Hard Combinatorial Problems: A Challenge for Satisfiability

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Joint work with Curtis Bright, Albert Heinle, Vijay Ganesh

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Satisfiability Checking and Symbolic Computation
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Talk Outline

1. Autocorrelation
2. Complementary Sequences
3. Hard Combinatorial Problems via Autocorrelation
4. SAT encodings of Autocorrelation
5. MathCheck
   https://sites.google.com/site/uwmathcheck
6. The new petaflop Canadian HPC landscape
7. Other significant Hard Combinatorial Problems
8. On-going and future work

Autocorrelation of finite sequences

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- The **periodic autocorrelation function** associated to a finite sequence $A = [a_0, \ldots, a_{n-1}]$ of length n is defined as

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where $k + s$ is taken modulo n, when $k + s \geq n$.

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- For sequences with complex number elements, a_{k+s} is replaced by $\overline{a_{k+s}}$.

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$$\begin{array}{lcl}
P_A(0) & = & {a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2 + {a_5}^2 + {a_6}^2 + {a_7}^2 \\
P_A(1) & = & a_1 a_2 + a_2 a_3 + a_3 a_4 + a_4 a_5 + a_5 a_6 + a_6 a_7 + a_7 a_1 \\
P_A(2) & = & a_1 a_3 + a_2 a_4 + a_3 a_5 + a_4 a_6 + a_5 a_7 + a_6 a_1 + a_7 a_2 \\
P_A(3) & = & a_1 a_4 + a_2 a_5 + a_3 a_6 + a_4 a_7 + a_5 a_1 + a_6 a_2 + a_7 a_3 \\
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Autoccorelation Properties

Circulant Matrices

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$$C(A) = \begin{bmatrix} a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_2 & a_3 & \dots & a_0 & a_1 \\ a_1 & a_2 & \dots & a_{n-1} & a_0 \end{bmatrix}$$

1. Consider a finite sequence $A = [a_0, \ldots, a_{n-1}]$ of length n and the circulant matrix $C(A)$ whose first row is equal to A. Then $P_A(i)$ is the inner product of the first row of $C(A)$ and the $i+1$ row of $C(A)$.

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   $\rightsquigarrow \quad P_A(1) + P_A(2) + \ldots + P_A(n-1) = 2e_2(a_0, \ldots, a_{n-1})$
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4. $\rightsquigarrow \quad N_A(s) + N_A(n-s) = P_A(s), s = 1, \ldots, n-1.$

Complementary Sequences

Definition

Let $\{A_i\}_{i=1,\dots,t}$ be t sequences of length v with complex elements. The sequences $\{A_i\}_{i=1,\dots,t}$ are called complementary, if

$$\sum_{i=1}^{t} PAF_{A_i} = [\alpha_0, \underbrace{\alpha, \dots, \alpha}_{v-1 \text{ terms}}]$$

with the convention:

$$PAF_{A_i} = [PAF_{A_i}(0), PAF_{A_i}(1), \dots, PAF_{A_i}(v-1)].$$

Algorithms and Metaheuristics for Combinatorial Matrices
Ilias S. Kotsireas
Handbook of Combinatorial Optimization, 2nd ed.
Pardalos, P. M., Du, D.-Z., Graham, R. L. (eds)
pp. 283-309, Springer 2013

Unified description of combinatorial objects

number/type of sequences	defining property	name
1 binary	aper. autoc. $0, \pm 1$	Barker sequences
1 ternary	per. autoc. 0	circulant weighing matrices
2 binary	aper. autoc. 0	Golay sequences
2 4-th roots	aper. autoc. 0	complex Golay sequences
2 binary	per. autoc. 0	Hadamard matrices
2 binary	per. autoc. 2	D-optimal matrices
2 binary	per. autoc. -2	Hadamard matrices
2 ternary	aper. autoc. 0	TCP
2 ternary	per. autoc. 0	Weighing matrices
3 binary	aper. autoc. const.	Normal sequences
4 binary	aper. autoc. 0	Base sequences
4 binary	aper. autoc. 0	Turyn type sequences
4 ternary	aper. autoc. 0	T-sequences
4 binary	per. autoc. 0	Williamson Hadamard
$2 \ldots 12$ binary	per. autoc. zero	PCS

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Definition

$PSD([a_1, \ldots, a_n], k)$ denotes the k-th element of the power spectral density sequence, i.e. the square magnitude of the k-th element of the discrete Fourier transform (DFT) sequence associated to $[a_1, \ldots, a_n]$.

