Enumeration of Complex Golay Pairs via Programmatic SAT

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July 17, 2018

SAT + CAS

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Brute force

SAT + CAS

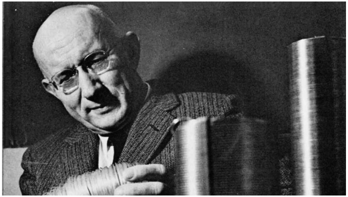
Brute force + Cleverness

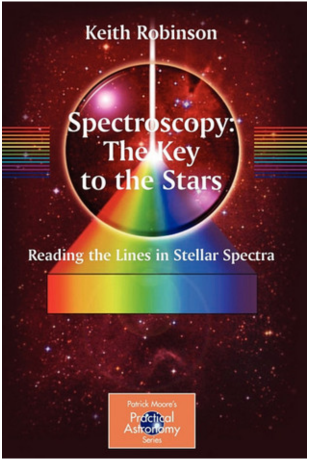
The research areas of SMT [SAT Modulo Theories] solving and symbolic computation are quite disconnected. [. . .] More common projects would allow to join forces and commonly develop improvements on both sides.

Dr. Erika Ábrahám
RWTH Aachen University
ISSAC 2015 Invited talk

Golay pairs

- Golay pairs, termed *complementary series* by Marcel Golay, were introduced in 1949 to solve a problem in multi-slit spectrometry.





Definition

- Let A and B be polynomials with ± 1 coefficients and degree $n-1$. They are a *Golay pair* if

$$|A(z)|^2 + |B(z)|^2 = 2n$$

for all z on the unit circle.

Example

- $A = 1 + z$ and $B = 1 - z$ are a Golay pair since for z on the unit circle we have

$$|1 + z|^2 + |1 - z|^2 = 4.$$

Norm test

- If A, B is a Golay pair then

$$|A(z)|^2 \leq 2n \quad \text{and} \quad |B(z)|^2 \leq 2n$$

for all z on the unit circle.

Sum-of-squares test

- If A, B is a Golay pair then

$$|A(z)|^2 + |B(z)|^2 = 2n$$

  is a decomposition of $2n$ into two *integer* squares when z is ± 1.

Alternate definition

- ± 1-sequences $A = [a_0, \ldots, a_{n-1}]$ and $B = [b_0, \ldots, b_{n-1}]$ are a *Golay pair* if

$$\sum_{k=0}^{n-s-1} a_k a_{k+s} + \sum_{k=0}^{n-s-1} b_k b_{k+s} = 0$$

for $s = 1, \ldots, n-1$.

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for $s = 1, \ldots, n-1$.

$N_A(s)$: A measure of how much A is correlated with itself with the first s entries removed.

Example

- $A = [1, 1, 1, -1]$ and $B = [1, 1, -1, 1]$ are a Golay pair since

$$\begin{aligned} N_A(1) + N_B(1) &= 1 + (-1) = 0 \\ N_A(2) + N_B(2) &= 0 + 0 = 0 \\ N_A(3) + N_B(3) &= (-1) + 1 = 0. \end{aligned}$$

Problem

- Golay found Golay pairs in lengths 2, 10, and 26.
- Golay pairs of length $2^a 10^b 26^c$ can be constructed using these "primitive" pairs but it is conjectured that Golay pairs exist in no other lengths.
- Borwein and Ferguson have searched lengths up to 100.

Peter Borwein and Ron Ferguson. A complete description of Golay pairs for lengths up to 100. *Mathematics of computation*, 2004.

Generalization

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- Sum-of-squares decomposition is now

$$\mathrm{Re}(A(z))^2 + \mathrm{Im}(A(z))^2 + \mathrm{Re}(B(z))^2 + \mathrm{Im}(B(z))^2 = 2n.$$

Example

- $A = [1, 1, -1]$ and $B = [1, i, 1]$ are a complex Golay pair since

$$N_A(1) + N_B(1) = 0 + 0 = 0$$
$$N_A(2) + N_B(2) = (-1) + 1 = 0.$$

Fiedler's theorem

- Let $A = A_{\text{even}} + A_{\text{odd}}$ be a decomposition of A into terms with even degree and terms with odd degree, e.g., $1 + z + z^2 = (1 + z^2) + z$.
- If A, B is a complex Golay pair then

$$|A_{\text{even}}(z)|^2 + |A_{\text{odd}}(z)|^2 + |B_{\text{even}}(z)|^2 + |B_{\text{odd}}(z)|^2 = 2n$$

for all z on the unit circle.

