New Infinite Families of Perfect Quaternion Sequences and Williamson Sequences

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Abstract

We present new constructions for perfect and odd perfect sequences over the quaternion group Q_8. In particular, we show for the first time that perfect and odd perfect quaternion sequences exist in all lengths 2^t for $t \geq 0$. In doing so we disprove the quaternionic form of Mow's conjecture that the longest perfect Q_8-sequence that can be constructed from an orthogonal array construction is of length 64. Furthermore, we use a connection to combinatorial design theory to prove the existence of a new infinite class of Williamson sequences, showing that Williamson sequences of length $2^t n$ exist for all $t \geq 0$ when Williamson sequences of odd length n exist. Our constructions explain the abundance of Williamson sequences in lengths that are multiples of a large power of two.

Index Terms

Perfect sequences, quaternions, Williamson sequences, odd perfect sequences, periodic autocorrelation, odd periodic autocorrelation, array orthogonality property

I. Introduction

Sequences that have zero correlation with themselves after a nontrivial cyclic shift are known as *perfect* [1]. Such sequences have a long history [2] and an amazing wealth of applications, for example, appearing in the 3GPP LTE standard [3]. Perfect sequences and their generalizations have been applied to spread spectrum multiple access systems [4], radar systems [5], fast start-up equalization [6], channel estimation and synchronization [7], peak-to-average power ratio reduction [8], image watermarking [9], constructing complementary sets [10], and constructing sequences with small aperiodic correlations [11]. As recounted by B. M. Popović [12]:

> These sequences usually have small aperiodic autocorrelation and ambiguity function sidelobes, so they are very useful in the pulse compression radars.

One of the first researchers to study perfect sequences was Heimiller [13] who in 1961 gave a construction for perfect sequences using matrices with array orthogonality. His construction generated perfect sequences of length p^2 over the complex pth roots of unity for any prime p. Shortly after

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Heimiller's paper was published, Frank and Zadoff published a response [14] pointing out that Frank had discovered the same construction a decade prior as an aircraft engineer at the Sperry Gyroscope Company. Moreover, Frank was granted a patent for a communication system based on his sequences [15].

Frank's construction generates perfect sequences of length n^2 over the complex nth roots of unity. It has been conjectured by Mow [16] (and sometimes referred to as the Heimiller–Frank conjecture) that a construction that generates longer perfect sequences of this form does not exist. Kuznetsov [17] states the conjecture as follows and points out the importance of the conjecture to applications that rely on perfect sequences:

> It has been conjectured that there are no perfect sequences longer than n^2 over the n-complex roots of unity [...]. This conjecture, if true, imposes significant limits for the lengths of perfect sequences over the roots of unity, restricting a potential for practical applications which require longer sequences.

Given the theoretical elegance of perfect sequences and their importance to many fields of engineering it would be extremely interesting and useful if a construction could be found that produces perfect sequences longer than n^2 using nth roots of unity. However, over 60 years of effort has failed to find such a construction. In light of this, several researchers have searched for perfect sequences over other alphabets such as the group of quaternions. Quaternions are generalizations of the complex numbers that include the additional numbers j and k that satisfy the relationships

$$i^2 = j^2 = k^2 = -1, \qquad ij = k, \qquad jk = i, \qquad ki = j.$$

Note that these relationships imply that quaternions are generally noncommutative.

Communication systems have been described that are based on the quaternions [18], [19] and standard mathematical techniques like the Fourier transform have been generalized to the quaternions for usage in signal processing [20] and image processing [21]. Perfect sequences over the quaternions were first studied by Kuznetsov [22]. Kuznetsov and Hall [23] showed that Mow's conjecture cannot be directly extended to these sequences by constructing a perfect sequence over a quaternion alphabet with 24 elements and whose length is over 5 billion. In 2012, Acevedo and Hall [24] constructed perfect sequences over the alphabet $\{\pm 1, \pm i, j\}$ in all lengths of the form $q + 1$ where $q \equiv 1 \pmod 4$ is a prime power. Most recently, Blake [25] has run extensive searches for perfect sequences over the basic quaternion alphabet $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ (see Section VI for more on previous work). The longest perfect Q_8-sequence generated from an orthogonal array construction that Blake found had length 64. Intriguingly, this length is exactly the square of the alphabet size of 8. This lead Blake to conjecture a quaternionic form of Mow's conjecture that the quadratic Frank–Heimiller bound applies to perfect sequences over Q_8 generated from an orthogonal array construction:

> We conjecture, as an extension of the Heimiller–Frank conjecture, that longer perfect sequences with the array orthogonality property over the unit quaternions do not exist.

In this paper we show that Blake's conjecture is false. This is accomplished via a new construction for perfect quaternion sequences that can be used to construct arbitrarily long perfect Q_8-sequences

(see Section V) and odd perfect Q_8-sequences (see Section IV). In particular, we construct the first infinite family of odd perfect quaternion sequences and show for the first time that perfect and odd perfect quaternion sequences of length 2^t exist for all t. This is starkly different from what happens if one restricts the alphabet to only include the purely real or complex elements of Q_8. It is known that the longest perfect $\{\pm 1\}$-sequence of length 2^t is of length four [26] and the longest perfect $\{\pm 1, \pm i\}$-sequence of length 2^t is of length sixteen [27]. Additionally, the longest odd perfect $\{\pm 1\}$-sequence has length two [28].

Recently Acevedo and Dietrich [29], [30] discovered a relationship between perfect symmetric sequences over the quaternions and Williamson sequences from combinatorial design theory [31] (see Section II for the definition of Williamson sequences). Using this relationship we construct new Williamson sequences in even lengths, including all powers of two. Prior to our construction Williamson sequences of length 2^t were only known to exist for $t \leq 6$ [32]. See Section III for our Williamson sequence construction and Section IV for a construction for a variant of Williamson sequences that our Williamson sequence construction relies on.

In 1944, when Williamson introduced the sequences that now bear his name [31] he showed that the existence of Williamson sequences of odd length n implied the existence of Williamson sequences of length $2n$. In 1970, twenty-six years later, Turyn [27] generalized Williamson's result by showing that the existence of Williamson sequences of odd length n implied the existence of Williamson sequences of lengths $2^t n$ for $t \leq 4$. Nearly fifty years after Turyn's result, Acevedo and Dietrich [29] improved this to $t \leq 6$ in 2019. In this paper we complete this process of generalizing Williamson's doubling result by showing the result in fact holds for all $t \geq 0$. In other words, the existence of Williamson sequences of odd length n implies the existence of Williamson sequences of length $2^t n$ for all t and this provides a large new infinite class of Williamson sequences.

Exhaustive searches for Williamson sequences [33], [34] have shown that they exist in all lengths $n < 65$ except for $n = 35$, 47, 53, and 59. They are generally very abundant in the even lengths (particularly in the lengths that are divisible by a large power of two) but not in the odd lengths. For example, over 50,000 inequivalent sets of Williamson sequences exist in length 64 but fewer than 100 sets of Williamson sequences exist in all odd lengths up to 65. Previously this dichotomy was unexplained but the constructions that we provide in this paper produce approximately 75% of the Williamson sequences that exist in all even lengths $n \leq 70$ (see Section VII).

II. Preliminaries

In this section we provide the preliminaries necessary to explain our construction for perfect quaternion sequences, odd perfect quaternion sequences, and Williamson sequences. First we define the concept of sequence perfection in terms of the amount of correlation that a sequence has with cyclically shifted copies of itself. Following this, we discuss the array orthogonality property used in several constructions for perfect sequences. We then define Williamson sequences and finally present Acevedo and Dietrich's equivalence between perfect quaternion sequences and Williamson sequences.

A. Complementary sequences and perfect sequences

The *aperiodic* crosscorrelation of two sequences $A = [a_0, \ldots, a_{n-1}]$ and $B = [b_0, \ldots, b_{n-1}]$ of length n is given by

$$C_{A,B}(t) \coloneqq \sum_{r=0}^{n-t-1} a_r b_{r+t}^*$$

where z^* denotes the conjugate of z, i.e., $(x + y)^* \coloneqq x - y$ where x is purely real and y is purely imaginary (or quaternionic) [35]. The aperiodic autocorrelation of A is given by $C_A(t) \coloneqq C_{A,A}(t)$. More generally, we also define the *periodic* and *odd periodic* (or *negaperiodic*) crosscorrelations by

$$R_{A,B}(t) \coloneqq C_{A,B}(t) + C_{B,A}(n-t)^* \qquad \text{and} \qquad \hat{R}_{A,B}(t) \coloneqq C_{A,B}(t) - C_{B,A}(n-t)^* \qquad \text{for } 0 \le t < n.$$

The periodic and odd periodic autocorrelations of A are given by $R_A(t) \coloneqq R_{A,A}(t)$ and $\hat{R}_A(t) \coloneqq \hat{R}_{A,A}(t)$ respectively. These functions may be extended to all integers t via the expressions

$$R_{A,B}(t) \coloneqq \sum_{r=0}^{n-1} a_r b_{r+t \bmod n}^* \qquad \text{and} \qquad \hat{R}_{A,B}(t) \coloneqq \sum_{r=0}^{n-1} (-1)^{\lfloor (r+t)/n \rfloor} a_r b_{r+t \bmod n}^*.$$

A set S of sequences of length n is called *complementary* if $\sum_{A \in S} C_A(t) = 0$ for all $1 \le t < n$.[1] Similarly, S is called is called *periodic complementary* if $\sum_{A \in S} R_A(t) = 0$ and *odd periodic complementary* (or *negacomplementary*) if $\sum_{A \in S} \hat{R}_A(t) = 0$ for all $1 \le t < n$. A single sequence A is called *perfect* if $R_A(t) = 0$ for all $1 \le t < n$ and *odd perfect* if $\hat{R}_A(t) = 0$ for all $1 \le t < n$.

