A NEW LOWER BOUND IN THE *ABC* CONJECTURE

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Abstract. We prove that there exist infinitely many coprime numbers a, b, c with $a+b=c$ and $c > \mathrm{rad}(abc)\exp(6.563\sqrt{\log c}/\log\log c)$. These are the most extremal examples currently known in the *abc* conjecture, thereby providing a new lower bound on the tightest possible form of the conjecture. This builds on work of van Frankenhuysen (1999) whom proved the existence of examples satisfying the above bound with the constant 6.068 in place of 6.563. We show that the constant 6.563 may be replaced by $4\sqrt{2\delta/e}$ where δ is a constant such that all full-rank unimodular lattices of sufficiently large dimension n contain a nonzero vector with ℓ_1 norm at most n/δ.

Key words. *abc* conjecture; good *abc* examples; *abc* conjecture lower bound.

MSC codes. 11D75, 11H06, 11G50, 11N25

1. Introduction. Three natural numbers a, b, c are said to be an *abc triple* if they do not share a common factor and satisfy the equation

$$a+b=c.$$

Informally, the *abc* conjecture says that large *abc* triples cannot be 'very composite', in the sense of *abc* having a prime factorization containing large powers of small primes. The *radical* of *abc* is defined to be the product of the primes in the prime factorization of *abc*, i.e.,

$$\mathrm{rad}(abc) \coloneqq \prod_{p|abc} p.$$

The *abc* conjecture then states that *abc* triples satisfy

$$c = O\big(\mathrm{rad}(abc)^{1+\epsilon}\big) \tag{1.1}$$

for every $\epsilon > 0$, where the implied big-O constant may depend on ϵ.

Presently, the conjecture is far from being proved; not a single ϵ is known for which (1.1) holds.[1] The best known upper bound is due to C. L. Stewart and K. Yu [11] and says that *abc* triples satisfy

$$c = O\big(\exp(\mathrm{rad}(abc)^{1/3}(\log \mathrm{rad}(abc))^3)\big).$$

On the other hand, Machiel van Frankenhuysen [12] proved in 1997 that there are infinitely many *abc* triples with

$$c > \mathrm{rad}(abc)\exp(k\sqrt{\log c}/\log\log c) \tag{1.2}$$

for $k = 4\sqrt{2}$, and two years later improved this to $k = 6.068$ using a sphere-packing idea credited to H. W. Lenstra. Such *abc* triples are exceptional in the sense that their radical is relatively small in comparison to c and they provide a lower bound on the best possible form of (1.1). Prior to van Frankenhuysen's result, C. L. Stewart and R. Tijdeman [10] proved that there are infinitely many *abc* triples satisfying (1.2) for all $k < 4$. We improve on these results by showing that there are infinitely many *abc* triples satisfying (1.2) with $k = 6.563$.

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[1] A proof of the *abc* conjecture is claimed by S. Mochizuki, but this has not been accepted by the general mathematical community. [9]

2. Preliminaries. Let S be a set of prime numbers. An S-unit is defined to be a rational number whose numerator and denominator in lowest terms are divisible by only the primes in S. That is, one has

$$S\text{-units} \coloneqq \left\{ \pm \prod_{p_i \in S} p_i^{e_i} : e_i \in \mathbb{Z} \right\}.$$

This generalizes the notion of units of $\mathbb{Z}$; in particular, the $\emptyset$-units are ± 1. The *height* of a rational number p/q in lowest terms is $h(p/q) \coloneqq \max\{|p|, |q|\}$. This provides a convenient way of measuring the 'size' of an S-unit. Finally, if $\boldsymbol{x} = (x_1, \ldots, x_n)$ is a vector in $\mathbb{R}^n$, we let

$$\|\boldsymbol{x}\|_k \coloneqq \Big(\sum_{i=1}^{n} |x_i|^k \Big)^{1/k}$$

be its standard ℓ_k norm. The existence of exceptional *abc* triples follows from some basic results in the geometry of numbers. In particular, from the existence of a short nonzero vector in a suitably chosen lattice.

2.1. The odd prime number lattice. The result involves in an essential way the *odd prime number lattice* L_n generated by the rows $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ of the matrix

$$\begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix} = \begin{bmatrix} \log 3 & & & & & \log 3 \\ & \log 5 & & & & \log 5 \\ & & \log 7 & & & \log 7 \\ & & & \ddots & & \\ & & & & \log p_n & \log p_n \end{bmatrix}$$

where p_i denotes the ith odd prime number. This lattice has a number of interesting applications. For example, it is used in Schnorr's factoring algorithm [8] and Micciancio's proof that approximating the shortest vector to within a constant factor is NP-hard under a randomized reduction [6]. There is an obvious isomorphism between the points of L_n and the positive $\{p_1, \ldots, p_n\}$-units given by

$$\sum_{i=1}^{n} e_i \boldsymbol{b}_i \leftrightarrow \prod_{i=1}^{n} p_i^{e_i}.$$

Furthermore, this relationship works well with a natural notion of size, as shown in the following lemma.

