Finding generalized near-repdigit squares

1 Introduction

A repdigit to base $b$ is a natural number which consists of a single repeated digit when expressed in base $b$. A $k$-digit base $b$ repdigit has the general form

$$\underbrace{aaa \ldots a}_{k} = a \frac{b^k - 1}{b - 1}$$

with the repeated digit $0 \leq a < b$.

Obláth’s problem is to determine all decimal repdigits which are also perfect powers. Trivially, the single-digit perfect powers 4, 8 and 9 satisfy this criteria. Obláth [1] proved that any other such number must be a repunit, that is, having every digit equal to 1. Bugeaud and Mignotte [2] completed his problem by showing a repunit in base 10 cannot be a perfect power.

Similarly, a near-repdigit to base $b$ is a number having all digits but one equal in base $b$. Generalizing this, we define an $n$-near-repdigit to be a number with all equal digits except for an $n$ digit streak of a different digit. A $k$-digit $n$-near-repdigit to base $b$ has the general form

$$\underbrace{aa \ldots aa}_{k-n-m} \underbrace{cc \ldots cc}_{n} \underbrace{aa \ldots aa}_{m} = a \frac{b^k - 1}{b - 1} + (c - a) b^n \frac{b^m - 1}{b - 1}$$

with $k \geq n + m$, $a \neq c$ and $0 \leq a, c < b$.

Gica and Panaitopol [3] determined all square near-repdigits in base 10. Our objective was to automate the process of finding all $n$-near-repdigits in any base $b$, for given $n$ and $b$. A Maple program was developed which enumerates the possible cases and then attempts to find an upper bound on the $n$-near-repdigit digit length. If successful, the output consists of every satisfying number written in base $b$ and an abbreviated justification why each case admits only these solutions. Any problem cases are also noted.

2 Enumerating possible cases

First, we consider all possibilities for the last $d$ digits (base $b$) of any square number. Every natural number can be written in the form $b^dk + r$, for some $k$ and $0 \leq r < b^d$. Since $(b^dk + r)^2 \equiv r^2 \pmod{b^d}$, the final $d$ digits of $(b^dk + r)^2$
and $r^2$ are the same. Thus by checking if the last $d$ digits of $r^2$ (for $0 \leq r < b^d$) are also final digits of some $n$-near-repdigit, we can determine the possible forms for any square $n$-near-repdigit.

Rather than checking $r^2$ for all $0 \leq r < b^d$, the search can be simplified if we know the possible $r$'s which yield acceptable $d-1$ final digits. We “extend” such an $r$ by one digit by forming $b^{d-1}k + r$ for each $0 \leq k < b$. Since every natural number can be written as $b^d k' + b^{d-1}k + r \equiv b^{d-1}k + r \pmod{b^d}$, the possible last $d$ digits for square $n$-near-repdigits of this form are the last $d$ digits of $(b^{d-1}k + r)^2$. We simply compute this for $0 \leq k < b$ and record for which $k$ (if any) yield digits of some $n$-near-repdigit. By starting from a small number of digits and applying this repeatedly, every possible case for the last $d$ digits may be found without searching over every $0 \leq r < b^d$.

Once the possibilities for the last $d$ digits have been found, it is often possible to immediately identify the repeating digit (denoted $a$), the differing digit (denoted $c$), and the position at which the streak of different digits begins (denoted $m$). Sometimes it is not possible to distinguish the repeating digit from the differing digit$^1$, but this problem can be solved by choosing a larger value for $d$. A more serious problem occurs when the digits in a possible case are all the same; if $d > n$ then this digit cannot be $c$ (it must be $a$), but it is unknown what $c$ and (more importantly) $m$ are (though $m \geq d$).

3 Solving cases

3.1 Trivial cases

The case where the repeated digit is 0 is easily solved, since then an $n$-near-repdigit has the form $cb^m b^{n-1}$, which is a perfect square when $m$ is even and $c b^{n-1}$ is a perfect square, or when $m$ is odd and $cb^{b-1}$ is a perfect square. Both cases are checked for each $0 < c < b$. When successful, they yield the family of solutions $cb^i b^{n-1}$ or $c b^i b^{n-1}$ for $i > 0$.

3.2 Remaining cases

We start knowing $b$ and $n$, and from Section 2 we found all possible combinations of $a$, $c$ and $m$. Thus, we now need to find values of $k$ such that (1) is a perfect square.

3.2.1 Simple test

Let $A_k$ represent the $k$-digit $n$-near-repdigit of the case we are testing. Writing $A_k$ in terms of $A_{k-1}$ we find the recurrence relation $A_k = b A_{k-1} + a - (c - a) b^m (b^n - 1)$. Thus for any $h$ the sequence $\{A_k \pmod{h}\}_{k=n+m}$ must be periodic with period at most $h$. By computing this sequence for various primes

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$^1$For example, if the last 4 digits of a 3-near-repdigit are 1222, it could be of the form 11 · · · 11222 or 22 · · · 221111222.
(or prime powers) $h$, it is sometimes possible to find a sequence whose period is entirely composed of quadratic nonresidues (mod $h$).

