Rational Cubics Through Non-Generic Points

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Joint work in progress with Fulvio Gesmundo and Avi Steiner
(assisted by *Salmon* the computer)

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Computer-Assisted Mathematics
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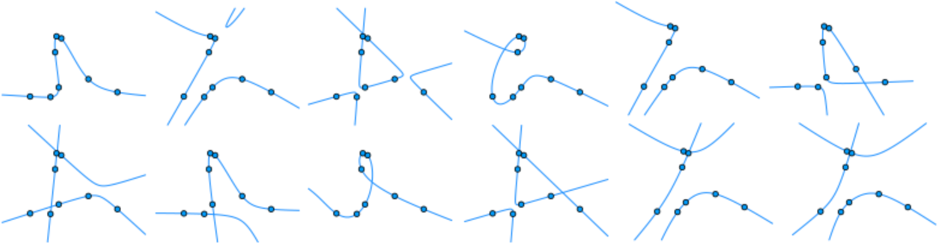
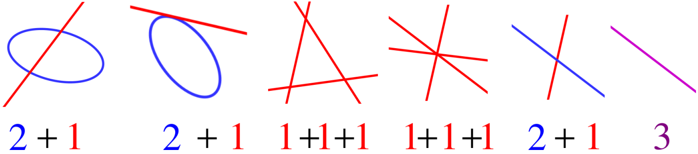
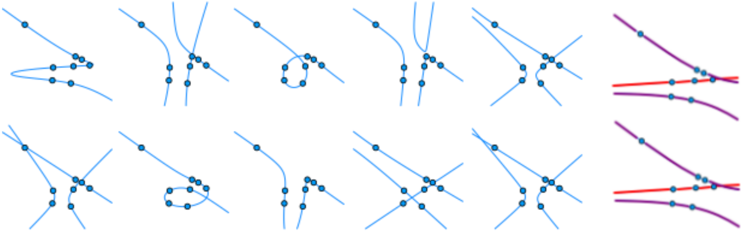
How many rational cubics go through 8 **generic** points?



Generically, no reducible cubic goes through them



How many rational cubics go through 8 (distinct) **non-generic** points?

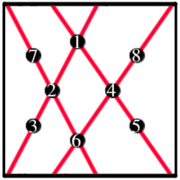


Big Ideas:

- Generic means no 3 points on a line and no 6 points on a conic
- Otherwise, there are reducible cubics through the configuration
- These reducible cubics steal from the count of 12 singular cubics

Quatroids

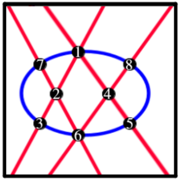
The combinatorial data describing how 8 points lie on lines and conics is a quatroid



Matroid

123 145

267 468



Quatroid

123 145

267 468

135678

Two Big Questions

1. What quatroids exist on eight points?
2. For a generic quatroid representation, how many rational cubics go through it?

Finding All Candidate Quatroids

Pairs $\mathcal{Q} = (\underbrace{\mathcal{I}}_{\text{triples}}, \underbrace{\mathcal{J}}_{\text{sextuples}})$ satisfying the obvious necessary conditions are candidates

Matroidal Condition

- The linear conditions must be satisfiable ($\mathcal{I}$ gives a realizable matroid)

Bézout Condition

- Any two triples (lines) intersect in at most one point
- Any two sextuples (conics) intersect in at most four points
- Any triple/sextuple (line/conic) pair intersect in at most two points

Algorithm

1. Start with a simple rank ≤ 2 matroid on 8 elements (databases exist)
2. Greedily append conic conditions while satisfying Bézout
3. Apply the symmetry group of $\mathfrak{S}_8$
4. Find representatives of each, or show none exists

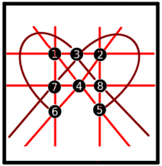
Theorem (Brysiewicz, Gesmundo, and Steiner (assisted by Salmon the Computer))

There are 780617 *candidate quatroids, appearing in* 126 *orbits* $\mathcal{Q}_1, \ldots, \mathcal{Q}_{126}$*:*

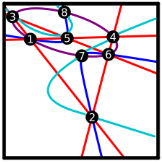
Orbit Size	1	8	28	35	56	70	105	168	210	280
# Orbits	3	2	2	1	3	1	1	3	2	3
Orbit Size	**420**	**560**	**840**	**1680**	**2520**	**3360**	**5040**	**6720**	**10080**	**20160**
# Orbits	2	1	13	4	10	13	17	6	22	17

