Exceptional examples in the *abc* conjecture

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Abstract

In this report we use arguments from the geometry of numbers to show that there are infinitely many coprime natural numbers a, b, c for which $a+b=c$ and $c > \operatorname{rad}(abc)\exp(k\sqrt{\log c}/\log\log c)$ for some constant k, and provide a connection between this constant and Hermite's lattice constant.

1 Introduction

Three natural numbers a, b, c are said to be an *abc triple* if they do not share a common factor and satisfy the equation

$$a+b=c.$$

Informally, the *abc* conjecture says that large *abc* triples cannot be 'very composite', in the sense of *abc* having a prime factorization containing large powers of small primes.

For the formal statement, we define the *radical* of *abc* to be the product of the primes in the prime factorization of *abc*,

$$\operatorname{rad}(abc) \coloneqq \prod_{p \mid abc} p.$$

The *abc* conjecture then states that *abc* triples satisfy

$$c = O\big(\operatorname{rad}(abc)^{1+\epsilon}\big) \tag{1}$$

for every $\epsilon > 0$, where the implied big-O constant may depend on ϵ.

Presently, the conjecture is far from being proved; not a single ϵ is known for which (1) holds. The best known result [5] says that *abc* triples satisfy

$$c = O\big(\exp(\operatorname{rad}(abc)^{1/3}(\log\operatorname{rad}(abc))^3)\big).$$

On the other hand, it is also known [2] that there are infinitely many *abc* triples for which

$$c > \operatorname{rad}(abc)\exp(6.07\sqrt{\log c}/\log\log c).$$

Such *abc* triples are exceptional in the sense that their radical is relatively small in comparison to c, and while this does not contradict (1), it provides a lower bound on its best possible form. Note that the function $\exp(k\sqrt{\log c}/\log\log c)$ grows faster than any power of $\log c$, but slower than any nontrivial power of c.

1.1 Preliminaries

Let S be a set of prime numbers. An S-unit is defined to be a rational number whose numerator and denominator in lowest terms are divisible by only the primes in S. That is, one has

$$S\text{-units} := \left\{ \pm \prod_{p_i \in S} p_i^{e_i} : e_i \in \mathbb{Z} \right\}.$$

This generalizes the notion of units of $\mathbb{Z}$; in particular, the $\emptyset$-units are ± 1.

The *height* of a rational number p/q in lowest terms is $h(p/q) := \max\{|p|, |q|\}$. This provides a convenient way of measuring the 'size' of an S-unit.

Finally, if $\boldsymbol{x} = (x_1, \ldots, x_n)$ is a vector in $\mathbb{R}^n$, we let

$$\|\boldsymbol{x}\|_k := \left(\sum_{i=1}^{n} |x_i|^k \right)^{1/k}$$

be its standard k-norm, with $k = 2$ if not explicitly specified.

2 The geometry of numbers

The existence of exceptional *abc* triples will follow from some basic results in the geometry of numbers. In particular, we will require the existence of a short nonzero vector in a suitably chosen lattice.

2.1 The odd prime number lattice

The result involves in an essential way the *odd prime number lattice* L_n generated by the rows $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ of the matrix

$$\begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix} = \begin{bmatrix} \log 3 & & & & \\ & \log 5 & & & \\ & & \log 7 & & \\ & & & \ddots & \\ & & & & \log p_n \end{bmatrix}$$

where p_i denotes the ith odd prime number.

The importance of this lattice stems from the fact its elements have a close link with the $\{p_1, \ldots, p_n\}$-units. In particular, there is an obvious isomorphism between the points of L_n and the positive $\{p_1, \ldots, p_n\}$-units, as given by

$$\sum_{i=1}^{n} e_i \boldsymbol{b}_i \leftrightarrow \prod_{i=1}^{n} p_i^{e_i}.$$

Furthermore, this relationship works well with a natural notion of 'size', as the following lemma shows.