Note that the periodic correlation values $R_A(t)$ of a sequence are preserved under the *cyclic shift* operator $[a_0, \ldots, a_{n-2}, a_{n-1}] \mapsto [a_{n-1}, a_0, \ldots, a_{n-2}]$ and the negaperiodic correlation values $\hat{R}_A(t)$ of a sequence are preserved under the *negacyclic shift* operator $[a_0, \ldots, a_{n-2}, a_{n-1}] \mapsto [-a_{n-1}, a_0, \ldots, a_{n-2}]$ (see [36], [37]). Therefore applying a cyclic shift to any sequence in a set of periodic complementary sequences or applying a negacyclic shift to any sequence in a set of negacomplementary sequences does not disturb the property of the set being periodic complementary or negacomplementary.

Let $(-1) * X$ denote the sequence whose rth entry is $(-1)^r x_r$, i.e., the *alternating negation* operation. A set of periodic complementary sequences of odd length n can be converted into a set of negacomplementary sequences and vice versa by applying the alternating negation operation (see [38]).

Lemma 1. *If n is odd then (A, B, C, D) are periodic complementary sequences of length n if and only if $((-1) * A, (-1) * B, (-1) * C, (-1) * D)$ are negacomplementary sequences.*

Example 1. $(+{-}{-}, +{-}{-}, +{-}{-}, +{+}{+})$ is a set of complementary sequences of length 3 and $(+{+}{-}, +{+}{-}, +{+}{-}, +{-}{+})$ is the set of negacomplementary sequences generated from it using Lemma 1. Note that we follow the convention of writing 1s by $+$ and -1s by $-$.

[1] Technically S should be defined to be a multiset but we follow standard convention and refer to it as a set.

B. Matrices with array orthogonality

The sequences in a set S are *periodically uncorrelated* if any two distinct sequences A, B in S satisfy $R_{A,B}(t) = 0$ for all $0 \le t < n$. If a set of periodic complementary sequences $S = \{S_1, \ldots, S_m\}$ (with each sequence of length n a multiple of m) are periodically uncorrelated then the $n \times m$ matrix whose columns are given by $S_1, \ldots, S_m$ is said to have *array orthogonality*.

Matrices with array orthogonality are important because they are used in many constructions for perfect sequences such as in the Frank–Heimiller and Mow constructions. In particular, the sequence formed by concatenating the rows of a matrix with array orthogonality is perfect. Additionally, matrices with array orthogonality are themselves perfect arrays. *Perfect arrays* are often studied as a way of generalizing the concept of perfection from sequences to matrices (e.g., see [39], [40]). They are defined to be $n \times m$ matrices $A = (a_{r,s})$ that satisfy

$$\sum_{r=0}^{n-1} \sum_{s=0}^{m-1} a_{r,s} a^*_{r+t \bmod n, s+t' \bmod n} = 0 \qquad \text{for all } (t,t') \neq (0,0) \text{ with } 0 \le t < n,\ 0 \le t' < m.$$

Example 2. The columns of the matrix

$$\begin{bmatrix} + & + \\ + & - \end{bmatrix}$$

have array orthogonality and therefore this is a perfect array. It generates the perfect sequence $[+++-]$.

C. Williamson and nega Williamson sequences

First, we describe the symmetry properties that are used in the definition of Williamson sequences and will be important in our construction for Williamson sequences. A sequence A of length n is called *symmetric* if $a_t = a_{n-t}$ for all $1 \le t < n$ and is *palindromic* if $a_t = a_{n-t-1}$ for all $0 \le t < n$. For example, $[x, y, z, y]$ is symmetric and $[y, z, y]$ is palindromic. Note that a sequence is symmetric if and only if the subsequence formed by removing its first element is palindromic. Additionally, we call a sequence *antipalindromic* if $a_t = -a_{n-t-1}$ for all $0 \le t < n$ and *antisymmetric* if $a_t = -a_{n-t}$ for all $1 \le t < n$. If $[X;Y]$ denotes sequence concatenation and $\tilde{X}$ denotes the reversal of X then $[X;\tilde{X}]$ is palindromic and $[X;-\tilde{X}]$ is antipalindromic.

A quadruple of symmetric $\{\pm 1\}$-sequences (A, B, C, D) are known as *Williamson* if they are periodic complementary, i.e., if $R_A(t) + R_B(t) + R_C(t) + R_D(t) = 0$ for all $1 \le t < n$. Additionally, we call a quadruple of $\{\pm 1\}$-sequences (A, B, C, D) *nega Williamson* if they are negacomplementary, i.e., $\hat{R}_A(t) + \hat{R}_B(t) + \hat{R}_C(t) + \hat{R}_D(t) = 0$ for all $1 \le t < n$.

As is conventional, we require by definition that Williamson sequences are symmetric. For nega Williamson sequences we do not require them to be symmetric. Instead, our work has discovered the importance of *palindromic* and *antipalindromic* nega Williamson sequences; see Section III for details.

Example 3. $(++, ++, +-, +-)$ are Williamson sequences of length 2. Similarly, $(+-, +-, +-, +-)$ are antipalindromic nega Williamson sequences and $(++, ++, ++, ++)$ are palindromic nega Williamson

sequences. $(++-+, ++-+, ++-+, ++-+)$ are Williamson sequences of length 4. Similarly, $(+-+-, +-+-, ++--, ++--)$ are antipalindromic nega Williamson sequences and $(+--+, +--+, ++++, ++++)$ are palindromic nega Williamson sequences.

D. The Acevedo–Dietrich construction

Let $Q_8 \coloneqq \{\pm 1, \pm i, \pm j, \pm k\}$ be the group generated by the fundamental unit quaternions and $Q_+ \coloneqq Q_8 \cup qQ_8$ where $q \coloneqq (1 + i + j + k)/2$ (note that Q_+ is a set of sixteen quaternions that is *not* a group). Acevedo and Dietrich [29, Theorem 2.4] show that there is an equivalence between Williamson sequences and perfect sequences over Q_+.

Theorem 1. *There is a one-to-one correspondence between sets of Williamson sequences of length n and symmetric perfect sequences of length n over Q_+.*

This correspondence is made explicit through the following mapping between the rth entries of a set of Williamson sequences (a_r, b_r, c_r, d_r) and the rth entry of the corresponding perfect sequence s_r:

a_r	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
b_r	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$
c_r	$-$	$-$	$-$	$+$	$-$	$-$	$+$	$+$
d_r	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$+$
s_r	1	i	j	k	q	qi	qj	qk

Additionally, we have the rule that if (a_r, b_r, c_r, d_r) maps to s_r then $(-a_r, -b_r, -c_r, -d_r)$ maps to $-s_r$.

Because of this theorem a construction for Williamson sequences of length n also produces perfect Q_+-sequences. In this paper we state our construction in terms of Williamson sequences with the understanding that it equivalently produces perfect Q_+-sequences. In Section V we also show how our construction can be used to produce perfect Q_8-sequences in many lengths including all powers of two.

If the entries of a set of Williamson sequences (A, B, C, D) of length n satisfy $a_r b_r c_r d_r = 1$ for all $0 \le r < n$ then the Acevedo–Dietrich construction produces perfect Q_8-sequences. For this reason we say that a quadruple of sequences has the Q_8-*property* if the entries of its sequences satisfy $a_r b_r c_r d_r = 1$ for all $0 \le r < n$. In his original paper [31] Williamson proved that the entries of all Williamson sequences in odd lengths n satisfy $a_r b_r c_r d_r = -a_0 b_0 c_0 d_0$ for $1 \le r < n$. As a consequence, no Williamson sequence of odd length $n > 1$ can have the Q_8-property. However, many Williamson sequences in even lengths have the Q_8-property. Exhaustive searches [34] have found Williamson sequences with the Q_8-property in all even lengths $n \le 2^5$ except for 6, 12, and 28.

Acevedo and Dietrich also use the periodic product construction of Lüke [41] that generates perfect sequences from shorter perfect sequences. Let $X \times Y$ be the sequence whose rth entry is $x_{r \bmod n} y_{r \bmod m}$ for $0 \le r < nm$ (where X has length n and Y has length m).

Theorem 2. *Suppose X and Y have coprime lengths n and m. If X is a perfect Q_8-sequence and Y is a perfect Q_+-sequence then $X \times Y$ is a perfect Q_+-sequence. Furthermore, if X and Y are symmetric then $X \times Y$ is symmetric.*

Theorems 1 and 2 immediately yield the following corollary.

Corollary 1. *If a perfect symmetric Q_8-sequence of length 2^t exists and Williamson sequences of odd length n exist then Williamson sequences of length $2^t n$ exist.*

Acevedo and Dietrich also provide examples of symmetric perfect Q_8-sequences for all $t \leq 6$. In this paper we extend their result by showing that perfect symmetric Q_8-sequences of length 2^t exist for all $t \geq 0$ and therefore show Williamson sequences exist in all lengths of the form $2^t n$ whenever Williamson sequences exist in odd length n.

Example 4. Using the Acevedo–Dietrich construction the Williamson sequences (++−+, ++−+, ++−+, ++−+) produce the perfect Q_8-sequence [−−+−], the Williamson sequences (++−−+, −+−−+, −++++, −++++) produce the perfect Q_+-sequence [q−jj−] and the Williamson sequences (++−−+−−+, ++−−+−−+, +++−+−++, +++−+−++) produce the perfect Q_8-sequence [−−j+−+j−]. (We denote i, j, k, and q by i, j, k, and q.)