- *Not representable over* $\mathbb{C}$*: Only* $\mathcal{Q}_{63}$
- *Not representable over* $\mathbb{R}$*:* $\mathcal{Q}_{41}$ *and* $\mathcal{Q}_{63}$
- *All others are representable over* $\mathbb{Q}$ *(proof by examples)*

Hence, there are 779777 (125 *orbits*) *quatroids represented by* 8 *distinct points.*



Quatroid 41
(MacLane)
Not all lines real



Candidate 63
(Fano Quatroid)
Aqua curve can't be (any) conic

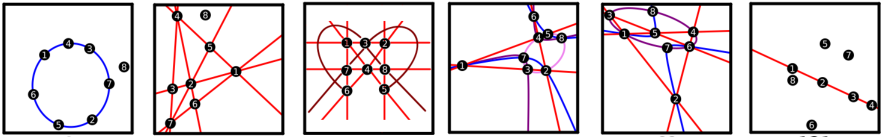
Theorem (Brysiewicz, Gesmundo, and Steiner (assisted by Salmon the Computer))

For each quatroid $\mathcal{Q}$, we determine the number $d_{\mathcal{Q}}$ of rational cubics through a generic point on the realization space of $\mathcal{Q}$. Those that have no rational cubics through them are those <u>*containing*</u> *a quatroid of the form*

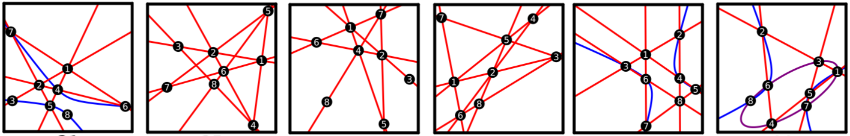
$$\mathcal{Q}_6, \mathcal{Q}_{32}, \mathcal{Q}_{41}, \mathcal{Q}_{59}, \mathcal{Q}_{62}$$

Those with $d_{\mathcal{Q}} = 1$ are

$$\mathcal{Q}_{31}, \mathcal{Q}_{35}, \mathcal{Q}_{36}, \mathcal{Q}_{40}, \mathcal{Q}_{53}, \mathcal{Q}_{66}$$



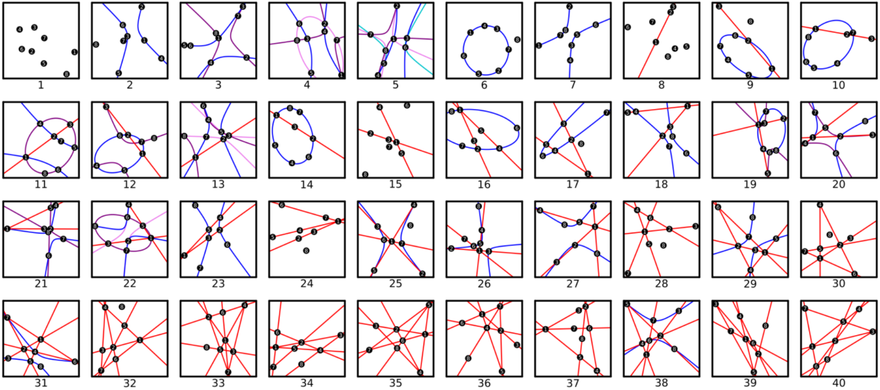
6 32 41 59 62 121



31 35 36 40 53 66

Proof: Lower bound (examples), upper bound (count the "steal")