LEMMA 2.1. *$\|\boldsymbol{x}\|_1 = 2 \log h(p/q)$ where $\boldsymbol{x} = \sum_{i=1}^{n} e_i \boldsymbol{b}_i$ and $p/q = \prod_{i=1}^{n} p_i^{e_i}$ is expressed in lowest terms.*

Proof. Without loss of generality suppose $p \geq q$. Then

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} \left| e_i \log p_i \right| + \left| \sum_{i=1}^{n} e_i \log p_i \right| = \log p + \log q + \log p - \log q = 2 \log p.$$

∎

2.2. The kernel sublattice. Let P be the set of positive $\{p_1, \ldots, p_n\}$-units, and consider the map ϕ reducing the elements of P modulo 2^m. Since each $p_1, \ldots, p_n$ is odd, $\phi\colon P \to (\mathbb{Z}/2^m\mathbb{Z})^*$ is well-defined. The odd prime number lattice L_n has

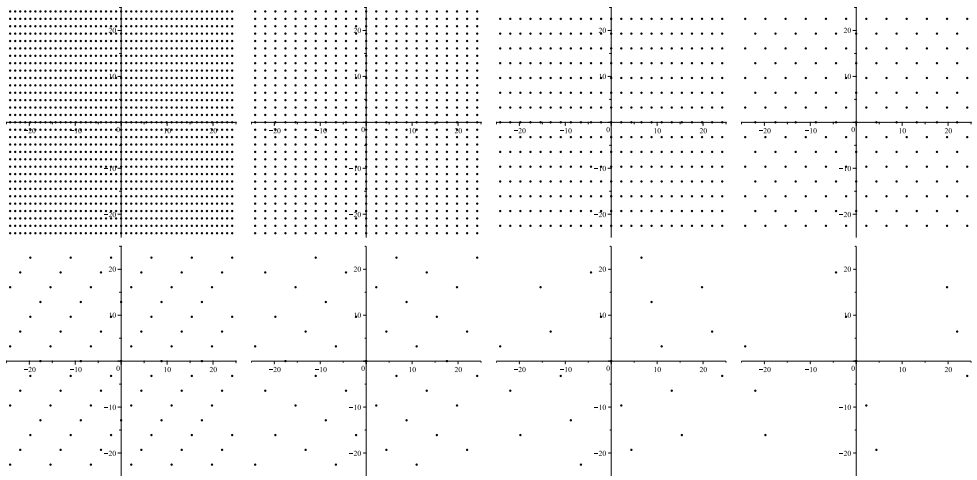

Fig. 1: Plots of $\{\,(x,y) : (x,y,z) \in L_{2,m}\,\}$ for $1 \le m \le 8$.

an important sublattice that we call the *kernel sublattice* $L_{n,m}$. It consists of those vectors whose associated $\{p_1,\dots,p_n\}$-units lie in the kernel of ϕ. Formally, we define

$$L_{n,m} \coloneqq \left\{ \sum_{i=1}^{n} e_i \boldsymbol{b}_i : \prod_{i=1}^{n} p_i^{e_i} \equiv 1 \pmod{2^m} \right\}.$$

Figure 1 plots the first two coordinates of vectors in the kernel sublattice for varying m.

LEMMA 2.2. *$L_{n,m}$ is a sublattice of L_n of index 2^{m-1} when $n \ge 2$.*

Proof. Note that $L_{n,m}$ is discrete and closed under addition and subtraction. $L_{n,m}$ also contains the n linearly independent vectors $\operatorname{ord}_{2^m}(p_i)\boldsymbol{b}_i$ for $1 \le i \le n$, so this demonstrates that $L_{n,m}$ is a full-rank sublattice of L_n.

Since 3 and 5 generate $(\mathbb{Z}/2^m\mathbb{Z})^*$, when $n \ge 2$ we have $\phi(P) = (\mathbb{Z}/2^m\mathbb{Z})^*$. Since $L_n \cong P$ and $L_{n,m} \cong \ker\phi$ it follows that $L_n/L_{n,m} \cong (\mathbb{Z}/2^m\mathbb{Z})^*$ by the first isomorphism theorem. Thus the index of $L_{n,m}$ in L_n is $|(\mathbb{Z}/2^m\mathbb{Z})^*| = 2^{m-1}$. ∎

2.3. Hermite's constant. The *Hermite constant* γ_n is defined to be the smallest positive number such that every lattice of dimension n and volume $\det(L)$ contains a nonzero vector $\boldsymbol{x}$ with

$$\|\boldsymbol{x}\|_2^2 \le \gamma_n \det(L)^{2/n}.$$

It is actually sufficient to only consider *unimodular* lattices in this definition, as an arbitrary lattice can be reduced to a unimodular lattice by scaling each vector by a factor $\det(L)^{-1/n}$. The first formulation of γ_n is more convenient for our purpose, as our construction will ultimately be based on the existence of a relatively short vector in a lattice with $\det(L) \ne 1$. The existence of such the constant γ_n was first shown by Hermite [3], who proved the exponential bound $\gamma_n \le \sqrt{4/3}^{n-1}$.