III. Constructions for Williamson sequences

Our main construction is based on the following simple sequence operations.

1) The *doubling* of X, denoted by $\mathrm{d}(X)$, i.e., $\mathrm{d}(X) \coloneqq [x_0, \ldots, x_{n-1}, x_0, \ldots, x_{n-1}]$.
2) The *negadoubling* of X, denoted by $\mathrm{n}(X)$, i.e., $\mathrm{n}(X) \coloneqq [x_0, \ldots, x_{n-1}, -x_0, \ldots, -x_{n-1}]$.
3) The *interleaving* of X and Y, denoted by $X \amalg Y$, i.e., $X \amalg Y \coloneqq [x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}]$.

We will use the following properties of these operations in our construction. For completeness, proofs of these properties are given in the appendix. In each property X and Y denote arbitrary sequences of the same length.

1) $R_{\mathrm{d}(X)}(t) = 2R_X(t)$.
2) $R_{\mathrm{n}(X)}(t) = 2\hat{R}_X(t)$.
3) $R_{\mathrm{d}(X),\mathrm{n}(Y)}(t) = 0$ and $R_{\mathrm{n}(Y),\mathrm{d}(X)}(t) = 0$.
4) $R_{X \amalg Y}(2t) = R_X(t) + R_Y(t)$.
5) $R_{X \amalg Y}(2t+1) = R_{X,Y}(t) + R_{Y,X}(t+1)$.
6) If X is symmetric then $\mathrm{d}(X)$ is symmetric.
7) If X is antipalindromic and of even length then $\mathrm{n}(X)$ is palindromic.
8) If X is symmetric and Y is palindromic then $X \amalg Y$ is symmetric.

Our main construction for Williamson sequences is given by the following theorem.

Theorem 3. *If (A, B, C, D) are Williamson sequences of even length n and (A', B', C', D') are antipalindromic nega Williamson sequences of length n then*

$$(\mathrm{d}(A) \amalg \mathrm{n}(A'), \mathrm{d}(B) \amalg \mathrm{n}(B'), \mathrm{d}(C) \amalg \mathrm{n}(C'), \mathrm{d}(D) \amalg \mathrm{n}(D'))$$

are Williamson sequences of length $4n$.

This theorem can be used to construct a large number of new Williamson sequences, assuming that sequences that satisfy the preconditions are known. For example, if N Williamson sequences of length n are known and M antipalindromic nega Williamson sequences of length n are known this theorem immediately implies that at least NM Williamson sequences of length $4n$ exist.

We now prove Theorem 3.

Proof. Let $X_{\text{new}} \coloneqq \mathrm{d}(X) \amalg \mathrm{n}(X')$ for $X \in \{A, B, C, D\}$. By construction it is clear that X_{new} is a $\{\pm 1\}$-sequence of length $4n$. Additionally, X_{new} is symmetric by property 8 since $\mathrm{d}(X)$ is symmetric by property 6 and $\mathrm{n}(X')$ is palindromic by property 7. It remains to show that

$$R_{A_{\text{new}}}(t) + R_{B_{\text{new}}}(t) + R_{C_{\text{new}}}(t) + R_{D_{\text{new}}}(t) = 0 \qquad \text{for } 1 \leq t < 4n.$$

Note that by properties 4 and 5 we have

$$R_{X_{\text{new}}}(t) = \begin{cases} R_{\mathrm{d}(X)}(t/2) + R_{\mathrm{n}(X')}(t/2) & \text{if } t \text{ is even,} \\ R_{\mathrm{d}(X),\mathrm{n}(X')}\left(\frac{t-1}{2}\right) + R_{\mathrm{n}(X'),\mathrm{d}(X)}\left(\frac{t+1}{2}\right) & \text{if } t \text{ is odd.} \end{cases}$$

By property 3 we have $R_{X_{\text{new}}}(t) = 0$ when t is odd. When t is even by properties 1 and 2 we have

$$R_{X_{\text{new}}}(t) = 2R_X(t/2) + 2\hat{R}_{X'}(t/2).$$

When $t = 2n$ we have $R_X(t/2) = n$ and $\hat{R}_{X'}(t/2) = -n$ (because X and X' both have length n) so $R_{X_{\text{new}}}(t) = 0$ in this case. Otherwise we have

$$\sum_{X=A,B,C,D} R_X(t/2) = 0 \quad \text{and} \quad \sum_{X=A,B,C,D} \hat{R}_{X'}(t/2) = 0 \qquad \text{for even } t \neq 2n \text{ with } 1 \leq t < 4n$$

since (A, B, C, D) are Williamson sequences and (A', B', C', D') are nega Williamson sequences. It follows that $\sum_{X=A,B,C,D} R_{X_{\text{new}}}(t) = 0$, as required. □

Example 5. Using the Williamson sequences $(++-+, ++-+, ++-+, ++-+)$ and antipalindromic nega Williamson sequences $(+-+-, +-+-, ++--, ++--)$ in Theorem 3 produces the Williamson sequences $(+++--++-+-++--++, +++--++-+-++--++, ++++--+-+-+--+++, ++++--+-+-+--+++)$.

Note that the assumption that n is even is essential to the theorem. If n is odd the constructed sequences will not be symmetric. Additionally, antipalindromic nega Williamson sequences do not exist in odd lengths $n > 1$, as we now show.

Lemma 2. *Antipalindromic nega Williamson sequences do not exist in odd lengths except for $n = 1$.*

Proof. Let (A, B, C, D) be a hypothetical set of antipalindromic nega Williamson sequences of odd length n. Note that a sequence X of odd length n is antipalindromic if and only if $X' := (-1) * X$ is antipalindromic. By Lemma 1, it follows that (A', B', C', D') are antipalindromic periodic complementary sequences. If $\mathrm{sum}(X)$ denotes the rowsum of X it is well-known (e.g., via the Wiener–Khinchin theorem or sequence compression [42]) that

$$\mathrm{sum}(A')^2 + \mathrm{sum}(B')^2 + \mathrm{sum}(C')^2 + \mathrm{sum}(D')^2 = 4n.$$

However, if X' is antipalindromic we have $\mathrm{sum}(X') = 1$ and the above sum of squared rowsums must be equal to four, implying that $n = 1$. □

Although the construction in Theorem 3 requires sequences of even lengths there is a variant of the construction that uses sequences of odd lengths. The proof is similar and uses the following additional properties.

9) If X is antisymmetric and of odd length then $\mathrm{n}(X)$ is symmetric.
10) If X is palindromic then $\mathrm{d}(X)$ is palindromic.

Theorem 4. *Let n be odd and let (A, B, C, D) and (A', B', C', D') be quadruples of sequences of length n. If (A, B, C, D) is the result of applying $(n-1)/2$ cyclic shifts to each member in a set of Williamson sequences and (A', B', C', D') is the result of applying $(n+1)/2$ negacyclic shifts to each member in a set of palindromic nega Williamson sequences then*

$$(\mathrm{n}(A') \amalg \mathrm{d}(A), \mathrm{n}(B') \amalg \mathrm{d}(B), \mathrm{n}(C') \amalg \mathrm{d}(C), \mathrm{n}(D') \amalg \mathrm{d}(D))$$

are Williamson sequences of length $4n$.

Proof. Since a symmetric sequence of odd length n becomes palindromic after $(n-1)/2$ cyclic shifts, (A, B, C, D) are palindromic. Since a palindromic sequence of odd length n becomes antisymmetric after $(n+1)/2$ negacyclic shifts, (A', B', C', D') are antisymmetric.

Let $X_{\mathrm{new}} := \mathrm{n}(X') \amalg \mathrm{d}(X)$ for $X \in \{A, B, C, D\}$. By construction it is clear that X_{new} is a $\{\pm 1\}$-sequence of length $4n$. Additionally, X_{new} is symmetric by property 8 since $\mathrm{n}(X')$ is symmetric by property 9 and $\mathrm{d}(X)$ is palindromic by property 10. The remainder of the proof now proceeds as in the proof of Theorem 3. □

Example 6. Note $(++{-}{-}+, -+{-}{-}+, -++++, -++++)$ are Williamson sequences and $(+{-}{-}{-}+, ++{-}++, +{-}{-}{-}+, +++++)$ are palindromic nega Williamson sequences. Then using $(A, B, C, D) = (-+++-, -+-+-, ++-++, ++-++)$ and $(A', B', C', D') = (++-+-, +--++, ++-+-, ---++)$ in Theorem 4 produces the following Williamson sequences of length 20:

$$(+-++-+++-----+++-++-, +--+--+++--+++--+--,$$
$$++++--++-+-+-++--+++, -+-+--++++++++--+-+).$$

IV. Nega Williamson sequences and odd perfect sequences

Although antipalindromic nega Williamson sequences do not exist in odd lengths larger than 1 we now present constructions showing that they exist in many even lengths. First, note that when the length is even there is a correspondence between palindromic and antipalindromic nega Williamson sequences. In other words, to use Theorem 3 it is sufficient to find palindromic nega Williamson sequences sequences.

Lemma 3. *There is a one-to-one correspondence between palindromic and antipalindromic nega Williamson sequences in even lengths.*

Proof. Applying $n/2$ negacyclic shifts to each sequence in a set of palindromic nega Williamson sequences of even length n yields a set of antipalindromic nega Williamson sequences. Similarly, a set of palindromic nega Williamson sequences can be produced from a set of antipalindromic nega Williamson sequences by applying the negacyclic shift operator $n/2$ times. □

Example 7. $(+--++-, +-+-+-, ++-+--, +++---)$ are antipalindromic nega Williamson sequences that may be converted into the palindromic nega Williamson sequences $(--++--, +-+++-+, -++++-, ++++++)$ using the translation in Lemma 3.