#	Type	Lines	Conics	Bez.	Codim	Orbit	d_Q
1	1.0	$\{\}$	$\{\}$	true	0	1	12
2	1.1	$\{\}$	$\{123456\}$	true	1	28	10
3	1.2	$\{\}$	$\{123456, 123478\}$	true	2	210	8
4	1.3	$\{\}$	$\{123456, 123478, 125678\}$	true	3	420	6
5	1.4	$\{\}$	$\{123456, 123478, 125678, 345678\}$	true	4	105	4
6	1.7	$\{\}$	$\{1234567\}$	false	2	8	NB
7	1.28	$\{\}$	$\{12345678\}$	false	3	1	NB
8	2.0	$\{123\}$	$\{\}$	true	1	56	10
9	2.1	$\{123\}$	$\{124567\}$	true	2	840	8
10	2.1	$\{123\}$	$\{145678\}$	true	2	168	9
11	2.2	$\{123\}$	$\{124567, 134568\}$	true	3	3360	6
12	2.2	$\{123\}$	$\{124567, 345678\}$	true	3	840	7
13	2.3	$\{123\}$	$\{124567, 134568, 234578\}$	true	4	3360	4
14	2.7	$\{123\}$	$\{1245678\}$	false	3	168	NB
15	3.0	$\{123, 145\}$	$\{\}$	true	2	840	8
16	3.1	$\{123, 145\}$	$\{124678\}$	true	3	3360	6
17	3.1	$\{123, 145\}$	$\{234567\}$	true	3	2520	6
18	3.1	$\{123, 145\}$	$\{234678\}$	true	3	3360	7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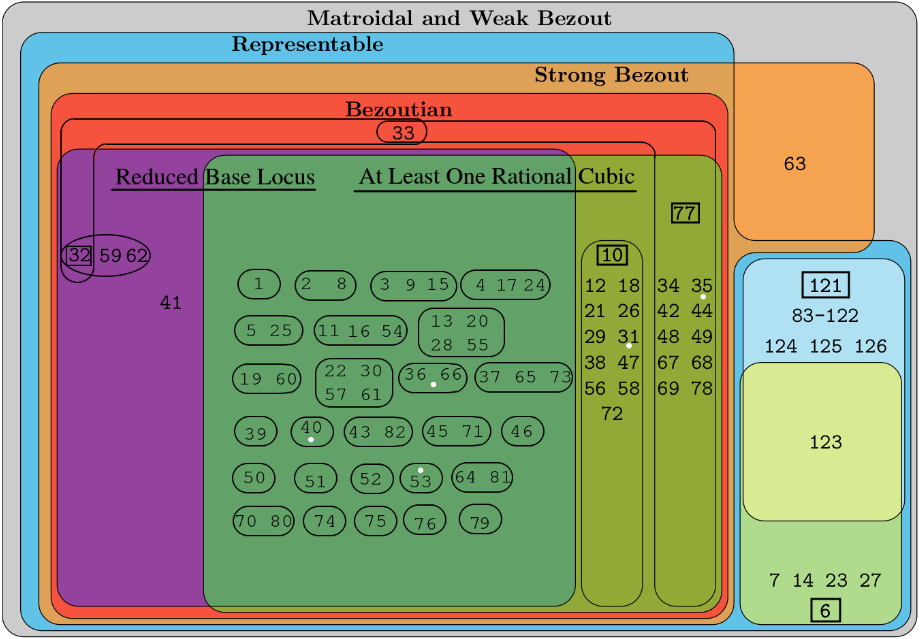
Quatroid #	Rational Represenative
1:	[5 0 -5 -7 -6 5 0 -5; -1 -4 2 7 3 2 1 -9; -2 -1 -3 8 6 9 -9 1]
2:	[24 0 0 24 -3 -1 -2 1; 0 24 0 24 -5 3 -3 -1; 0 0 24 24 -1 1 -4 -4]
3:	[-2 2 -1 -1 1 2 2 1; 2 -1 2 -1 2 -2 1 -2; -1 -1 -2 2 2 -1 2 -1]
4:	[2 1 -2 1 1 1 2 -2; -2 1 -1 1 2 -2 -1 1; 2 -1 -1 2 1 -1 0 1]
5:	[1 -1 0 1 2 2 0 -1; 2 0 1 1 1 -2 0 2; 0 -2 2 2 -2 -2 -1 0]
6:	[-2 0 2 0 -2 1 1 1; 2 2 1 -2 -1 2 -2 -1; -1 -2 0 -1 -1 1 1 -2]
7:	[1 -15 -8 0 -3 0 5 -4; 0 8 -6 -2 -4 2 0 -3; 1 17 10 2 5 2 -5 -5]
8:	[4 -2 -9 7 -4 2 9 -9; -6 1 -9 -3 -1 9 0 -6; 3 4 0 5 -8 -6 -5 1]
9:	[24 0 0 24 -1 3 -4 -5; 0 24 0 24 3 -1 4 5; 0 0 24 24 5 4 5 4]
10:	[24 0 0 24 -1 2 -2 1; 0 24 0 24 -4 4 4 3; 0 0 24 24 2 -3 -3 -3]
11:	[4 5 -2 -1 2 2 -4 2; 4 -5 2 -5 5 5 -4 2; 1 -5 -2 -2 -2 2 5 4]
12:	[-3 3 2 -3 -2 3 3 -2; 0 3 2 3 2 0 4 -2; 5 -1 -2 -1 5 4 1 -5]
13:	[-2 -1 1 0 -1 2 0 -1; 0 0 2 -2 -2 -2 2 0; -1 -2 2 -1 -1 -2 -2 1]
14:	[2 2 -2 -1 -1 2 2 -1; -2 -1 -1 1 -1 1 -1 2; 0 -2 0 2 1 -1 -1 1]
15:	[2 -1 -10 -2 6 10 3 1; 2 7 -10 -2 -1 -3 5 7; 7 0 9 -4 2 2 6 -10]
16:	[24 0 0 24 1 5 -5 -5; 0 24 0 24 0 -3 3 -2; 0 0 24 24 -5 -5 -1 4]
17:	[24 0 0 24 -1 -3 -5 -5; 0 24 0 24 -4 5 -2 0; 0 0 24 24 1 4 0 1]
18:	[24 0 0 24 2 1 -4 -2; 0 24 0 24 -3 4 -1 -2; 0 0 24 24 -3 2 1 5]
19:	[24 0 0 24 5 5 -1 -4; 0 24 0 24 3 -4 -5 2; 0 0 24 24 4 4 5 -3]
20:	[1 2 1 -2 2 -1 -1 2; -1 2 0 1 -1 -2 1 0; 1 0 0 2 1 1 2 -1]
21:	[2 2 -1 -1 2 1 1 -1; 1 0 -2 2 2 -1 1 1; 1 1 1 -2 -1 0 1 -1]
22:	[1 -2 -2 -2 2 0 -1 0; 0 1 2 -1 0 -1 1 2; -1 2 0 2 2 -2 2 -1]
23:	[2 0 1 2 -2 -1 2 0; 2 -2 1 -2 -1 1 1 -2; -1 -1 0 1 2 2 -1 0]