Lemma 1. *$\|\boldsymbol{x}\|_1 \geq \log h(p/q)$ where $\boldsymbol{x} = \sum_{i=1}^{n} e_i \boldsymbol{b}_i$ and $p/q = \prod_{i=1}^{n} p_i^{e_i}$ is expressed in lowest terms.*

Proof. By the definition of the one-norm, we have:

$$\begin{aligned}
\|\boldsymbol{x}\|_1 &= \sum_{i=1}^{n} \left| e_i \log p_i \right| \\
&= \sum_{e_i > 0} e_i \log p_i - \sum_{e_i < 0} e_i \log p_i \\
&= \log p + \log q \\
&= \log h(p/q) + \log \min\{p, q\}
\end{aligned}$$

The result follows since $\log \min\{p, q\}$ is nonnegative. $\square$

It will also be important to know the determinant (or volume) of the lattice L_n. Since its defining basis was a diagonal matrix, the lattice determinant is simply the product of the entries on the diagonal of the basis matrix.

Lemma 2. *The determinant of the lattice L_n is $\prod_{i=1}^{n} \log p_i$.*

2.2 The kernel sublattice

Let P be the set of positive $\{p_1, \ldots, p_n\}$-units, and consider the map φ reducing the elements of P modulo 2^m. Since each $p_1, \ldots, p_n$ is odd, $\varphi\colon P \to (\mathbb{Z}/2^m\mathbb{Z})^*$ is well-defined. The odd prime number lattice L_n has an important sublattice which we call the *kernel sublattice* $L_{n,m}$. It consists of those vectors whose associated $\{p_1, \ldots, p_n\}$-units lie in the kernel of φ. Formally, we define

$$L_{n,m} \coloneqq \left\{ \sum_{i=1}^{n} e_i \boldsymbol{b}_i : \prod_{i=1}^{n} p_i^{e_i} \equiv 1 \pmod{2^m} \right\}.$$

Figure 1 demonstrates the kernel sublattice in two dimensions for varying m.

Lemma 3. *$L_{n,m}$ is a sublattice of L_n with index 2^{m-1} when $n \geq 2$.*

Proof. Note that $L_{n,m}$ is discrete (as it is a subset of L_n) and closed under addition and subtraction since if $\sum_{i=1}^{n} e_i \boldsymbol{b}_i$, $\sum_{i=1}^{n} f_i \boldsymbol{b}_i \in L_{n,m}$ then

$$\prod_{i=1}^{n} p_i^{e_i \pm f_i} \equiv \prod_{i=1}^{n} p_i^{e_i} \cdot \prod_{i=1}^{n} p_i^{\pm f_i} \equiv 1 \pmod{2^m}.$$

$L_{n,m}$ also contains the n linearly independent vectors $\operatorname{ord}_{2^m}(p_i)\boldsymbol{b}_i$ for $1 \leq i \leq n$, so this demonstrates that $L_{n,m}$ is a full-rank sublattice of L_n.

Since 3 and 5 generate $(\mathbb{Z}/2^m\mathbb{Z})^*$, when $n \geq 2$ we have $\varphi(P) = (\mathbb{Z}/2^m\mathbb{Z})^*$. Since $L_n \cong P$ and $L_{n,m} \cong \ker \varphi$ it follows that

$$L_n / L_{n,m} \cong (\mathbb{Z}/2^m\mathbb{Z})^*$$

by the first isomorphism theorem. Thus the index of $L_{n,m}$ in L_n is $|(\mathbb{Z}/2^m\mathbb{Z})^*| = 2^{m-1}$. $\square$