We now provide a construction that shows that infinitely many palindromic nega Williamson sequences exist. Recall that $\tilde{X}$ denotes the reverse of the sequence X and $[X;Y]$ denotes the concatenation of X and Y. Note that if X and Y have length n then $C_{[X;Y]}(t) = C_X(t) + C_Y(t) + C_{Y,X}(n-t)^*$ and $C_{[X;Y]}(2n-t) = C_{X,Y}(n-t)$ for $0 \leq t \leq n$. First we prove a simple lemma.

Lemma 4. *Applying the negadoubling operator to the sequences in a complementary set produces a negacomplementary set.*

Proof. Note that if X is a sequence of length n then $\hat{R}_{\mathrm{n}(X)}(t) = 2C_X(t)$ for $0 \leq t \leq n$; by definition we have $\hat{R}_{\mathrm{n}(X)}(t) = C_{\mathrm{n}(X)}(t) - C_{\mathrm{n}(X)}(2n-t)^*$ and when $0 \leq t \leq n$ we have $C_{\mathrm{n}(X)}(t) = C_X(t) + C_{-X}(t) + C_{-X,X}(n-t)^* = 2C_X(t) - C_X(n-t)^*$ and $C_{\mathrm{n}(X)}(2n-t) = C_{X,-X}(n-t) = -C_X(n-t)$. Thus $\hat{R}_{\mathrm{n}(X)} = 2C_X(t) - C_X(n-t)^* + C_X(n-t)^* = 2C_X(t)$.

Suppose S is a complementary set of sequences of length n. Using the above property we have $\sum_{A\in S} \hat{R}_{\mathrm{n}(A)}(t) = 2\sum_{A\in S} C_A(t)$ for all $0 \leq t < n$. Since S is complementary this implies $\sum_{A\in S} \hat{R}_{\mathrm{n}(A)}(t) = 0$ for all $1 \leq t \leq n$. Using the symmetry $\hat{R}_X(t) = \hat{R}_X(-t)^*$ shows that this also holds for all $n < t < 2n$. □

Example 8. Using the set of complementary sequences $(+++, +--, +-+, ++-)$ with Lemma 4 produces the set of negacomplementary sequences $(+++---, +---++, +-+-+-, ++---+)$.

Theorem 5. *If A and B are complementary $\{\pm 1\}$-sequences (i.e., (A, B) is a Golay pair) then*

$$([A; B; \tilde{B}; \tilde{A}], [\tilde{B}; \tilde{A}; A; B], [-\tilde{B}; \tilde{A}; A; -B], [-A; B; \tilde{B}; -\tilde{A}])$$

and

$$([A; B; \tilde{B}; \tilde{A}; A; B; \tilde{B}; \tilde{A}], [A; B; -\tilde{B}; -\tilde{A}; -A; -B; \tilde{B}; \tilde{A}],$$
$$[A; -B; \tilde{B}; -\tilde{A}; -A; B; -\tilde{B}; \tilde{A}], [A; -B; -\tilde{B}; \tilde{A}; A; -B; -\tilde{B}; \tilde{A}])$$

are sets of palindromic nega Williamson sequences with the Q_8-property.

Proof. The fact that the sequences are palindromic is immediate from the manner in which they were constructed. Note that $(\tilde{A}, \tilde{B})$ is a Golay pair since (A, B) is a Golay pair. Thus $(A, B, \tilde{B}, \tilde{A})$ is a complementary set and since

$$\begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix}$$

is an orthogonal matrix the sequences

$$([A; B; \tilde{B}; \tilde{A}], [A; B; -\tilde{B}; -\tilde{A}], [A; -B; \tilde{B}; -\tilde{A}], [A; -B; -\tilde{B}; \tilde{A}]) \qquad (*)$$

form a complementary set of sequences by [43, Thm. 7]. Since they are complementary they are also negacomplementary and applying $2n$ negacyclic shifts (where A and B are of length n) to the second and third sequences and negating the fourth shows the first set in the theorem is negacomplementary.

Since $(*)$ are complementary, by Lemma 4 applying the negadoubling operator to these sequences will produce negacomplementary sequences. In other words,

$$([A; B; \tilde{B}; \tilde{A}; -A; -B; -\tilde{B}; -\tilde{A}], [A; B; -\tilde{B}; -\tilde{A}; -A; -B; \tilde{B}; \tilde{A}],$$
$$[A; -B; \tilde{B}; -\tilde{A}; -A; B; -\tilde{B}; \tilde{A}], [A; -B; -\tilde{B}; \tilde{A}; -A; B; \tilde{B}; -\tilde{A}])$$

are negacomplementary. Applying $4n$ negacyclic shifts (where A and B are of length n) to the first and last sequences shows the second set in the theorem is negacomplementary.

Lastly, we show that the produced sequences X, Y, U, and V have the Q_8-property. In the first set we have $x_r = a_r$, $y_r = b_{n-r+1}$, $u_r = -y_r$, and $v_r = -x_r$ for all $0 \le r < n$ and in the second set we have $x_r = y_r = u_r = v_r = a_r$ for all $0 \le r < n$. In each case $x_r y_r u_r v_r = 1$ for all $0 \le r < n$ and more generally we have $x_r y_r u_r v_r = 1$ for all $0 \le r < 4n$ in the first set (and $0 \le r < 8n$ in the second set). □

Example 9. Using the Golay pair $(++, +-)$ the first set of palindromic nega Williamson sequences generated by Theorem 5 is $(+++--+++, -+++++-, +-++++-+, --+--+--)$ and the second set is $(+++--++++++--+++, +++-+------+-+++, ++-+-+----+-+-++, ++-++-++++-++-++)$.

Golay sequences, originally defined by Golay [44], are known to exist in lengths 2, 10, and 26 [45]. Since Turyn has shown that Golay pairs in lengths n and m can be composed to form Golay pairs in length nm [46] they also exist in all lengths of the form $2^a5^b13^c$ with $a \geq b + c$. Thus, Theorem 5 implies that palindromic nega Williamson sequences with the Q_8-property exist in all lengths of the form $2^a5^b13^c$ with $a \geq b+c+2$. In particular, taking $b = c = 0$ gives that palindromic nega Williamson sequences with the Q_8-property exist in all lengths that are powers of two (since palindromic nega Williamson sequences with the Q_8-property of length 2 exist, see Example 3). These are not the only lengths in which palindromic nega Williamson sequences exist, however. Palindromic nega Williamson sequences in many other lengths may be constructed using Lüke's product construction with odd perfect sequences.

First, note that in odd lengths there is an equivalence between Williamson sequences and palindromic nega Williamson sequences.

Lemma 5. *There is a one-to-one correspondence between palindromic nega Williamson sequences and Williamson sequences in odd lengths.*

Proof. Let (A, B, C, D) be a set of Williamson sequences of odd length n. By Lemma 1, $((-1) * A, (-1) * B, (-1) * C, (-1) * D)$ will be a set of nega Williamson sequences. The sequences in this set will be antisymmetric since if X is symmetric and of odd length then $(-1) * X$ is antisymmetric. Applying $(n-1)/2$ negacyclic shifts to each sequence in this set produces a set of palindromic nega Williamson sequences. Similarly, arbitrary palindromic nega Williamson sequences of odd length can be transformed into Williamson sequences by applying the inverse of the above transformations. □

Example 10. The set of Williamson sequences $(-++--++, ---++--, -+----+, -+----+)$ generates the set of palindromic nega Williamson sequences $(++--++, --+-+--, +----+, +----+)$ and vice versa using the transformation in Lemma 5.

Since Williamson sequences are known to exist for all lengths $n < 35$ this implies that palindromic nega Williamson sequences exist for all odd lengths up to 33.

Next, we note that odd perfect sequences in even lengths can be composed with odd perfect sequences in odd lengths to generate longer odd perfect sequences. Let $X \hat{\times} Y$ be the sequence whose rth entry is $(-1)^{\lfloor r/n \rfloor + \lfloor r/m \rfloor} x_{r \bmod n} y_{r \bmod m}$ for $0 \leq r < nm$ (where X has length n and Y has length m).

Lemma 6. *Suppose X and Y have coprime lengths n and m, one of which is even. If X is odd perfect and Y is odd perfect then $X \hat{\times} Y$ is odd perfect. Furthermore, if X and Y are palindromic then $X \hat{\times} Y$ is antipalindromic.*

Proof. The fact that $X \hat{\times} Y$ is odd perfect follows from $\hat{R}_{X \hat{\times} Y}(t) = \hat{R}_X(t)\hat{R}_Y(t)$ as given in [28, Eq. 9]. Suppose $0 \leq r < nm$ is arbitrary. Since X and Y are palindromic we have

$$x_{nm-r-1 \bmod n} y_{nm-r-1 \bmod m} = x_{n-r-1 \bmod n} y_{m-r-1 \bmod m} = x_{r \bmod n} y_{r \bmod m}.$$

Thus the $(nm-r-1)$th entry of $X \hat{\times} Y$ is

$$(-1)^{\lfloor (nm-r-1)/n \rfloor + \lfloor (nm-r-1)/m \rfloor} x_{r \bmod n} y_{r \bmod m} = (-1)^{n+m+\lfloor -(r+1)/n \rfloor + \lfloor -(r+1)/m \rfloor} x_{r \bmod n} y_{r \bmod m}$$

Using the fact that $\lfloor -(r+1)/s \rfloor = -(\lfloor r/s \rfloor + 1)$ for positive integers r, s and that $n+m$ is odd this becomes

$$(-1)^{1+\lfloor r/n \rfloor + \lfloor r/m \rfloor} x_{r \bmod n} y_{r \bmod m}$$

which is the negative of the rth entry of $X \hat{\times} Y$ as required. □

Example 11. The palindromic odd perfect sequences $X = [++]$ and $Y = [+\mathtt{q}+]$ used with Lemma 6 produces the antipalindromic odd perfect sequence $[+\mathtt{q}-+\mathtt{Q}-]$. (We use $\mathtt{Q}$ to denote $-q$.)