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The DFT sequence associated to $[a_1, \ldots, a_n]$ is defined as

$$DFT_{[a_1,\ldots,a_n]} = [\,\mu_0, \ldots, \mu_{n-1}\,], \text{ with } \mu_k = \sum_{i=0}^{n-1} a_{i+1}\,\omega^{ik}, \; k = 0, \ldots, n-1$$

where $\omega = e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$ is a primitive n-th root of unity.

PSD criterion

Williamson Hadamard matrices: 4 complementary sequences of length n, (odd)
PAF constant: 0, PSD constant: $4n$.

$$PAF(A,s)+PAF(B,s)+PAF(C,s)+PAF(D,s)=0, \quad s=1,\ldots,\frac{n-1}{2}$$

$$PSD(A,s)+PSD(B,s)+PSD(C,s)+PSD(D,s)=4n, \quad s=1,\ldots,\frac{n-1}{2}$$

if for a certain sequence $A=[a_1,\ldots,a_n]$ there exists $s\in\{1,\ldots,n-1\}$ with the property that $PSD(A,s)>4n$, then this sequence cannot be used to construct 4 such complementary sequences

Important Consequence: we can now decouple the PAF equations, roughly corresponding to cutting down the complexity by four.

Claude Monet
Haystacks, End of Summer, (Meules, fin de l'été), 1891.
Oil on canvas. Musée d'Orsay, Paris, France.

Compression of complementary sequences

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Definition

Let $A = [a_0, a_1, \ldots, a_{v-1}]$ be a complex sequence of length $v = dm$. Set $a_j^{(d)} = a_j + a_{j+d} + \ldots + a_{j+(m-1)d}$, for $j = 0, \ldots, d-1$. Then we say that the sequence $A^{(d)} = [a_0^{(d)}, a_1^{(d)}, \ldots, a_{d-1}^{(d)}]$ of length d is the m-compression of A.

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PhD thesis of Yoseph Strassler, (1997), Bar-Ilan University, Israel.

Example

$A = CW(24, 9) =$
$[0, 0, 0, -1, -1, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 1, 0, 0, -1, -1]$

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$m = 2, \quad d = 12, \quad \rightsquigarrow \quad A^{(12)} = [0, 0, 0, -2, 0, 0, 0, 1, 0, 0, 0, -2]$

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$m = 2, \quad d = 12, \quad \rightsquigarrow \quad A^{(12)} = [0, 0, 0, -2, 0, 0, 0, 1, 0, 0, 0, -2]$
$m = 3, \quad d = 8, \quad \rightsquigarrow \quad A^{(8)} = [1, 0, 1, -1, -1, 0, -1, -2]$

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- Assume that $v = dm$ and set $a_{ij}^{(d)} = a_{i,j} + a_{i,j+d} + \cdots + a_{i,j+(m-1)d}, i = 1, \ldots, t, j = 0, \ldots, d-1.$

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$$\sum_{i=1}^{t} PAF_{A_i^{(d)}} = [\alpha_0 + (m-1)\alpha, \underbrace{m\alpha, \ldots, m\alpha}_{d-1 \text{ terms}}]$$

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$$\sum_{i=1}^{t} PAF_{A_i^{(d)}} = [\alpha_0 + (m-1)\alpha, \underbrace{m\alpha, \ldots, m\alpha}_{d-1 \text{ terms}}]$$

$$\sum_{i=1}^{t} PSD_{A_i^{(d)}} = [\beta_0, \underbrace{\beta, \ldots, \beta}_{d-1 \text{ terms}}]$$

Periodic Golay pairs of length 68

Consider the following two sequences of length 34 each, with $\{-2, 0, +2\}$ elements:

$$A^{(34)} = [0, 0, 0, 2, 0, 0, -2, 0, 0, 0, 2, -2, 0, 0, -2, 0, 0, 2, 0, 0, 0, 2, 2, -2, 0, 0, -2, 0, 0, 2, 0, 2, 0, 2]$$
$$B^{(34)} = [0, 0, -2, 2, 0, 2, 0, -2, -2, 0, 2, 2, 0, 2, -2, 0, 2, 0, -2, 2, 0, 2, 2, 0, 2, 0, 2, 2, 0, -2, 2, 0, -2, -2]$$

These two sequences satisfy the following properties:

1. $\mathrm{PAF}(A^{(34)}, s) + \mathrm{PAF}(B^{(34)}, s) = 0, s = 1, \ldots, 33;$
2. $\mathrm{PSD}(A^{(34)}, s) + \mathrm{PSD}(B^{(34)}, s) = 2 \cdot 68 = 136, s = 1, \ldots, 33;$
3. $\mathrm{PSD}(A^{(34)}, 17) = 100$ and $\mathrm{PSD}(B^{(34)}, 17) = 36;$
4. $\sum_{i=1}^{34} A_i^{(34)} = 6$ and $\sum_{i=1}^{34} B_i^{(34)} = 10; \quad 6^2 + 10^2 = 2 \cdot 68$
5. The total number of 0 elements in $A^{(34)}$ and $B^{(34)}$ is 34;
6. The total number of ± 2 elements in $A^{(34)}$ and $B^{(34)}$ is 34;
7. $A^{(34)}$ contains 21 zeros and $B^{(34)}$ contains 13 zeros.