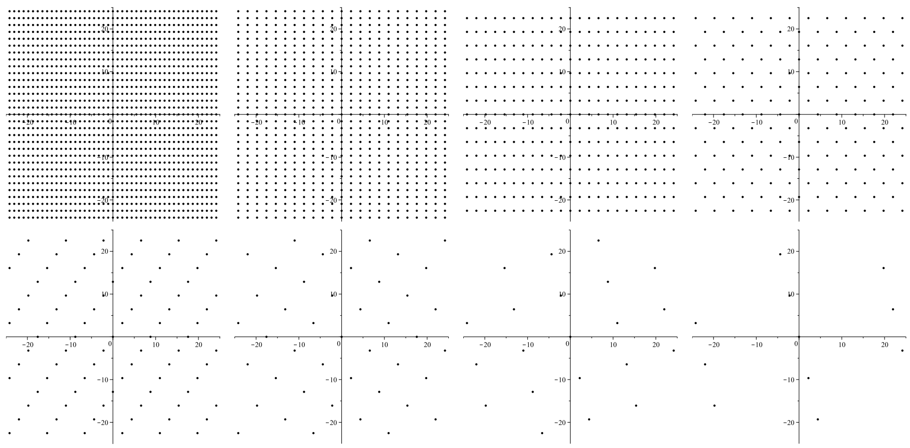

Figure 1: $L_{2,m}$ for $1 \le m \le 8$

Since $\det(L_{n,m}) = |L_n/L_{n,m}| \cdot \det(L_n)$, the following corollary follows by Lemmas 2 and 3.

Corollary 1. $\det(L_{n,m}) = 2^{m-1} \prod_{i=1}^{n} \log p_i$ *when* $n \ge 2$.

2.3 Hermite's constant

The *Hermite constant* γ_n is defined to be the smallest positive number such that every lattice of dimension n contains some nonzero vector $\boldsymbol{x}$ which satisfies

$$\|\boldsymbol{x}\|^2 \le \gamma_n \det(L)^{2/n}.$$

The existence of such a constant was first shown by Hermite, who proved the exponential bound $\gamma_n \le \sqrt{4/3}^{n-1}$. However, it is now known that γ_n essentially grows linearly in n. By Minkowski's theorem applied to a sphere of sufficiently large volume, it follows that

$$\gamma_n \le 4\omega_n^{-2/n} \sim \frac{2n}{\pi e} \approx 0.234n$$

where ω_n is the volume of the n-dimensional unit sphere.

Improving on this, Kabatiansky & Levenshtein [3] showed that

$$\gamma_n \le \frac{2n}{4^{0.599}\pi e} \approx 0.102n$$

for sufficiently large n.

Since we are interested in the one-norm instead of the usual Euclidean norm, so we define the related constants δ_n by the smallest positive number such that every full-rank lattice of dimension n contains some nonzero vector $\boldsymbol{x}$ which satisfies

$$\|\boldsymbol{x}\|_1 \le \delta_n \det(L)^{1/n}.$$

By Minkowski's theorem applied to a generalized octahedron (a 'sphere' in the one-norm), one has that every full-rank lattice of dimension n contains some nonzero vector $\boldsymbol{x}$ which satisfies

$$\|\boldsymbol{x}\|_1 \leq (n! \det(L))^{1/n},$$

from which it follows that

$$\delta_n \leq (n!)^{1/n} \sim \frac{n}{e} \approx 0.368n.$$

It is also possible to use the bounds on γ_n to derive bounds on δ_n. By the relationship between the one-norm and two-norm, we have that every lattice of dimension n contains some nonzero vector $\boldsymbol{x}$ which satisfies

$$\|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2 \leq \sqrt{n\gamma_n} \det(L)^{1/n}.$$

Therefore $\delta_n \leq \sqrt{n\gamma_n}$ and so by the result of Kabatiansky–Levenshtein,

$$\delta_n \leq \sqrt{\frac{2n^2}{4^{0.599}\pi e}} \approx 0.320n$$

for large enough n.

