MathCheck: A Math Assistant Combining SAT with Computer Algebra Systems

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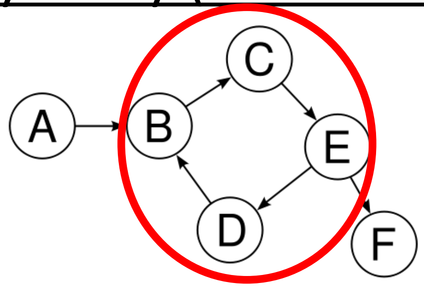
Problem Statement

Many problems have an underlying Boolean structure, but are **not easily expressed** using standard SAT/SMT solvers.

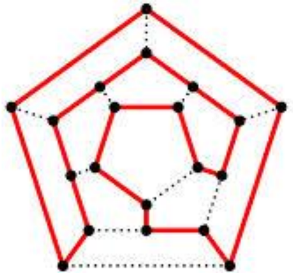
Acyclicity (Gebser'14)



Constrained Clustering (Métivier'12)

	A	B	C	D	E	F
t_1	×		×	×		
t_2		×			×	
t_3	×		×	×		
t_4		×	×	×		×
t_5		×			×	
t_6		×			×	
t_7		×		×		×

Hamiltonicity (Velev'09)

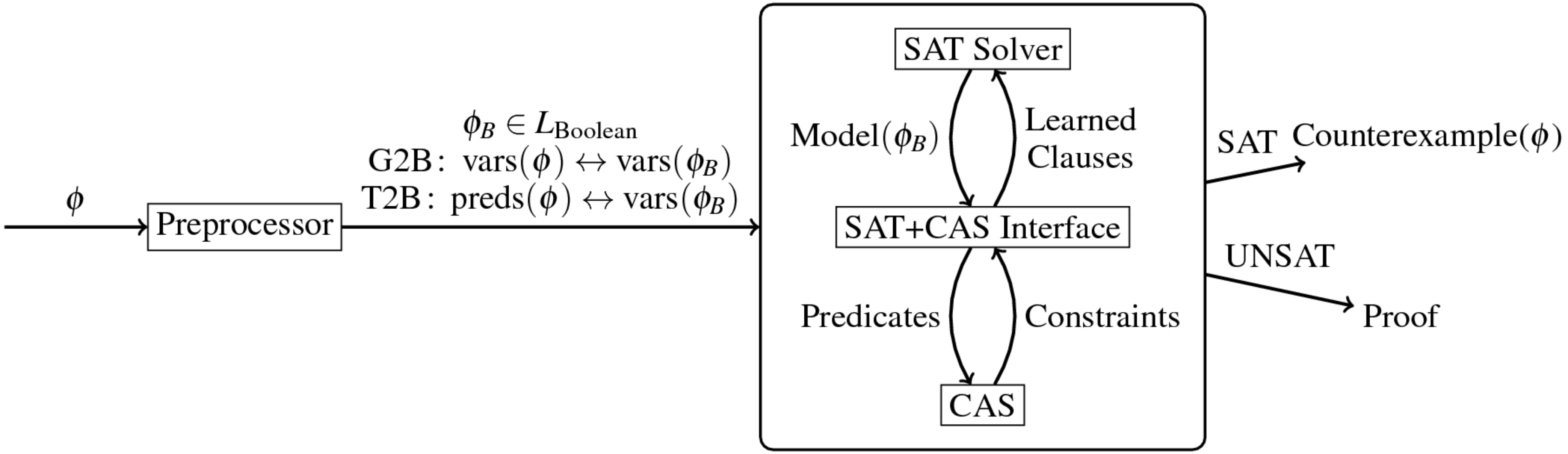


Finite domain search + complex predicates.

