Abstract

This note describes the best matrix conjecture from combinatorial design theory and the latest results that are known on the conjecture. In particular, examples of best matrices in orders $r^2 + r + 1$ for all $r$ up to and including 6 are given.

1 Introduction

Best matrices were introduced by Georgiou, Koukouvinos, and Seberry [2001] and further studied by Koukouvinos and Stylianou [2008] and Đoković [2009]. A quadruple of matrices $A$, $B$, $C$, $D$ are known as best matrices if they are square matrices of order $n$ with ±1 entries and satisfy the following axioms:

1. $A - I, B - I, C - I$ are skew matrices and $D$ is a symmetric matrix.
3. $AA^T + BB^T + CC^T + DD^T$ is the scalar matrix $4nI$.

Note that a matrix $X$ is symmetric if $X = X^T$, a matrix $X$ is skew if $X = -X^T$, and two matrices $X, Y$ commute if $XY = YX$. Best matrices can be used to generate skew Hadamard matrices via a construction introduced by Goethals and Seidel [1970]. In particular, if $A, B, C, D$ are best matrices then the Goethals–Seidel array

$$\begin{pmatrix}
A & BR & CR & DR \\
-BR & A & -D^T R & C^T R \\
-CR & D^T R & A & -B^T R \\
-DR & -C^T R & B^T R & A
\end{pmatrix}$$
gives a skew Hadamard matrix of order $4n$ where $R$ is the exchange matrix (anti-diagonal identity matrix) of order $n$.

Furthermore, $X$ is circulant if its $(i, j)$ entry is the same as its $(i + 1, j + 1)$ entry for all indices $i$ and $j$ (reducing mod $n$ if necessary). For the purposes of this note we will only consider circulant best matrices. In this case condition (2) is always satisfied.

Georgiou, Koukouvinos, and Seberry [2001] show that if circulant best matrices exist in odd order $n$ then $n$ must be of the form $(m^2 + 3)/4$ for odd $m$. In other words, letting $m = 2r + 1$ we have that $n = r^2 + r + 1$ and the possible values for $n$ are

$$\{1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, \ldots\}.$$  

Georgiou, Koukouvinos, and Seberry [2001] found that best matrices exist for all $r \leq 5$ and for many years no additional best matrices were found. Recently the situation changed as Đoković and Kotsireas [2018] found that best matrices also exist for $r = 6$, i.e., in order $n = 43$.

We call the best matrix conjecture the conjecture that best matrices exist in all orders of the form $r^2 + r + 1$. The conjecture is currently open for each $r \geq 7$.

## 2 Examples

We now explicitly give examples of best matrices for $r = 1, 2, \ldots, 6$. The first five examples were found by Georgiou, Koukouvinos, and Seberry [2001] and the sixth was found by Đoković and Kotsireas [2018]. In each example the four matrices $A$, $B$, $C$, $D$ are drawn using a different colour. The coloured squares represent 1 and the grey squares represent $-1$.

![Example matrices](image)

Case $r = 1$: Best matrices of order 3.
Case $r = 2$: Best matrices of order 7.

Case $r = 3$: Best matrices of order 13.

Case $r = 4$: Best matrices of order 21.
Case $r = 5$: Best matrices of order 31.

Case $r = 6$: Best matrices of order 43.

References