Frank Fiedler. Small Golay sequences. *Advances in mathematics of communications*, 2013.

Preprocessing: Enumerate A_{even} and A_{odd}

- We will find lists of the A_{even} and A_{odd} which pass the norm tests

$$|A_{\text{even}}(z)|^2 \leq 2n \quad \text{and} \quad |A_{\text{odd}}(z)|^2 \leq 2n$$

  for $M = 2^{14}$ equally-spaced points on the unit circle.
- Can compute via brute force for $n \approx 30$.

Stage 1: Enumerate possibilities for A

- For all A_{even} and A_{odd} found in the preprocessing, we form $A = A_{\text{even}} + A_{\text{odd}}$ and filter those which fail either the norm test or the sums-of-squares test. That is, those for which

$$\mathrm{Re}(A(z))^2 + \mathrm{Im}(A(z))^2 + x^2 + y^2 = 2n$$

has no integer solutions x, y for when z is ± 1 or $\pm i$.

Stage 2: Construct B from A

- Given A, generate a SAT instance which encodes the property of (A, B) being a complex Golay pair.
- Let $v_0, \ldots, v_{2n-1}$ be variables which represent the entries of B under the following encoding scheme:

v_{2k}	v_{2k+1}	b_k
F	F	1
F	T	-1
T	F	i
T	T	$-i$

SAT instance

- How to encode the property of A, B being a complex Golay pair into a SAT instance?
- That is, $N_A(s) + N_B(s) = 0$ for $s = 1, \ldots, n-1$.
- We use a SAT solver custom-tailored to this problem which can *programmatically* learn logical facts.

Example

- If $A = [1, 1, -1]$ then $N_A(1) = 0$ and $N_A(2) = -1$.
- Say during the search the SAT solver tries assigning

$$\begin{array}{cccccc} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 \\ \mathrm{F} & \mathrm{F} & ? & ? & \mathrm{T} & \mathrm{T} \end{array}$$

- $B = [1, ?, -i]$ and $N_A(2) + N_B(2) = -1 + i \neq 0$, so we can learn the clause which says at least one of these variables must be assigned differently:

$$v_0 \lor v_1 \lor \neg v_4 \lor \neg v_5.$$

A product theorem

- We proved that if A, B is a complex Golay pair then $a_k a_{n-k-1} b_k b_{n-k-1} = \pm 1$ for $k = 0, \ldots, n-1$.
- From this we deduce if exactly one of $\{b_k, b_{n-k-1}\}$ is real. If so, we learn the following:

$$v_{2k} \vee v_{2(n-k-1)}$$

$$\neg v_{2k} \vee \neg v_{2(n-k-1)}$$

Implementation

- We implemented this algorithm using C and C++ to do the enumerations, MAPLE to form the sum-of-squares decompositions, and FFTW to compute the values of $A(z)$ at equally-spaced points along the unit circle.

Results

- We split the enumeration work across 25 Intel Xeon 2.1 GHz processors and enumerated all complex Golay pairs up to length 25 in 40 realtime hours.
- There are no complex Golay pairs in lengths 23 or 25 but there are 786,432 complex Golay pairs of length 24 (1056 up to an equivalence).
- Available on the MathCheck website:

  `https://sites.google.com/site/uwmathcheck/`

Future optimizations?

- Could the norm test could be done more efficiently by computing the maximum of $|A(z)|^2$ for z on the unit circle?
- Could we make the SAT solver more efficient by encoding other theorems about complex Golay sequences?

Conclusion

- The SAT+CAS paradigm is very general and can be applied to problems in a large number of domains.
- Especially good for problems that require CAS functions as well as some kind of brute-force search.
- Pro: Make use of the immense amount of engineering effort that has gone into CAS and SAT solvers.
- Con: Can be difficult to split the problem in a way that takes advantage of this.