We now show that infinitely many palindromic odd perfect Q_8-sequences exist using a variant of Theorem 5 and the Acevedo–Dietrich correspondence.

Theorem 6. *If (A, B) is a Golay pair then*

$$P := [-A; jB; k\tilde{B}; i\tilde{A}; iA; kB; j\tilde{B}; -\tilde{A}]$$

is a palindromic odd perfect Q_8-sequence.

Proof. This follows by a direct but tedious calculation of $\hat{R}_P(t)$; we give an example of how this may be done. Let $P = [P'; \tilde{P'}]$ have length $8n$. We find for $0 \le t \le 4n$ that

$$\begin{aligned}\hat{R}_P(t) &= C_{[P';\tilde{P'}]}(t) - C_{[P';\tilde{P'}]}(8n-t)^* \\ &= C_{P'}(t) + C_{\tilde{P'}}(t) + C_{\tilde{P'},P'}(4n-t)^* - C_{P',\tilde{P'}}(4n-t)^*\end{aligned}$$

Suppose that $1 \le t \le n$. Then $C_{P'}(t) = \phi(t) + \psi(n-t)^*$ where

$$\begin{aligned}\phi(t) &:= C_{-A}(t) + C_{jB}(t) + C_{k\tilde{B}}(t) + C_{i\tilde{A}}(t) \\ \psi(t) &:= C_{jB,-A}(t) + C_{k\tilde{B},jB}(t) + C_{i\tilde{A},k\tilde{B}}(t).\end{aligned}$$

Since multiplying a sequence by a constant does not change its autocorrelation values and since $(A, B, \tilde{A}, \tilde{B})$ are complementary we find that $\phi(t) = 0$. Furthermore, using the fact $C_{xA,yB}(t) = xy^* C_{A,B}(t)$ and $C_{\tilde{A},\tilde{B}}(t) = C_{B,A}(t)$ (since A and B have real entries) we find

$$\psi(t) = -jC_{B,A}(t) + iC_{\tilde{B},B}(t) + jC_{B,A}(t) = iC_{\tilde{B},B}(t).$$

Additionally, $C_{\tilde{P'}}(t) = C_{P'}(t)^*$, $C_{P',\tilde{P'}}(4n-t) = C_{-A,-\tilde{A}}(n-t) = C_{A,\tilde{A}}(n-t)$, and $C_{\tilde{P'},P'}(4n-t) = C_{iA,i\tilde{A}}(n-t) = C_{A,\tilde{A}}(n-t)$. Then

$$\hat{R}_P(t) = (iC_{\tilde{B},B}(n-t))^* + iC_{\tilde{B},B}(n-t) + C_{A,\tilde{A}}(n-t)^* - C_{A,\tilde{A}}(n-t)^* = 0.$$

Similarly one can show $\hat{R}_P(t) = 0$ for $n < t \le 4n$ from which the symmetry $\hat{R}_X(t) = \hat{R}_X(-t)^*$ implies $\hat{R}_P(t) = 0$ for all $1 \le t < 8n$. □

Example 12. Using the Golay pair $(++, +-)$ with Theorem 6 produces the palindromic odd perfect sequence [--jJKkiiiikKJj--]. (We denote $-i$ by I, $-j$ by J, and $-k$ by K.)

In particular, palindromic odd perfect sequences exist in all lengths that are a power of two since Theorem 6 implies they exist in all lengths of the form $2^a5^b13^c$ with $a \geq b + c + 3$ and they exist in the lengths 2 and 4 as shown by the examples [++] and [+ii+]. Using the fact that palindromic nega Williamson sequences exist in all odd lengths up to 33 we used the Acevedo–Dietrich correspondence to construct palindromic odd perfect sequences in all odd lengths up to 33. Furthermore, using the fact that palindromic odd perfect Q_8-sequences exist in all lengths that are powers of two we used Lemma 6 to construct palindromic odd perfect sequences in all even lengths up to 68 (see the appendix for an explicit list). The Acevedo–Dietrich correspondence applied to these sequences gives palindromic nega Williamson sequences in all even lengths up to 68. It is conceivable that palindromic odd perfect sequences and palindromic nega Williamson sequences actually exist in all even lengths.

V. Perfect quaternion sequences

We now use our constructions for Williamson sequences and palindromic nega Williamson sequences to show that Williamson sequences and perfect Q_8-sequences exist in all lengths 2^t with $t \geq 0$.

Theorem 7. *Symmetric perfect sequences over Q_8 exist for all lengths 2^t.*

Proof. First, note that Golay sequences exist in all lengths 2^t. By Theorem 5 and Lemma 3 it follows that antipalindromic nega Williamson sequences with the Q_8-property exist in all lengths 2^t for $t \geq 3$ (examples are also known for smaller t, see Example 3). Theorem 3 then implies that if Williamson sequences with the Q_8-property exist in length 2^t they also exist in length 2^{t+2} for all $t \geq 1$. Additionally, it is known that Williamson sequences with the Q_8-property exist in lengths 2 and 4 (see below). By induction, Williamson sequences with the Q_8-property exist in all lengths 2^t for $t \geq 0$. By the Acevedo–Dietrich construction symmetric perfect Q_8-sequences exist in all lengths 2^t as well. □

Example 13. We use the base Golay pair $(+, +)$, the base Williamson sequences $(+, +, +, +)$, $(+-, +-, ++, ++)$, and the base nega Williamson sequences $(++, ++, ++, ++)$. Additionally, we use Golay's interleaving doubling construction [44] to generate larger Golay pairs via the mapping $(A, B) \mapsto (A \amalg B, A \amalg -B)$. From Theorem 4 we generate the Williamson sequences $(++-+, ++-+, ++-+, ++-+)$, from Theorem 5 and Lemma 3 we generate the antipalindromic nega Williamson sequences $(++++, ++++, -++-, -++-)$, and from Theorem 3 we generate the Williamson sequences $(+-++++-, +-++++-, +--++--, +--++--)$. Continuing in this way and using the Acevedo–Dietrich construction produces perfect Q_8-sequences of lengths 2^t for all $t \geq 0$. We denote $-i$ by I, $-j$ by J, and $-k$ by K and explicitly give the sequences produced with this construction for $t \leq 7$:

$$[-], \quad [-j], \quad [--+-], \quad [-+j---j+],$$

[−+−J+j−−−−−j+J−+], [−i++jJ−K−k−jj−+I−I+−jj−k−K−Jj++i],

[−+++−iJI+kjK−J−J−j−j−kjK+iJI−−+−−−+−−IJi+Kjk−j−j−J−J−Kjk+IJi−+++],

[−iii+i+Ij+J+−−K+−Jkj−JjJjk−K+KIK−kIk+k−Kjjjj−Jkj−−K+−−J−ji+I+IiI
−IiI+I+ij−J−−+K−−jkJ−jjjjK−k+kIk−KIK+K−kjJjJ−jkJ−+K−−+J+jI+i+iii]

The perfect sequences generated by Theorem 7 for $t \geq 7$ are counterexamples to the quaternionic form of Mow's conjecture presented by Blake [25] because they can be generated using an orthogonal matrix construction, as we now show.

Theorem 8. *Let $n \geq 32$ and let M be the $(n/4)\times 4$ matrix containing the entries of a perfect sequence P generated by Theorem 7 using the second set from Theorem 5. Write the entries of P in M from left to right and top to bottom (i.e., $M_{i,j} = P_{4i+j}$). Then the columns of M have the array orthogonality property.*

Proof. Let M_0, M_1, M_2, M_3 denote the columns of M as sequences. We must show that $\sum_{r=0}^{3} R_{M_r}(t) = 0$ for all $1 \leq t < n/4$ and that $R_{M_r,M_s}(t) = 0$ for all t and $0 \leq r < s < 4$.

By construction $P = (M_0 \text{ш } M_2) \text{ш } (M_1 \text{ш } M_3)$ is a perfect sequence and therefore

$$R_{(M_0 \text{ш} M_2)\text{ш}(M_1 \text{ш} M_3)}(t) = 0 \qquad \text{for all } 1 \leq t < n.$$

By property 4 in Section III this implies $R_{M_0 \text{ш} M_2}(t) + R_{M_1 \text{ш} M_3}(t) = 0$ for all $1 \leq t < n/2$ and $\sum_{r=0}^{3} R_{M_r}(t) = 0$ for all $1 \leq t < n/4$.

Since P was generated by applying the Acevedo–Dietrich construction to the sequences generated in Theorem 3 we have $P = \mathrm{d}(A) \text{ш } \mathrm{n}(B)$ for some perfect symmetric A and antipalindromic B of even length $n/4$. Let X' denote the sequence formed by the even entries of X and let X'' denote the sequence formed by the odd entries of X. Then we have

$$M_0 = \mathrm{d}(A'), \quad M_1 = \mathrm{n}(B'), \quad M_2 = \mathrm{d}(A''), \quad M_3 = \mathrm{n}(B'').$$

By property 3 of Section III this representation yields $R_{M_0,M_1}(t) = R_{M_0,M_3}(t) = R_{M_1,M_2}(t) = R_{M_2,M_3}(t) = 0$ for all t and only the crosscorrelation of the pairs (M_0, M_2) and (M_1, M_3) are left to consider. Since $A = A' \text{ш } A''$ is perfect and was generated by applying Theorem 3 we have that A' is of the form $\mathrm{d}(C_1)$ for some C_1 and A'' is of the form $\mathrm{n}(C_2)$ for some C_2. Property 3 of Section III then yields $R_{M_0,M_2}(t) = 2R_{A',A''}(t) = 2R_{\mathrm{d}(C_1),\mathrm{n}(C_2)}(t) = 0$ for all t.