Goals

- Computer algebra systems (CAS) contain SOTA algorithms for solving complex properties
- SAT solvers are one of the best general approaches for finite domain search
- **Goal 1**: incorporate algorithms from a CAS with a SAT solver for:
    - Counterexample Construction for Math Conjectures
    - Bug finding
- **Goal 2**: design an easily extensible language/API for such a system
    - Current focus is on graph theory

DPLL(CAS) Architecture

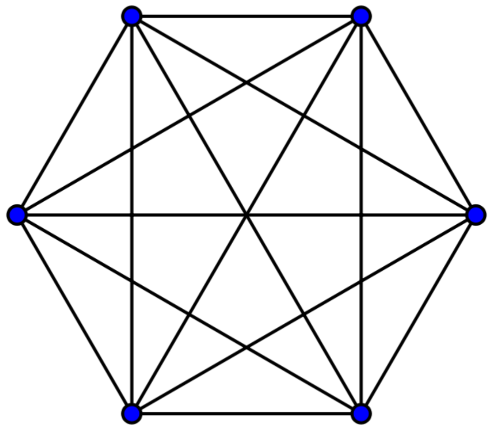


Extensibility preferred to a "one-algorithm-fits-all" approach.

Graph Variable Representation

graph x(6)

- One Boolean per each potential vertex
- One Boolean per each potential edge

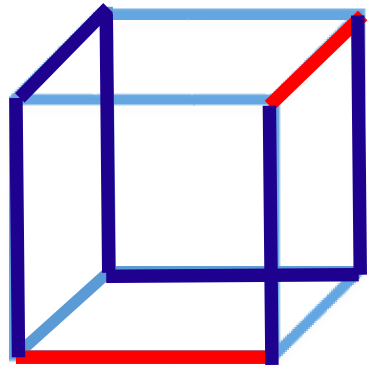


- Mapping between graph components and Booleans to facilitate defining SAT-based graph constraints

Case Study: Ruskey-Savage Conjecture

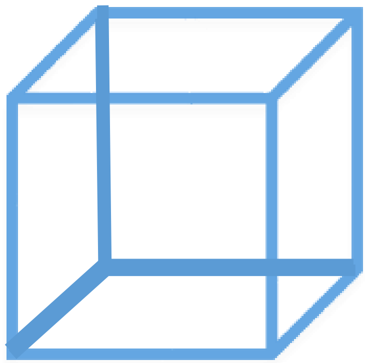
Conjecture: For every $d \geq 2$, any matching of the hypercube Q_d extends to a Hamiltonian cycle.

- Matching – independent set of edges that share no vertices
  - Maximal – cannot add edges without violating the matching property
  - Perfect – it covers all vertices
- Hamiltonian cycle – cycle that touches every vertex
- Previously shown true for $d \leq 4$



Q_3

Case Study Specification ($d = 5$)



graph x(32)

sage.CubeGraph G(5)

//$\forall x. matching(x, G) \Rightarrow extends\_to\_hamiltonian(x, G)$

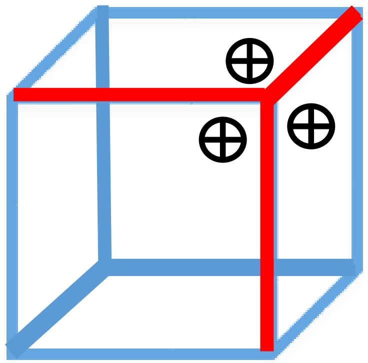
assert(matching(x,G) ∧

imperfect_matching(x,G) ∧

maximal_matching(x,G)),

query(extends_to_Hamiltonian_cycle(x,G))

Case Study Specification ($d = 5$)



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Blasted to SAT

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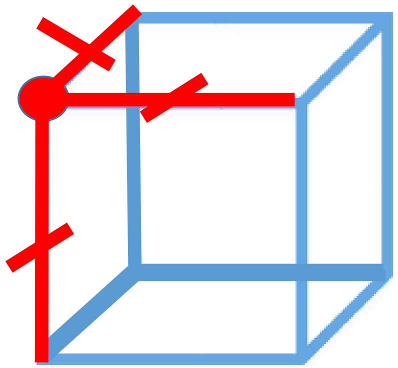