### 3.2.2 Main test setup

Multiplying (1) by $b - 1$, we find that we want to solve the Diophantine equation

$$(b - 1)x^2 = ab^k - a + (c - a)b^m(b^n - 1)$$

for $x$ and $k$. We split this into two cases, for even and odd $k$ (say $k = 2M$ and $k = 2M + 1$). To ensure that the coefficient on $x^2$ is a perfect square we multiply by $b - 1$ again (this step can be skipped if $\sqrt{b - 1} \in \mathbb{Z}$), and then make the following substitutions:

$$y = (b - 1)x$$
$$z = b^M$$
$$N = (b - 1)((c - a)b^m(b^n - 1) - a)$$

Then we want to find the solutions $(y, z)$ for the pair of equations

$$y^2 - a(b - 1)z^2 = N$$
$$y^2 - a(b - 1)b^2z^2 = N,$$

thus reducing the problem to solving equations of the form

$$x^2 - Dy^2 = N. \quad (2)$$

#### 3.2.3 Solving $x^2 - Dy^2 = N$

When $D$ is a perfect square then (2) is a difference of squares and factorizes

$$(x + \sqrt{D}y)(x - \sqrt{D}y) = N,$$

thus all solutions can be found by examining the divisors of $N$. If $N = d_1d_2$ then we have $x = (d_1 + d_2)/2$ and $y = (d_1 - d_2)/2$, although this will only be an integer solution when $d_1 \equiv d_2 \pmod{2}$.

When $D$ is not a perfect square then (2) is a generalized form of Pell’s equation

$$x^2 - Dy^2 = 1, \quad (3)$$

which plays an important role when solving (2). Note that if $(x, y)$ is a solution of (2) and $(u, v)$ is a solution of (3), then $(ux + vyD, uy + vx)$ is also a solution of (2). Using this fact, solutions of (2) can be partitioned into equivalence classes: we say that $(x_1, y_1) \equiv (x_2, y_2)$ if there exists some solution $(u, v)$ of (3) such that $(ux_1 + vy_1D, uy_1 + vx_1) = (x_2, y_2)$.

We define the primitive solution to be the smallest positive solution of (3); it may be computed by examining convergents to $\sqrt{D}$ as described in [4] and [5]. Also, for each class of solutions of (2) we define the fundamental solution to be the solution $(x, y)$ with the smallest $y \geq 0$ in the class. The following theorem demonstrates how all solutions to (2) may be determined once all fundamental solutions are known.
Theorem 1 Let \((p_x, p_y)\) be the primitive solution of \(x^2 - Dy^2 = 1\), and \((f_x, f_y)\) be a fundamental solution of \(x^2 - Dy^2 = N\). Define the pair of linear recurrence relations:

\[
\begin{align*}
X_i &= 2p_x X_{i-1} - X_{i-2} \\
Y_i &= 2p_y Y_{i-1} - Y_{i-2}
\end{align*}
\]  

with initial conditions

\[
\begin{align*}
X_0 &= f_x \\
X_1 &= p_x f_x + p_y f_y D \\
Y_0 &= f_y \\
Y_1 &= p_x f_y + p_y f_x
\end{align*}
\]

Then all solutions to \(x^2 - Dy^2 = N\) in the class of \((f_x, f_y)\) are given by \(\pm (X_i, Y_i)\) for \(i \in \mathbb{Z}\).

\((X_i, Y_i)\) is well-defined for \(i < 0\) since rearranging (1) yields

\[
\begin{align*}
X_i &= 2p_x X_{i+1} - X_{i+2} \\
Y_i &= 2p_y Y_{i+1} - Y_{i+2}
\end{align*}
\]

Theorem 2 Let \((p_x, p_y)\) be the primitive solution of \(x^2 - Dy^2 = 1\), and \((f_x, f_y)\) be a fundamental solution of \(x^2 - Dy^2 = N\). Then

\[
\begin{align*}
0 &\leq f_y \leq \sqrt{\frac{N(p_x - 1)}{2D}} \quad \text{if } N > 0, \\
\sqrt{\frac{|N|}{D}} &\leq f_y \leq \sqrt{\frac{|N|(p_x + 1)}{2D}} \quad \text{if } N < 0.
\end{align*}
\]

Then for all integers \(y\) within these bounds we check if \((\sqrt{Dy^2 + N}, y)\) is an integer solution, in which case \((\pm \sqrt{Dy^2 + N}, y)\) are both fundamental solutions (unless they belong to the same class—in which case the class is called ambiguous and we choose the fundamental solution to have \(x > 0\)).
3.2.4 Comparing $y^2 - Dz^2 = N$ solutions with $z = b^M$

After solving (2) we want to be able to check which solutions actually correspond with $n$-near-repdigits. Because of our chosen substitution, $z$ must be some power of $b$. We will attempt to show that $z$ cannot be a power of $b$ when $z$ is larger than some explicitly computed bound.