Lastly, we must show $R_{M_1,M_3}(t) = 0$. Suppose the antipalindromic B was generated from Theorem 5 using the Golay pair (D, E). An analysis of the Acevedo–Dietrich construction shows that we have

$$B' = [iD; j\tilde{E}; D; -k\tilde{E}], \qquad B'' = [kE; -\tilde{D}; -jE; -i\tilde{D}].$$

We have $\hat{R}_{B',B''}(t) = C_{B',B''}(t) - C_{B'',B'}(n/8-t)^*$ by definition. Suppose that $0 \le t \le n/32$ so that $C_{B'',B'}(n/8-t) = C_{kE,-k\tilde{E}}(n/32-t) = -C_{E,\tilde{E}}(n/32-t)$ and $C_{B',B''}(t) = \phi(t) + \psi(n/32-t)^*$ where

$$\phi(t) \coloneqq C_{iD,kE}(t) + C_{j\tilde{E},-\tilde{D}}(t) + C_{D,-jE}(t) + C_{-k\tilde{E},-i\tilde{D}}(t),$$
$$\psi(t) \coloneqq C_{-\tilde{D},iD}(t) + C_{-jE,j\tilde{E}}(t) + C_{-i\tilde{D},D}(t).$$

Note that $C_{\tilde{E},\tilde{D}}(t) = C_{D,E}(t)$ since D and E contain real entries. Then

$$\phi(t) = jC_{D,E}(t) - jC_{\tilde{E},\tilde{D}}(t) + jC_{D,E}(t) - jC_{\tilde{E},\tilde{D}}(t) = 0,$$
$$\psi(t) = -iC_{\tilde{D},D}(t) - C_{E,\tilde{E}}(t) + iC_{\tilde{D},D}(t) = -C_{E,\tilde{E}}(t).$$

Finally, $R_{M_1,M_3}(t) = 2\hat{R}_{B',B''}(t) = 2(-C_{E,\tilde{E}}(n/32-t)^* - (-C_{E,\tilde{E}}(n/32-t))^*) = 0$ for $0 \le t \le n/32$. A similar calculation shows $\hat{R}_{B',B''}(t) = 0$ for $n/32 < t < n/8$ from which it follows that $R_{M_1,M_3}(t) = 0$ for all t. □

Example 14. For $n = 16$ we use the perfect sequence found in Example 13 and the matrix it generates has the array orthogonality property as well. Otherwise we give the matrices constructed using Theorem 8 for $n = 32$, 64, and 128. The transpose of the matrices are displayed to save space.

$$\begin{bmatrix} -+-+ \\ +j-J \\ ---- \\ J-j+ \end{bmatrix} \begin{bmatrix} -j-j-j-j \\ IJ-kij+K \\ +--++--+ \\ K+jik-JI \end{bmatrix} \begin{bmatrix} --+---+---+---+- \\ IKj+-jKiikJ-+JkI \\ +Jj--jJ++Jj--jJ+ \\ IkJ+-JkiiKj-+jKI \end{bmatrix} \begin{bmatrix} -+j---j+-+j---j+-+j---j+-+j---j+ \\ IIKKJj-+--jjkKIiiikkjJ+-++JJKkiI \\ IKJ+-jkiikj-+JKIIKJ+-jkiikj-+JKI \\ IikKJJ++-+Jjkkiii IKkjj--+-jJKKII \end{bmatrix}$$

VI. Previous Work

Despite an enormous amount of work the conjecture that the longest perfect sequences over the complex nth roots of unity have length n^2 remains open, though some special cases of this conjecture have been resolved. For example, consider the case when $n = 2$. In this case the Frank–Heimiller construction generates the sequence $[1, 1, 1, -1]$ and it is conjectured that no perfect binary sequence of length longer than four exists. In the context of difference sets, Turyn [26] showed that the length of any longer perfect binary sequence must have the form $4m^2$ for odd m. Despite this progress the general conjecture even for $n = 2$ remains open [47].

In the case $n = 4$ the conjecture states that the longest perfect sequence over the alphabet $\{\pm 1, \pm i\}$ has length 16. Turyn [27] showed that the length of any longer perfect $\{\pm 1, \pm i\}$-sequence cannot be of the form $2p^k$ for any prime p and integer k. Most recently, Ma and Ng [48] showed many restrictions on the length of perfect sequences over the pth roots of unity for prime p. In particular, they showed that no perfect sequences of length $2p^{k+2}$ or p^{k+3} exist for $k \ge 0$.

Several other constructions for perfect sequences over complex roots of unity have been found since the construction of Frank and Heimiller. In 1972, Chu described a method [49] of producing perfect sequences of any length n. Shortly after Chu's paper was published, Frank published a response [50]

pointing out that Zadoff had discovered the same construction and had been granted a patent for a communication system based on his construction [51]. Other variants of Zadoff and Chu's sequences have been described by Alltop [52] and Lewis and Kretschmer [53]. In 1983, Milewski [54] found a new construction for perfect sequences of length n^{2t+1} over n^{t+1}th roots of unity for all $n, t \geq 1$. In 2004, Liu and Fan [55] found a new construction for perfect sequences of length n over nth roots of unity when n is a multiple of four. In 2014, Blake and Tirkel [56] gave a construction for perfect sequences of length $4mn^{t+1}$ over $2mn^t$th roots of unity for $m, n, t \geq 1$.

Blake has run extensive searches for perfect sequences over complex roots of unity and quaternions [25] and has found a number of perfect sequences over the nth roots of unity that cannot be generated using matrices with array orthogonality (like those from the Frank–Heimiller and Mow constructions). The sequences that he found are counterexamples to the conjecture that Mow's unified construction produces all perfect sequences over nth roots of unity [16] but are shorter than n^2 and thus are not counterexamples to Mow's original conjecture [57] (also generalized by Lüke, Schotten, and Hadinejad-Mahram [28] to odd perfect sequences).

Apparently no infinite family of odd perfect Q_8-sequences has been previously constructed.[2] It is known that no infinite family of odd perfect sequences can exist over the alphabet $\{\pm 1\}$. Lüke, Schotten, and Hadinejad-Mahram show that the longest odd perfect $\{\pm 1\}$-sequence has length two and they conjecture that the longest odd perfect sequence over the alphabet $\{\pm 1, \pm i\}$ has length four [28]. However, Lüke has constructed *almost* perfect and odd perfect $\{\pm 1, \pm i\}$-sequences A (with $R_A(t) = 0$ or $\hat{R}_A(t) = 0$ for all $1 \leq t < n$ except $t = n/2$) in all lengths $q + 1$ where $q \equiv 1 \pmod 4$ is a prime power [58].

Some work has also been done on perfect quaternion arrays. In 2013, Acevedo and Jolly [59] constructed perfect Q_8-arrays of size $2\times(p+1)/2$ for primes $p \equiv 1 \pmod 4$. Furthermore, they extended a construction of Arasu and de Launey [60] to produce perfect Q_8-arrays of size $2p \times p(p+1)/2$ for primes $p \equiv 1 \pmod 4$. Additionally, Blake [25] found a construction for perfect Q_8-arrays of size $2^t \times 2^t$ with $2 \leq t < 7$.

There does not seem to be much prior work on nega Williamson sequences, though Xia, Xia, Seberry, and Wu [38] study them under the name "4-suitable negacyclic matrices". They give constructions for them in terms of Golay pairs, Williamson sequences, and base sequences. However, we could find no prior constructions that specifically generated palindromic or antipalindromic nega Williamson sequences. Lüke presents a method [61] of constructing pairs of negacomplementary binary sequences, also studied under the name "associated pairs" by Ito [62] and "negaperiodic Golay pairs" by Balonin and Đoković [63]. Lüke and Schotten [64] also give a construction for odd perfect almost binary sequences that Wen, Hu, and Jin [65] use to construct negacomplementary binary sequences. Jin et al. [66] give necessary conditions for the existence of negacomplementary sequences and a doubling

[2] A construction given in [25] uses the term 'odd-perfect' but with an alternative meaning that $R_A(t) = 0$ for odd t.

construction for negacomplementary sequences. Yang, Tang, and Zhou [67] show that a $\{\pm 1\}$-sequence A of length n satisfies $\max_{1\leq t<n}|\hat{R}_A(t)| - 1 \geq (n-1) \bmod 2$ and give constructions for sequences that are optimal, i.e., ones that meet the bound exactly.

Williamson sequences have been quite well-studied since introduced by Williamson 75 years ago [31]. They are often presented using matrix notation and known as "Williamson matrices" since one of their traditional applications has been to construct Hadamard matrices. In this formulation Williamson matrices are defined to be circulant (i.e., each row is the cyclic shift of the previous row) and Williamson sequences are simply the first rows of Williamson matrices. Baumert and Hall [68] performed an exhaustive search for Williamson sequences of odd length $n \leq 23$ and presented a doubling construction for a generalization of Williamson matrices that are symmetric but not circulant. Turyn [69] constructed Williamson sequences in all lengths $(q+1)/2$ where $q \equiv 1 \pmod 4$ is a prime power, Whiteman [70] constructed Williamson sequences in all lengths $q(q+1)/2$ where $q \equiv 1 \pmod 4$ is a prime power, and Spence [71] constructed Williamson sequences in all lengths $q^t(q+1)/2$ where $q \equiv 1 \pmod 4$ is a prime power and $t \geq 0$.