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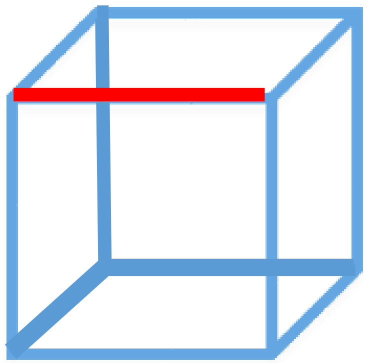

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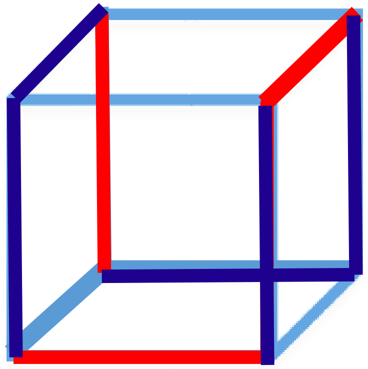

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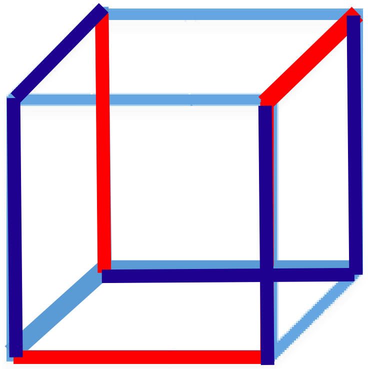
maximal_matching(x,G)),

query(extends_to_Hamiltonian_cycle(x,G))

Blasted to SAT

Checked with SAGE

Case Study Specification ($d = 5$)



graph x(32)

sage.CubeGraph G(5)

$//\forall x. matching(x, G) \Rightarrow extends\_to\_hamiltonian(x, G)$

assert(matching(x,G) ∧ ~10 LOC

imperfect_matching(x,G) ∧ ~5 LOC

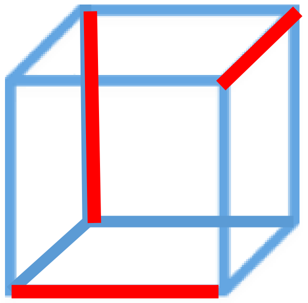
maximal_matching(x,G)), ~5 LOC

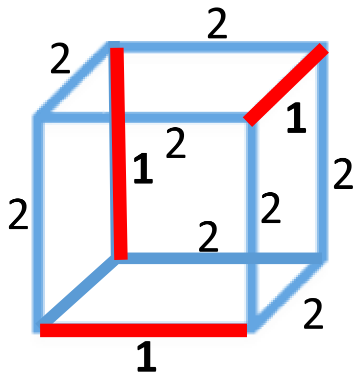
Blasted to SAT

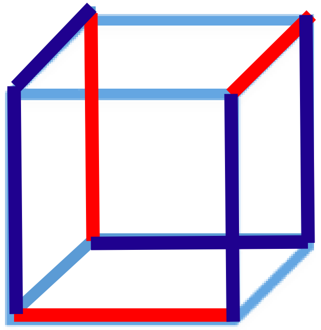
query(extends_to_Hamiltonian_cycle(x,G)) ~25 LOC