Let $z_1(h)$ be the sequence $\{b^i \mod h\}_{i=0}^{\infty}$, this sequence remains the same while solving each $n$-near-repdigit case. Also, let $z_2(h)$ be the sequence $\{Z_i \mod h\}_{i=0}^{\infty}$, where $Z_i$ represents the recurrence relation in Theorem 1 applied to $y^2 - Dz^2 = N$, the relevant equation for whichever case is currently trying to be solved. It is obvious that both $z_1(h)$ and $z_2(h)$ must be periodic for all $h$; the following propositions are more exact.

**Proposition 1** Let $\{p_i\}$ be the prime factors of $b$; write $h = h' \prod p_i^{e_i}$, where $\gcd(h, h') = 1$. Then the pre-period of $z_1(h)$ is given by $\max\{e_i\}$ and the period of $z_1(h)$ is given by $\text{ord}_{h'}(b)$.

**Proposition 2** There is no pre-period of $z_2(h)$ for any $h$, and the period of $z_2(h)$ is no larger than $2h$.

*Proof.* Let $s$ and $t$ be the pre-period and period, respectively, of $z_1(h)$. Then $s \geq 0$ and $t > 0$ are the smallest integers such that $b^{s+t} \equiv b^s \pmod{h'}$. This is equivalent to the system of simultaneous congruences:

\[
\begin{align*}
  b^{s+t} &\equiv b^s \pmod{h'} \\
  b^{s+t} &\equiv b^s \pmod{p_i^{e_i}}
\end{align*}
\]

Dividing by coprime factors, this reduces to:

\[
\begin{align*}
  b^t &\equiv 1 \pmod{h'} \\
  p_i^{s+t} &\equiv p_i^t \pmod{p_i^{e_i}}
\end{align*}
\]

Since $t > 1$ the last congruence implies $p_i^s \equiv 0 \pmod{p_i^{e_i}}$. The smallest $s$ such that this is true for all $\{p_i\}$ will be $\max\{e_i\}$. And the smallest $t$ such that $b^t \equiv 1 \pmod{h'}$ is $\text{ord}_{h'}(b)$ by definition.

For the pre-period of $z_2(h)$, as already pointed out, the recurrence $Z_i$ is bi-directional so that you can determine a term in $z_2(h)$ based on terms that succeed it in the sequence. Thus, the periodicity that occurs must also extend backwards.

Let $k(h)$ be the period of $z_2(h)$. If we write $h = \prod q_i^{f_i}$ where $\{q_i\}$ are the prime factors of $h$, then $k(h) = \text{lcm}\{k(q_i^{f_i})\}$. The proof is similar to some found in [6].

Once we have found the pre-period and period length for both $z_1(h)$ and $z_2(h)$, all that remains is to compare the appropriate terms in the two sequences and find some $h$ such that the terms found in the repeating portion of $z_1(h)"

\[\text{For conciseness, only the congruence for the general case } p_i \text{ is shown, but it should be understood that there will be a single congruence for each prime factor.}\]
and those in the repeating portion of $z_2(h)$ have no elements in common. The entire pre-period of $z_1(h)$ also must be checked for solutions. Although finding a suitable $h$ is not difficult for most equations, unfortunately no $h$ was found in some cases, for example in $(3x)^2 - 3(4^{4\text{th}})^2 = 33$.

4 Example Results

All solutions to the near-repdigit square problem:

BASE 2:
only the trivial family $10^{2i}$ for $i > 0$

BASE 6:
13, 24, 41, 121, 144, 244, 441, 4424
and the trivial families $10^{2i}$, $4 \cdot 10^{2i}$ for $i > 0$

BASE 10:
16, 25, 36, 49, 64, 81, 121, 144, 225, 441, 484, 676, 1444, 44944
and the trivial families $10^{2i}$, $4 \cdot 10^{2i}$, $9 \cdot 10^{2i}$ for $i > 0$

All solutions to the 2-near-repdigit square problem:

BASE 2:
100, 1001, 11001, 1111001

BASE 6:
100, 144, 244, 400, 441, 3344, 11441

BASE 10:
100, 144, 225, 400, 441, 900, 7744, 11881, 55225

All solutions to the 3-near-repdigit square problem:

BASE 2:
110001

BASE 6:
none

BASE 10:
1444

References