However, better bounds on δ_n are known. Blichfeldt [1] showed that

$$\delta_n \leq \sqrt{\frac{4(n+1)(n+2)}{3\pi(n+3)}} \left(\frac{2(n+1)}{n+3}\left(\frac{n}{2}+1\right)!\right)^{1/n} \sim \sqrt{\frac{2}{3\pi e}} n \approx 0.279n,$$

where $x! := \Gamma(x+1)$. Improving this, Rankin [4] showed that

$$\delta_n \leq \left(\frac{2-x}{1-x}\right)^{x-1} \left(\frac{1+xn}{x \cdot x!^n}(xn)!\right)^{1/n} n^{1-x} \sim \left(\frac{2-x}{1-x}\right)^{x-1} \frac{(x/e)^x}{x!} n$$

for any $x \in [1/2, 1]$. This attains a minimum for $x \approx 0.645$, so that the expression on the right becomes approximately $0.273n$.

For convenience, we define δ to be a constant so that $\delta_n \leq n/\delta$ holds for all sufficiently large n. In light of Rankin's result, we can take $\delta \approx 3.659$.

3 Exceptional *abc* triples

The real importance of the kernel sublattice is that it lets us show the existence of *abc* triples in which c is fairly large relative to $\operatorname{rad}(abc)$. The following lemma is the first result along these lines.

Lemma 4. *For all $m \geq 1$ and sufficiently large n, there exists an abc triple satisfying*

$$\frac{2^{m-1}}{\prod_{i=1}^n p_i} \operatorname{rad}(abc) \leq c \qquad \text{and} \qquad \log c \leq \frac{n}{\delta}\left(2^{m-1}\prod_{i=1}^n \log p_i\right)^{1/n}.$$

Proof. By the definition of δ applied to the kernel sublattice $L_{n,m}$, we have that for all sufficiently large n there exists some nonzero $\boldsymbol{x} \in L_{n,m}$ with

$$\|\boldsymbol{x}\|_1 \leq \frac{n}{\delta} \det(L_{n,m})^{1/n}. \tag{2}$$

Say $\boldsymbol{x} = \sum_{i=1}^n e_i \boldsymbol{b}_i$, and let $\prod_{i=1}^n p_i^{e_i} = p/q$ be expressed in lowest terms. By construction of the kernel sublattice, we have that

$$p/q \equiv 1 \pmod{2^m}.$$

Let $c \coloneqq \max\{p, q\}$, $b \coloneqq \min\{p, q\}$, and $a \coloneqq c - b$, so that a, b, c form an *abc* triple. Furthermore, we see that

$$c \equiv b \pmod{2^m}$$

so that $c = b + k2^m$ for some positive integer $k \leq c/2^m$. Note that a is divisible by 2 and any other prime which divides it also divides k, so that $\operatorname{rad}(a) \leq 2k \leq c/2^{m-1}$. Furthermore, by construction of b and c, $\operatorname{rad}(bc) \leq \prod_{i=1}^n p_i$. Thus

$$\operatorname{rad}(abc) \leq \frac{c}{2^{m-1}} \prod_{i=1}^n p_i$$

from which the first bound follows. The second bound follows using our previous results:

$$\begin{aligned}
\log c &= \log h(p/q) && \text{since } c = h(p/q) \\
&\leq \|\boldsymbol{x}\|_1 && \text{by Lemma 1} \\
&\leq \tfrac{n}{\delta} \det(L_{n,m})^{1/n} && \text{by (2)} \\
&= \tfrac{n}{\delta} \big(2^{m-1} \textstyle\prod_{i=1}^n \log p_i\big)^{1/n} && \text{by Corollary 1}
\end{aligned}$$

□

4 Asymptotic formulae

Let $x \coloneqq p_n$ and let $\pi(x)$ be the prime counting function, so that $n = \pi(x) - 1$. The prime number theorem states that $\pi(x) \sim \operatorname{li}(x)$, and by the asymptotic expansion of the logarithmic integral,

$$n = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + O\Big(\frac{x}{\log^4 x}\Big). \tag{3}$$

Rearranging this to find an expression for x, we have

$$\begin{aligned}
x &= n \log x - \frac{x}{\log x} - \frac{2x}{\log^2 x} + O\Big(\frac{x}{\log^3 x}\Big) \\
&= n \log x - n - \frac{x}{\log^2 x} + O\Big(\frac{x}{\log^3 x}\Big).
\end{aligned}$$

An alternative form of the prime number theorem states $\sum_{p\le x} \log p \sim x$, so we have that

$$\sum_{i=1}^{n} \log p_i = x + O\Big(\frac{x}{\log^3 x}\Big).$$

Putting these together, we derive the following lemma.