Checked with SAGE

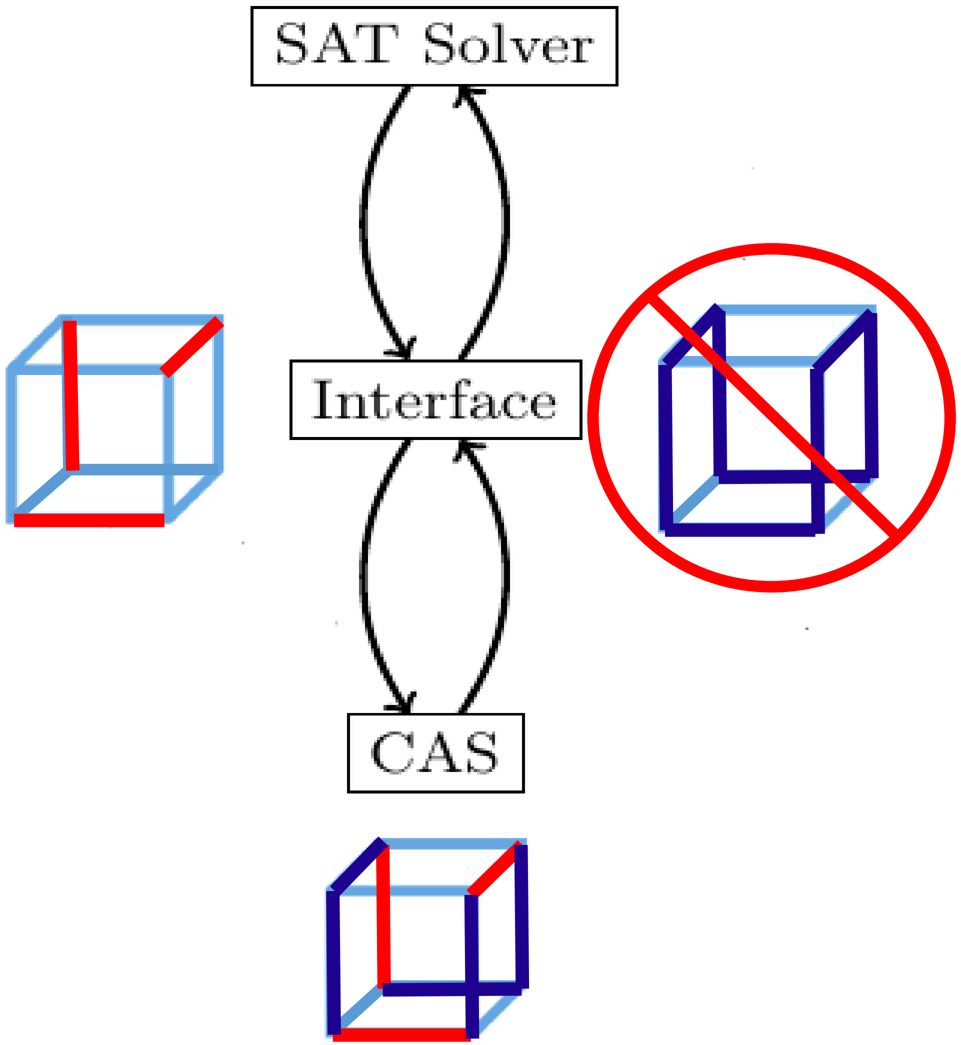
```
ExtendsToHamiltonian()
    x ← s.getGraph(G)
    q ← CubeGraph(5)
    for e in q.edges() do
        if e in g
            q.setEdgeLabel(e, 1)
        else
            q.setEdgeLabel(e, 2)
        ⟨cycle, weight⟩ ← TSP(q)
        if weight == 2 · q.order() − |x|
            return True
        else
            return False
```







Case Study Approach



- Unsat after ~8 hours on laptop (Conjecture holds for $\boldsymbol{d = 5}$)
- For a pure SAT encoding, we need encode non-trivial Hamiltonicity constraints

A Sage-only approach…

- Without SAT, we need a problem-specific search routine

	#Checks of extends_to_Hamiltonian_cycle
Matchings	13,803,794,944
Imperfect Matchings	4,619,529,024
Maximal Imperfect Matchings	6,911,604
SAT Approach	**384,000**

- A Sage-only approach is:
    - Potentially less efficient
    - Potentially more error-prone