Computer searches have determined that Williamson sequences in odd lengths are particularly rare. Following Baumert and Hall's exhaustive search, Baumert [72] found a new set of Williamson sequences in length 29, Sawade [73] found eight new sets in lengths 25 and 27, Yamada [74] found one new set in length 37, Koukouvinos and Kounias [75] found four new sets in length 33, Đoković [76]–[78] found six new sets in the lengths 25, 31, 33, 37, and 39, van Vliet found one new set in length 51 (unpublished but appears in [33]), Holzmann, Kharaghani, and Tayfeh-Rezaie [33] found one new set in length 43, and Bright, Kotsireas, and Ganesh [34] found one new set in length 63. Đoković [77] also found that no Williamson sequences exist in length 35 and Holzmann, Kharaghani, and Tayfeh-Rezaie [33] found that no Williamson sequences exist in lengths 47, 53, and 59.

In even lengths Williamson sequences are much more common, as first shown by an exhaustive search up to length 18 by Kotsireas and Koukouvinos [79]. Non-exhaustive searches were performed up to length 34 by Bright et al. [80], and up to length 42 by Zulkoski et al. [81]. Bright [82] completed an exhaustive search up to length 44 and Bright, Kotsireas, and Ganesh [34] completed an exhaustive search in the even lengths up to 70. Acevedo and Dietrich's construction [29] can be used to generate Williamson sequences in many lengths including 70.

VII. Conclusion

We have shown that perfect and odd perfect Q_8-sequences exist in all lengths that are a power of two. Acevedo and Dietrich [30] summarize the knowledge (as of 2018) of perfect Q_8-sequences as follows:

> Currently there exists only one infinite family of perfect sequences over the quaternions (of magnitude one)...

This infinite family was found by Acevedo and Hall [24] who gave a construction for perfect Q_8-sequences in lengths of the form $q+1$ where $q \equiv 1 \pmod 4$ is a prime power. Thus, our construction

for lengths of the form 2^t is the second known infinite family of perfect Q_8-sequences. Additionally, our construction for odd perfect Q_8-sequences is the first known infinite family of odd perfect Q_8-sequences. Because our perfect sequences can be constructed using matrices with array orthogonality (as shown in Theorem 8) they disprove Blake's conjecture [25, Conjecture 8.2.1] that the longest perfect Q_8-sequences generated from an orthogonal array construction have length 64. Theorem 8 also implies the existence of perfect Q_8-arrays of sizes $2^t \times 4$ for all $t \geq 2$ and the construction of Acevedo and Jolly [59] then implies the existence of perfect Q_8-arrays of size $2^t p \times 4p$ when $p = 2^{t+2} - 1$ is prime.

Furthermore, we generalize Williamson's doubling construction [31] from 1944, showing that the existence of Williamson sequences of odd length n implies not only the existence of Williamson sequences of length $2n$ but also implies the existence of Williamson sequences of length $2^t n$ for all $t \geq 1$. We have also shown the importance of nega Williamson sequences, a class of sequences defined by Xia et al. [38] in 2006. In particular, we have demonstrated the importance of palindromic nega Williamson sequences.

Lastly, our constructions provide an explanation for the abundance of Williamson sequences in lengths that are divisible by a large power of two. Prior to this work it was noticed that Williamson sequences are much more abundant in these lengths. For example, fewer than 100 sets of Williamson sequences are known to exist in the odd lengths, but an exhaustive computer search [34] found 130,739 sets of Williamson sequences in the even lengths up to 70. We found that it was possible to generate 95,759 (about 75%) of these sets using Theorems 3 and 4. Thus, these theorems provide an explanation for the existence of many Williamson sequences. However, they still do not explain the existence of Williamson sequences in all even lengths. In particular, they only work in lengths that are multiples of four.

It would also be interesting to find a construction that works in the even lengths that are not multiples of four. Williamson's doubling result can be used for lengths $2n$ but requires that Williamson sequences of odd length n exist. Acevedo and Dietrich's construction can be used in certain cases even if Williamson sequences of length n do not exist, assuming n is composite. For example, the Acevedo–Dietrich construction implies that Williamson sequences of length 70 exist since Williamson sequences of length 7 exist and Williamson sequences with the Q_8-property of length 10 exist. However, we could only generate about 40% of the Williamson sequences in length 70 using the Acevedo–Dietrich construction suggesting that there is another construction for Williamson sequences that is currently unknown.

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APPENDIX

A. Proofs

We now give proofs of the properties from Section III. In each case we let $X = [x_0, \ldots, x_{n-1}]$ and $Y = [y_0, \ldots, y_{n-1}]$ be arbitrary sequences of length n.

1) $R_{\mathrm{d}(X)}(t) = 2R_X(t)$.
The rth entry of $\mathrm{d}(X)$ is $x_{r \bmod n}$. Then
$$R_{\mathrm{d}(X)}(t) = \sum_{r=0}^{2n-1} x_{r \bmod n} x^*_{r+t \bmod n} = 2R_X(t).$$

2) $R_{\mathrm{n}(X)}(t) = 2\hat{R}_X(t)$.
The rth entry of $\mathrm{n}(X)$ is $(-1)^{\lfloor r/n \rfloor} x_{r \bmod n}$. Then
$$\begin{aligned} R_{\mathrm{n}(X)}(t) &= \sum_{r=0}^{2n-1} (-1)^{\lfloor r/n \rfloor} x_{r \bmod n} (-1)^{\lfloor (r+t)/n \rfloor} x^*_{r+t \bmod n} \\ &= \sum_{r=0}^{n-1} x_r (-1)^{\lfloor (r+t)/n \rfloor} x^*_{r+t \bmod n} + \sum_{r=0}^{n-1} (-x_r)(-1)^{\lfloor (r+n+t)/n \rfloor} x^*_{r+t \bmod n} \\ &= 2\hat{R}_X(t). \end{aligned}$$

3) $R_{\mathrm{d}(X),\mathrm{n}(Y)}(t) = 0$ and $R_{\mathrm{n}(Y),\mathrm{d}(X)}(t) = 0$.
The rth entry of $\mathrm{d}(X)$ is $x_{r \bmod n}$ and the rth entry of $\mathrm{n}(Y)$ is $(-1)^{\lfloor r/n \rfloor} y_{r \bmod n}$. Then
$$\begin{aligned} R_{\mathrm{d}(X),\mathrm{n}(Y)}(t) &= \sum_{r=0}^{2n-1} x_{r \bmod n} (-1)^{\lfloor (r+t)/n \rfloor} y^*_{r+t \bmod n} \\ &= \sum_{r=0}^{n-1} x_r (-1)^{\lfloor (r+t)/n \rfloor} y^*_{r+t \bmod n} + \sum_{r=0}^{n-1} x_r (-1)^{\lfloor (r+n+t)/n \rfloor} y^*_{r+t \bmod n} \\ &= \hat{R}_{X,Y}(t \bmod n) - \hat{R}_{X,Y}(t \bmod n) = 0. \end{aligned}$$
The second property follows because $R_{X,Y}(t) = R_{Y,X}(-t)^*$ for all X, Y, and t.

4) $R_{X \shuffle Y}(2t) = R_X(t) + R_Y(t)$.
The $(2r)$th entry of $X \shuffle Y$ is x_r and the $(2r+1)$th entry is y_r. Then
$$R_{X \shuffle Y}(2t) = \sum_{\substack{r=0 \\ r \text{ even}}}^{2n-1} x_{r/2} x^*_{r/2+t \bmod n} + \sum_{\substack{r=0 \\ r \text{ odd}}}^{2n-1} y_{(r-1)/2} y^*_{(r-1)/2+t \bmod n} = R_X(t) + R_Y(t).$$

5) $R_{X \shuffle Y}(2t+1) = R_{X,Y}(t) + R_{Y,X}(t+1)$.

The $(2r)$th entry of $X \text{ш} Y$ is x_r and the $(2r+1)$th entry is y_r. Then

$$R_{X\text{ш}Y}(2t+1) = \sum_{\substack{r=0 \\ r \text{ even}}}^{2n-1} x_{r/2} y^*_{r/2+t \bmod n} + \sum_{\substack{r=0 \\ r \text{ odd}}}^{2n-1} y_{(r-1)/2} x^*_{(r+1)/2+t \bmod n} = R_{X,Y}(t) + R_{Y,X}(t+1).$$

6) If X is symmetric then $\mathrm{d}(X)$ is symmetric.
Note that $x_{2n-r \bmod n} = x_{n-r \bmod n} = x_{r \bmod n}$. Thus the $(2n-r)$th entry of $\mathrm{d}(X)$ is equal to the rth entry, as required.

7) If X is antipalindromic and of even length then $\mathrm{n}(X)$ is palindromic.
Let Y be the first half of X (i.e., $X = [y_0, \ldots, y_{n/2-1}, -y_{n/2-1}, \ldots, -y_0]$) so that

$$\mathrm{n}(X) = [y_0, \ldots, y_{n/2-1}, -y_{n/2-1}, \ldots, -y_0, -y_0, \ldots, -y_{n/2-1}, y_{n/2-1}, \ldots, y_0]$$

is a palindrome.