Lemma 5. $\sum_{i=1}^{n} \log p_i = n \log p_n - n - p_n/\log^2 p_n + O(p_n/\log^3 p_n)$

Additionally, we've seen that to estimate $\det(L_{n,m})$ we need an estimate of the product of $\log p_i$ over the first n odd primes. The following lemma gives an asymptotic formula for the logarithm of this quantity.

Lemma 6. $\sum_{i=1}^{n} \log\log p_i = n \log\log p_n - p_n/\log^2 p_n + O(p_n/\log^3 p_n)$

Proof. By Abel's summation formula with $f(k) := \log\log k$ and

$$a_k := \begin{cases} 1 & \text{if } k \text{ is an odd prime} \\ 0 & \text{otherwise} \end{cases}$$

for k up to $x := p_n$, we have

$$\sum_{i=1}^{n} \log\log p_i = n \log\log x - \int_2^x \frac{\pi(t)-1}{t\log t}\,\mathrm{d}t.$$

We have $\pi(t) - 1 = t/\log t + O(t/\log^2 t)$ by the prime number theorem, so that

$$\int_2^x \frac{\pi(t)-1}{t\log t}\,\mathrm{d}t = \int_2^x \frac{\mathrm{d}t}{\log^2 t} + O\Big(\int_2^x \frac{\mathrm{d}t}{\log^3 t}\Big).$$

The first integral on the right works out to

$$\int_2^x \frac{\mathrm{d}t}{\log^2 t} = \mathrm{li}(x) - \frac{x}{\log x} + O(1) = \frac{x}{\log^2 x} + O\Big(\frac{x}{\log^3 x}\Big)$$

by the asymptotic expansion of the logarithmic integral. The second integral on the right can split in two (around $\sqrt{x}$) and then estimated by

$$\int_2^{\sqrt{x}} \frac{\mathrm{d}t}{\log^3 t} + \int_{\sqrt{x}}^x \frac{\mathrm{d}t}{\log^3 t} \le \frac{\sqrt{x}}{\log^3 2} + \frac{x-\sqrt{x}}{\log^3 \sqrt{x}} = O\Big(\frac{x}{\log^3 x}\Big).$$

Putting everything together gives

$$\sum_{i=1}^{n} \log\log p_i = n\log\log x - \frac{x}{\log^2 x} + O\Big(\frac{x}{\log^3 x}\Big).$$

□

5 Optimal choice of m

The first bound in Lemma 4 allows us to show the existence of infinitely many abc triples whose ratio of c to $\operatorname{rad}(abc)$ grows arbitrarily large. Using the second bound, we can even show that this ratio grows faster than some function of c. The construction allows us the freedom to choose m in terms of n, though it is not immediately clear how to choose m optimally, i.e., to maximize the ratio $c/\operatorname{rad}(abc)$.

For convenience, let $R := \frac{n}{\delta}\det(L_{n,m})^{1/n}$ denote the upper bound on the second inequality in Lemma 4. Then $2^{m-1} = (\delta R/n)^n / \prod_{i=1}^n \log p_i$, so the bounds of Lemma 4 can be rewritten in terms of R:

$$\begin{gathered}\frac{(\delta R/n)^n}{\prod_{i=1}^n p_i \log p_i}\operatorname{rad}(abc) \le c \\ \log c \le R\end{gathered} \tag{4}$$

The question now becomes how to choose R in terms of n so that $c/\operatorname{rad}(abc)$ is maximized.