Matroidal and Weak Bezout
Representable
Strong Bezout
Bezoutian
33
Reduced Base Locus
At Least One Rational Cubic
63
77
32 59 62
41
1
2 8
3 9 15
4 17 24
5 25
11 16 54
13 20 28 55
19 60
22 30 57 61
36 66
37 65 73
39
40
43 82
45 71
46
50
51
52
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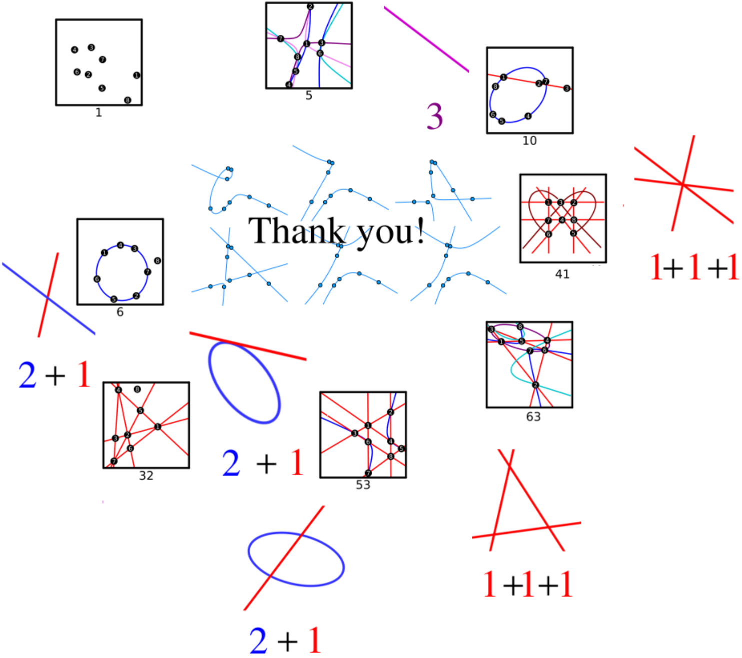
64 81
70 80
74
75
76
79
10
12 18 21 26 29 31 38 47 56 58 72
34 35 42 44 48 49 67 68 69 78
121
83–122
124 125 126
123
7 14 23 27
6


Thank you!
1
5
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41
1+1+1
6
2 + 1
32
2 + 1
53
63
1+1+1
2 + 1