Periodic Golay pairs of length 68

$A^{(34)}$ and $B^{(34)}$ are the 2-compressed sequences of two $\{-1,+1\}$ sequences of length 68 each, that form a particular **periodic Golay pair of length** 68:

$$
\begin{array}{ccl}
A & = & --++-+-+-++--+--++---++------+-+++ \\
  &   & ++-++---+-+-+--+-++++++-++-+++++-+ \\
B & = & ---+++---+++++--++-+-+++++++--+--- \\
  &   & ++-+-++---++-+-++--++++-+-+++-++--
\end{array}
$$

$\rightsquigarrow$ Hadamard matrices of order $2 \cdot 68$

Reference

Djokovic, Dragomir; Kotsireas, Ilias; Recoskie, Daniel; Sawada, Joe
Charm bracelets and their application to the construction of periodic Golay pairs. Discrete Appl. Math. 188 (2015), 32-40.

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https://docs.computecanada.ca/wiki/Graham

2015 http://top500.org/

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TIANHE-2	(MILKYWAY-2)
Site:	National Super Computer Center, Guangzhou
Cores:	3,120,000
Linpack Perf (Rmax)	33,862.7 TFlop/s
Theoretical Peak (Rpeak)	54,902.4 TFlop/s
Memory:	1,024,000 GB
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Compiler:	icc

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2007: open problem, 2^{50} ops $\rightsquigarrow$ 2015: ex. search in 10 minutes

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SAT encodings of Autocorrelation

D-optimal designs

S. Arunachalam, I. Kotsireas
Hard satisfiable 3-SAT instances via autocorrelation.
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SC 2014 Vienna

Proceedings of SAT COMPETITION 2014
Solver and Benchmark Descriptions
Anton Belov, Daniel Diepold, Marijn J.H. Heule, and Matti Järvisalo (editors)

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Williamson Hadamard matrices

Curtis Bright
Computational Methods for Combinatorial and Number Theoretic Problems
PhD Thesis, 2017, University of Waterloo

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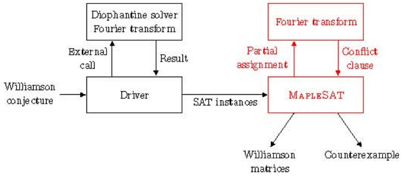
- We tell the SAT solver to learn the constraint

$$\neg(\{p_0 = 1\} \wedge \{p_2 = 1\})$$

Solution: Programmatic SAT

- A *programmatic* SAT solver[5] contains a special *callback* function which periodically examines the current partial assignment while the SAT solver is running.
- If it can determine that the partial assignment cannot be extended into a satisfying assignment then a conflict clause is generated encoding that fact.

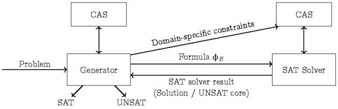


[5] V. Ganesh et al., LYNX: A programmatic SAT solver for the RNA-folding problem, SAT 2012.

MathCheck

https://sites.google.com/site/uwmathcheck

The MATHCHECK2 System



MathCheck main reference

Zulkoski, Edward; Bright, Curtis; Heinle, Albert; Kotsireas, Ilias; Czarnecki, Krzysztof; Ganesh, Vijay
Combining SAT solvers with computer algebra systems to verify combinatorial conjectures
J. Automat. Reason. 58 (2017), no. 3, pp. 313–339

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4. Complex Golay conjecture (2002): Complex Golay sequences do not exist for order 23.
   MathCheck: Confirmation of the conjecture

Other significant Hard Combinatorial Problems

- Marijn J. H. Heule (2018). Schur Number Five. Proceedings of AAAI-18, pp. 6598–6606.
- Marijn J. H. Heule (2018). Computing Small Unit-Distance Graphs with Chromatic Number 5. To appear in Geombinatorics XXVIII(1)
- Marijn J. H. Heule, Oliver Kullmann, and Armin Biere (2018). Cube and Conquer for Satisfiability. Handbook of Parallel Constraint Reasoning, Chapter 2, pp. 31-59.
- Marijn J. H. Heule (2017). Avoiding Triples in Arithmetic Progression. Journal of Combinatorics 8(3): 391–422

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> Is this now the limit of what we can do? It may very well be, but certainly advances will not be made by people who think they cannot succeed.
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