Taking the logarithm of the first inequality in (4) gives

$$n\log\Big(\frac{\delta R}{n}\Big) - \sum_{i=1}^n \log p_i - \sum_{i=1}^n \log\log p_i + \log\operatorname{rad}(abc) \le \log c.$$

Using the asymptotic formulae in Lemmas 5 and 6, this becomes

$$n\log\Big(\frac{e\delta R}{np_n\log p_n}\Big) + \frac{2p_n}{\log^2 p_n} + O\Big(\frac{p_n}{\log^3 p_n}\Big) + \log\operatorname{rad}(abc) \le \log c. \tag{5}$$

By the prime number theorem (3) the leftmost term becomes

$$n\log\bigg(\frac{e\delta R}{p_n^2\big(1+1/\log p_n + O(1/\log^2 p_n)\big)}\bigg),$$

and with $\log(1+1/x) = 1/x + O(1/x^2)$ as $x \to \infty$, this becomes

$$n\log\Big(\frac{e\delta R}{p_n^2}\Big) - \frac{n}{\log p_n} + O\Big(\frac{n}{\log^2 p_n}\Big).$$

Using (3) again on the last two terms and putting this back into (5), we get

$$n\log\Big(\frac{e\delta R}{p_n^2}\Big) + \frac{p_n}{\log^2 p_n} + O\Big(\frac{p_n}{\log^3 p_n}\Big) + \log\operatorname{rad}(abc) \le \log c, \tag{6}$$

and our goal becomes to choose R to maximize $n\log(e\delta R/p_n^2)$.

Clearly, we must take $R > p_n^2/(e\delta)$ for the logarithm to be positive. In order to maximize the leading factor n in terms of R, we will choose R in terms of n as asymptotically slow-growing as possible (subject to the above constraint). With the choice $R \coloneqq kp_n^2$ for some constant k we have that $n\log(e\delta R/p_n^2)$ simplifies down to

$$n\log(e\delta k) \sim \frac{p_n}{\log p_n}\log(e\delta k) = \frac{\sqrt{R/k}}{\log\sqrt{R/k}}\log(e\delta k) \sim \frac{2\sqrt{R/k}}{\log R}\log(e\delta k).$$

For fixed R this is maximized when $k \coloneqq e/\delta$. Using $R = ep_n^2/\delta$ in our previous result (6),

$$2n + \frac{p_n}{\log p_n} + O\Big(\frac{p_n}{\log^3 p_n}\Big) + \log\operatorname{rad}(abc) \le \log c.$$

By the prime number theorem (3),

$$\frac{2p_n}{\log p_n} + \frac{3p_n}{\log^2 p_n} + O\Big(\frac{p_n}{\log^3 p_n}\Big) + \log\operatorname{rad}(abc) \le \log c.$$

Rewriting in terms of R,

$$\frac{2\sqrt{\delta R/e}}{\log\sqrt{\delta R/e}} + \frac{3\sqrt{\delta R/e}}{\log^2\sqrt{\delta R/e}} + O\Big(\frac{\sqrt{R}}{\log^3 R}\Big) + \log\operatorname{rad}(abc) \le \log c.$$

Simplifying,

$$\frac{4\sqrt{\delta R/e}}{\log(\delta R/e)} + \frac{12\sqrt{\delta R/e}}{\log^2(\delta R/e)} + O\Big(\frac{\sqrt{R}}{\log^3 R}\Big) + \log\operatorname{rad}(abc) \le \log c.$$

Using $1/(x+y) = 1/x - y/x^2 + O(x^{-3})$ as $x \to \infty$ this gives

$$\frac{4\sqrt{\delta R/e}}{\log R} + \frac{(12 - 4\log(\delta/e))\sqrt{\delta R/e}}{\log^2 R} + O\Big(\frac{\sqrt{R}}{\log^3 R}\Big) + \log\operatorname{rad}(abc) \le \log c.$$

Using that $\delta < e^4$ the second term on the left is positive, and so for sufficiently large R the middle two terms are necessarily positive. Therefore for sufficiently large R this can be simplified to

$$\frac{4\sqrt{\delta R/e}}{\log R} + \log\operatorname{rad}(abc) \le \log c.$$

Using that $\log c \le R$ from (4) and the increasing monotonicity of $\sqrt{R}/\log R$ for sufficiently large R, we finally achieve that

$$\frac{4\sqrt{(\delta/e)\log c}}{\log\log c} + \log\operatorname{rad}(abc) \le \log c.$$

Taking the exponential, this proves the following theorem.

