.

Learning method: Ruskey–Savage conjecture



The CAS tries to extend the matching M to a Hamiltonian cycle...



...and if successful a conflict clause is learned.

Lam's problem: Third case

The final case was solved in 1989 by Lam et al. using 26 months on a supermini computer and 3 months on a supercomputer.

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Ultimate goal: Complete a SAT+CAS verification of this case and search for larger projective planes—little is known about projective planes of orders 11 and 12.

Verifiability

All previous searches were unverifiable. They require trusting:

- The hardware, compiler, and operating system used.
- The search code used.
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This is a lot to trust. *Our searches found bugs in prior searches.*

SAT certification

In contrast, SAT solvers provide unsatisfiability certificates.

Our searches reduce necessary trust to the SAT encoding, the CAS-derived clauses, and a small trusted proof verifier.

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Ultimate goal: Generate a complete nonexistence proof entirely from the projective plane axioms.

More SAT+CAS applications

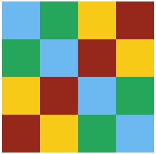
New and emerging areas of application include circuit minimization, verification, cryptography, and program synthesis.

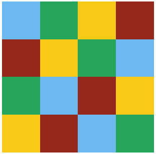
I will outline just two promising applications:

- Mutually orthogonal Latin squares
- The Hadwiger–Nelson problem

Latin squares

An $n \times n$ matrix whose entries contain n symbols is a *Latin square* if each row and column contains exactly one of each symbol.



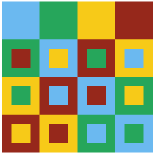


Two Latin squares of order four.

Applications to experimental design, statistics, codes, ...

Orthogonal Latin squares

Two Latin squares are *orthogonal* if all n^2 pairs of entries appear in their superposition.



Pair of orthogonal Latin squares of order four.

Mutually orthogonal Latin squares

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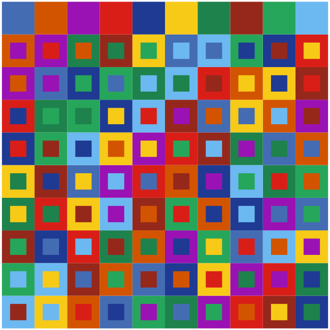
Mourtos estimated solving the first open case (order ten) would require 273 years using integer and constraint programming.

A SAT approach to Latin squares

A SAT approach solves the order six case about 500 times faster than Mourtos' method and is able to find a pair of orthogonal Latin squares of order ten about 20% faster:



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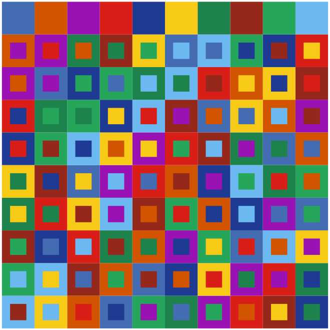

Ultimate goal: Prove or disprove the conjecture that a triple of mutually orthogonal Latin squares of order ten do not exist.

Hadwiger–Nelson problem

How many colours are needed to colour the plane so that no two points separated a distance of 1 are the same colour?

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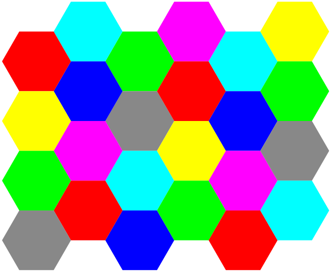
At most 7:



Hadwiger–Nelson problem

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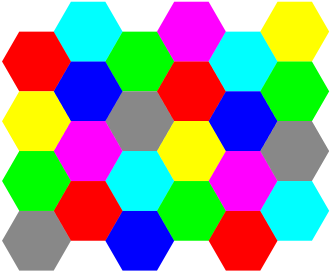
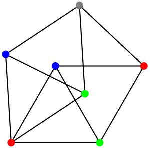


At least 4:



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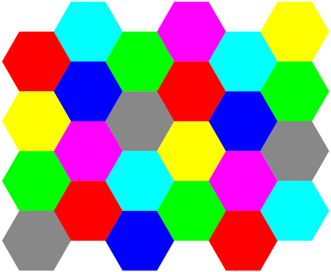
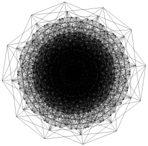


At least 5:



Ultimate goal: Improve these bounds and find the exact answer.

Conclusion

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Wide application: *Many* mathematical problems stand to benefit from faster search tools.

Bang for your buck: Requires knowledge of SAT and CAS, but generally simpler to write and verify than a special-purpose search.

Thank you and I'm happy to answer any questions.

`curtisbright.com`

Selected References:

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