8) If X is symmetric and Y is palindromic then $X \text{ш} Y$ is symmetric.
First, we show the even entries of $X \text{ш} Y$ satisfy the symmetric property. The $(2r)$th entry of $X \text{ш} Y$ is x_r and the $(2n-2r)$th entry of $X \text{ш} Y$ is x_{n-r} (for $r \neq 0$). Since X is symmetric $x_{n-r} = x_r$ showing the $(2r)$th and $(2n-2r)$th entries are equal.
Second, we show the odd entries of $X \text{ш} Y$ satisfy the symmetric property. The $(2r+1)$th entry of $X \text{ш} Y$ is y_r and the $(2n-2r-1)$the entry of $X \text{ш} Y$ is y_{n-r-1}. Since Y is palindromic $y_{n-r-1} = y_r$ showing the $(2r+1)$th and $(2n-2r-1)$th entries are equal.

9) If X is antisymmetric and of odd length then $\mathrm{n}(X)$ is symmetric.
Let Y be the first half of the symmetric part of X (i.e., $X = [x_0, y_0, \ldots, y_{(n-1)/2}, -y_{(n-1)/2}, \ldots, -y_0]$) so that

$$\mathrm{n}(X) = [x_0, y_0, \ldots, y_{(n-1)/2}, -y_{(n-1)/2}, \ldots, -y_0, -x_0, -y_0, \ldots, -y_{(n-1)/2}, y_{(n-1)/2}, \ldots, y_0]$$

is symmetric.

10) If X is palindromic then $\mathrm{d}(X)$ is palindromic.
Note that $x_{2n-r-1 \bmod n} = x_{n-r-1 \bmod n} = x_{r \bmod n}$. Thus the $(2n-r-1)$th entry of $\mathrm{d}(X)$ is equal to the rth entry, as required.

B. List of odd perfect quaternion sequences

We now give palindromic odd perfect Q_+-sequences in all lengths $n < 70$ except for 35, 47, 53, 59, 65, and 67. The sequences in odd lengths were constructed using Lemma 5 with a previously known set of Williamson sequences of length n. The sequences in even lengths were constructed using Lemma 6, Theorem 6, and Lemma 3 and are new to the best of our knowledge, though perfect quaternion sequences may be constructed in these lengths using the results of Acevedo and Dietrich [29].

The sequences are denoted by P_n where n is the length of the sequence. The symbols + and − denote 1 and −1, capitalization denotes negation of an entry, and an overlined entry denotes left multiplication by q, i.e., the symbol $\overline{\text{I}}$ denotes the entry $-qi$.

P_{1} = [+]

P_{2} = [++]

P_{3} = [I−I]

P_{4} = [+ii+]

P_{5} = [J+−+J]

P_{6} = [i−II−i]

P_{7} = [j−J−J−j]

P_{8} = [−jkiikj−]

P_{9} = [KjI+−+IjK]

P_{10} = [J−−+JJ+−−J]

P_{11} = [i+K−J−J−K+i]

P_{12} = [+−i+KIIK+i−+]

P_{13} = [KiJKji−ijKJiK]

P_{14} = [J−J−j−jj−j−J−J]

P_{15} = [Jj−JJ++−++JJ−jJ]

P_{16} = [−−jJKkiiiikKJj−−]

P_{17} = [+iK++jjJ−Jjj++Ki+]

P_{18} = [KJi+−−ijKKji−−+iJK]

P_{19} = [kKj−JJK++−++KJJ−jKk]

P_{20} = [k−−iKj+kiJJik+jKi−−k]

P_{21} = [++IIi++i+−−+−i++iII++]

P_{22} = [I+K+j−J+k+ii+k+J−j+K+I]

P_{23} = [j+k−i−−KJiI−IiJK−−i−k+j]

P_{24} = [+JkI−kjk−JIiiIJ−kjk−IkJ+]

P_{25} = [kJJIikk+j−+i−i+−j+kkiIJJk]

P_{26} = [KIjKjI−ijkjiKKijkji−IjKjIK]

P_{27} = [IikJII−j+K−JK−KJ−K+j−IIJkiI]

P_{28} = [k−jkk+jKij−KIjjIK−jiKj+kkj−k]

P_{29} = [+iIIKkjJkIIKKK−KKKIIkJjkKIIi+]

P_{30} = [jj−jj++−+Jj+jJJj+jJ+−−++jj−jj]

P_{31} = [−II+i−I−−−IiIii−iiIiI−−−I−i+II−]

P_{32} = [−−−+jJjjkkKkIiiiiiiIkKkkjjJj+−−−]

P_{33} = [J+IjKKkjkk+Iik−i−i−kiI+kkjkKKjI+J]

P_{34} = [+Ik++JJJ−jJj+−ki++ik−+jJj−JJJ++kI+]

P_{36} = [JJIik+iKJKJ−I−++kKKk++−I−JKJKi+kiIJJ]

P_{37} = [jikIIJI+KkJ−JJ+−K+−+K−+JJ−JkK+IJIIkij]

P_{38} = [KKj+jJK−−−+−kJJ+JKkkKJ+JJk−+−−−KJj+jKK]

P_{39} = [Jkk−J+I+KkJ+++kjKJj−jJKjk+++JkK+I+J−kkJ]

P_{40} = [kKI+j−KKII+−−+JIKijjjjjjiKIJ−−+IIKK−j+IKk]

P_{41} = [iKkKikIIKKikiKiiIikk−kkiIiiKikiKKIIkiKkKi]

P_{42} = [+−iIi−−i++−−+I−+iii++++iii+−I+−−++i−−iIi−+]

P_{43} = [j−IJJjIKKK+j+k−+KijK−−+−KjiK+−k+j+KKKIjJJI−j]

P_{44} = [−+kik−JIj+I−iK+kkj−jiiiij−jkk+Ki−I+jIJ−kik+−]

P_{45} = [kkkKkIiKIkikKKIKIiKKii$\bar{+}$iiKKiIKIKKkikIKiIkKkkk]

P_{46} = [J+k+I−−kjiI$\bar{+}$iiJk+−i+K+jj+K+i−+kJii$\bar{+}$Iijk−−I+k+J]

P_{48} = [+$\bar{K}$JJ$\bar{I}$Ki$\bar{+}$Ii$\bar{I}$KJ$\bar{j}$−+$\bar{K}$+j$\bar{J}$kk$\bar{+}$ii$\bar{+}$kk$\bar{J}$j+$\bar{K}$+−$\bar{j}$JK$\bar{I}$iI$\bar{+}$iK$\bar{I}$JJ$\bar{K}$+]

P_{49} = [kkiIIkIIiIIIkiKkikkIIKkK$\bar{+}$KkKIIkkikKikIIIiIIkIIikk]

P_{50} = [kjjIiKK+j+−i$\bar{+}$I−−j−KkiijJkkJjiikK−j−−I$\bar{+}$i−+j+KKiIjjk]

P_{51} = [JJ−j+−+−j++j+J+++j−jJ−−+j$\bar{+}$j+−−Jj−j+++J+j++j−+−+j−JJ]

P_{52} = [JIJjki$\bar{−}$+KkJ−jkiKjjI$\bar{K}$+JKK−KK−KKJ+$\bar{K}$IjjKikj−JkK+$\bar{−}$ikjJIJ]

P_{54} = [iikjiI−J−K−jk$\bar{−}$Kj+K+J+IIjKiIIiKjII+J+K+jK$\bar{−}$kj−K−J−Iijkii]

P_{55} = [JJJ++jJ++−+−++jj−j+−J+−−j+j$\bar{+}$j+j−−+J−+j−jj++−+−++Jj++JJJ]

P_{56} = [kK+$\bar{+}$JjikIiI$\bar{}$j−+IIK$\bar{J}$++j−KK$\bar{K}$iJJJiK$\bar{K}$K−j++$\bar{J}$KII+−j$\bar{I}$iIkijJ$\bar{+}$+Kk]

P_{57} = [Ii++I+−++−iiI−−−+i+I−i−−II−I$\bar{+}$I−II−−i−I+i+−−−Iii−++−+I++iI]

P_{58} = [+IiIKKJJkiiKKk$\bar{−}$KKkiIkjJkKiii++iiiKkJjkIikKK$\bar{−}$kKKiikJJKKIiI+]

P_{60} = [Kj+kk−+$\bar{K}$i+jkiJJkK+JKi+$\bar{+}$IIjJIkJJkIJjII$\bar{+}$+iKJ+KkJJikj+i$\bar{K}$+−kk+jK]

P_{61} = [KkKKIiKiKIIkkKiIIKIIKkiKKiiiIK$\bar{+}$KIiiiKKikKIIKIIiKkkIIKiKiIKKkK]

P_{62} = [+II−I−I++−IIiii$\bar{+}$IiIIi−−+i−i−iI−−Ii−i−i+−−iIIiI$\bar{+}$iiiII−++I−I−II+]

P_{63} = [−iiiIi+jjjJ+kiKJ−Kk−ik−Ij−−−+iI$\bar{−}$Ii+−−−jI−ki−kK−JKik+Jjjj+iIiii−]

P_{64} = [−−−+−−+−jJjjjJJJKKKkkkKkiIiiIiiiiiiIiiIikKkkkKKKJJJjjjJj−+−−+−−−]

P_{66} = [J−ijKkKjkK−IiK+i$\bar{+}$I+kii−kkJKKKJi+JJ+iJKKKJkk−iik+I$\bar{+}$i+KiI−KkjKkKji−J]

P_{68} = [IIKiijJk$\bar{K}$jjki+k+I+IJI−jkK$\bar{+}$jKK−+j−++−j+−KKj$\bar{+}$Kkj−IJI+I+k+ikjj$\bar{K}$kJjiiKII]

P_{69} = [−−−−−j−JJj−−+j−+−JJJj−j+Jj−jj−Jj+−$\bar{+}$−+jJ−jj−jJ+j−jJJJ−+−j+−−jJJ−j−−−−−]