Theorem 1. *There are infinitely many abc triples which satisfy*

$$\exp\Big(\frac{4\sqrt{(\delta/e)\log c}}{\log\log c}\Big)\operatorname{rad}(abc) \le c.$$

Using Rankin's result on δ, the constant in the exponent becomes approximately 4.641.

6 Improvements

One way of viewing the prime number lattice is to think of each coordinate as encoding a different valuation. Indeed, the ith entry of $\boldsymbol{x} \in L_n$ is $-\log|p/q|_{p_i}$, where p/q is the S-unit associated to $\boldsymbol{x}$ and $|\cdot|_{p_i}$ is the p_i-adic valuation. From this viewpoint, one might wonder if there would be any advantage to include the real valuation as well. To do this, we can add an extra column to the basis defining L_n:

$$B := \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix} = \begin{bmatrix} \log 3 & & & & & k \log 3 \\ & \log 5 & & & & k \log 5 \\ & & \log 7 & & & k \log 7 \\ & & & \ddots & & \vdots \\ & & & & \log p_n & k \log p_n \end{bmatrix}.$$

Scaling the final column by the constant $k > 0$ allows us to increase the 'weight' of the real valuation, should that prove desirable. Of course, the modification of the lattice L_n requires us to update the previous lemmas.

Modified Lemma 1. *If $k \geq 1$ then $\|\boldsymbol{x}\|_1 \geq 2 \log h(p/q)$ where $\boldsymbol{x} = \sum_{i=1}^n e_i \boldsymbol{b}_i$ and $p/q = \prod_{i=1}^n p_i^{e_i}$ is expressed in lowest terms.*

Proof. For simplicity, say $p > q$ (the other case is analogous). By the definition of the one-norm, we have:

$$\begin{aligned} \|\boldsymbol{x}\|_1 &= \sum_{i=1}^n \left| e_i \log p_i \right| + \left| k \sum_{i=1}^n e_i \log p_i \right| \\ &= \log p + \log q + k(\log p - \log q) \\ &= (k+1) \log p - (k-1) \log q \\ &= 2 \log p + (k-1)(\log p - \log q) \end{aligned}$$

The result follows since $(k-1)(\log p - \log q)$ is nonnegative. $\square$

We remark that with $k = 1$ the inequality in this lemma becomes an equality. Taking $k > 1$ could potentially improve the bound, however in that case one would need a nontrivial lower bound on $\log p - \log q$, which is not straightforward.

The addition of another column in the lattice does cause its volume to increase, however ultimately the amount it increases will be irrelevant asymptotically.

Modified Lemma 2. *The volume of the lattice L_n is $\sqrt{nk^2 + 1} \prod_{i=1}^n \log p_i$.*

Proof. By definition, we have $\det(L_n) = \sqrt{\det(BB^T)}$. Factoring out $\log p_i$ from row i of B and column i of B^T for each $1 \leq i \leq n$ and letting $\boldsymbol{k} \in \mathbb{R}^n$ be the column vector consisting of all k entries, we get

$$\det(L_n) = \sqrt{\det\left(\left[\, I_n \mid \boldsymbol{k} \,\right] \left[\frac{I_n}{\boldsymbol{k}^T} \right] \right)} \cdot \prod_{i=1}^{n} \log p_i$$
$$= \sqrt{\det(I_n + \boldsymbol{k}\boldsymbol{k}^T)} \cdot \prod_{i=1}^{n} \log p_i.$$

From Sylvester's determinant theorem one has

$$\det(I_n + \boldsymbol{k}\boldsymbol{k}^T) = \det(I_1 + \boldsymbol{k}^T\boldsymbol{k}) = 1 + nk^2,$$

and it follows that

$$\det(L_n) = \sqrt{nk^2 + 1} \cdot \prod_{i=1}^{n} \log p_i.$$

□

Adapting the proof of Lemma 4 to the modified lattice presents a problem, since the new lattice is not full-rank, and the definition of δ was with respect to full-rank lattices. At first glance this shouldn't be a problem, since one could apply a rotation to the lattice of rank n to zero out its final column, and then consider the rotated lattice as a full-rank lattice in $\mathbb{R}^n$.

However, the act of rotation does not preserve distances in the one-norm, so the short vector which must exist in the full-rank rotation of $L_{n,m}$ may not actually be short in $L_{n,m}$ itself. Conceivably, one could measure how much the rotation could affect the one-norm of the short vector, and still apply the previous argumentation to derive a modified Lemma 4. Another possibility is to work with the two-norm, which is preserved by rotation.

Toward this end, define γ to be a constant so that $\gamma_n \leq n/\gamma^2$ holds for all sufficiently large n. By Kabatiansky–Levenshtein we can take $\gamma \approx 3.13$. The updated Lemma 4 can then be stated as follows.

Modified Lemma 4. *For all $m \geq 1$ and sufficiently large n, there exists an abc triple satisfying*

$$\frac{2^{m-1}}{\prod_{i=1}^{n} p_i} \operatorname{rad}(abc) \leq c \qquad \text{and} \qquad 2 \log c \leq \frac{n}{\gamma} \left(2^{m-1} \sqrt{n+1} \prod_{i=1}^{n} \log p_i \right)^{1/n}.$$

Proof. By the definition of γ applied to the kernel sublattice $L_{n,m}$ (which has rank n, and can be thought of as a rotated lattice in $\mathbb{R}^n$), we have that for all sufficiently large n there exists some nonzero $\boldsymbol{x} \in L_{n,m}$ with

$$\|\boldsymbol{x}\|^2 \leq \frac{n}{\gamma^2} \det(L_{n,m})^{2/n}.$$

By the relationship between the one-norm and two-norm, we have

$$\|\boldsymbol{x}\|_1 \leq \sqrt{n} \|\boldsymbol{x}\|_2 \leq \frac{n}{\gamma} \det(L_{n,m})^{1/n}.$$

The proof now proceeds exactly as before; we take $k \coloneqq 1$ in the definition of L_n so that by the modified Lemmas 1 and 2 we have

$$2 \log h(p/q) = \|\boldsymbol{x}\|_1 \qquad \text{and} \qquad \det(L_{n,m}) = 2^{m-1}\sqrt{n+1}\prod_{i=1}^{n} \log p_i. \qquad \square$$

The optimal choice of m also proceeds as before, but (4) now becomes:

$$\frac{(\gamma R/n)^n}{\sqrt{n+1}\prod_{i=1}^{n} p_i \log p_i} \operatorname{rad}(abc) \le c$$
$$2 \log c \le R$$

The extra factor of $(n+1)^{-1/2}$ in the first inequality has no affect in the argument since $-\frac{1}{2}\log(n+1)$ is $O(p_n/\log^3 p_n)$. Fortunately, the extra factor of 2 in the second inequality improves the constant in the exponent of the final result by a factor of $\sqrt{2}$. Taking $R \coloneqq ep_n^2/\gamma$ and using that $2\gamma < e^4$ one derives the following theorem.

Theorem 2. *There are infinitely many abc triples which satisfy*

$$\exp\Big(\frac{4\sqrt{(2\gamma/e)\log c}}{\log\log c}\Big) \operatorname{rad}(abc) \le c.$$

Using the Kabatiansky–Levenshtein result on γ, the constant in the exponent becomes slightly larger than 6.07.

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