It is now known that γ_n grows linearly in n. By Minkowski's theorem [2] applied to a sphere of sufficiently large volume, it follows that

$$\gamma_n \le 4\omega_n^{-2/n} \sim \frac{2n}{\pi e} \approx 0.234n$$

where ω_n is the volume of the n-dimensional unit sphere. Improving on this, Kabatiansky and Levenshtein [5] showed for sufficiently large n that

$$\gamma_n \leq \frac{2n}{4^{0.599}\pi e} \approx 0.102n.$$

We are interested in the "Manhattan distance" ℓ_1 norm instead of the usual Euclidean norm, so we define the related constants δ_n by the smallest positive number such that every full-rank lattice of dimension n contains a nonzero vector $\boldsymbol{x}$ with

$$\|\boldsymbol{x}\|_1 \leq \delta_n \det(L)^{1/n}.$$

By Minkowski's theorem applied to a generalized octahedron (a 'sphere' in the ℓ_1 norm), one has that every full-rank lattice of dimension n contains a nonzero vector $\boldsymbol{x}$ with

$$\|\boldsymbol{x}\|_1 \leq (n! \det(L))^{1/n},$$

from which it follows that

$$\delta_n \leq (n!)^{1/n} \sim \frac{n}{e} \approx 0.368n.$$

It is also possible to use bounds on γ_n to derive bounds on δ_n. By the relationship between the ℓ_1 norm and ℓ_2 norm, we have that every lattice of dimension n contains a nonzero vector $\boldsymbol{x}$ with

$$\|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2 \leq \sqrt{n\gamma_n} \det(L)^{1/n}.$$

Therefore $\delta_n \leq \sqrt{n\gamma_n}$, and by the result of Kabatiansky–Levenshtein,

$$\delta_n \leq \sqrt{\frac{2n^2}{4^{0.599}\pi e}} \approx 0.320n$$

for large enough n.

However, better bounds on δ_n are known. Blichfeldt [1] showed that

$$\delta_n \leq \sqrt{\frac{4(n+1)(n+2)}{3\pi(n+3)}} \left(\frac{2(n+1)}{n+3}\left(\frac{n}{2}+1\right)!\right)^{1/n} \sim \sqrt{\frac{2}{3\pi e}}n \approx 0.279n,$$

where $x! \coloneqq \Gamma(x+1)$. Improving this, Rankin [7] showed that

$$\delta_n \leq \left(\frac{2-x}{1-x}\right)^{x-1} \left(\frac{1+xn}{x \cdot x!^n}(xn)!\right)^{1/n} n^{1-x} \sim \left(\frac{2-x}{1-x}\right)^{x-1} \frac{(x/e)^x}{x!} n$$

for all $x \in [1/2, 1]$. This attains a minimum for $x \approx 0.645$, so that the expression on the right becomes approximately $0.273n$. For convenience, we define δ to be a constant so that $\delta_n \leq n/\delta$ holds for all sufficiently large n. In light of Rankin's result, we can take $\delta \approx 3.659$.

2.4. A full-rank kernel sublattice. Since $L_{n,m} \in \mathbb{R}^{n+1}$ is of dimension n (i.e., not full-rank) it is awkward to use Rankin's result on $L_{n,m}$ directly. The basis matrix of $L_{n,m}$ cannot simply be rotated to embed it in $\mathbb{R}^n$, since rotation does not preserve the ℓ_1 norm. To circumvent this and work with a full-rank lattice we adjoin the new basis vector $\boldsymbol{b}_{n+1} = [0, \ldots, 0, \alpha]$ to L_n to form a full-rank lattice $\overline{L}_n$ (and similarly a full-rank lattice $\overline{L}_{n,m}$) where $\alpha > 0$ is a fixed constant.

LEMMA 2.3. *The volume of* $\overline{L}_{n,m}$ *is* $2^{m-1}\alpha\prod_{i=1}^{n}\log p_i$ *when* $n \geq 2$.

Proof. The basis matrix of L_n adjoined with $\boldsymbol{b}_{n+1}$ is an upper-triangular matrix, so $\det(\overline{L}_n) = \alpha\prod_{i=1}^{n}\log p_i$. The index of $\overline{L}_{n,m}$ in $\overline{L}_n$ is 2^{m-1} when $n \geq 2$ by the same argument as in Lemma 2.2. Thus $\det(\overline{L}_{n,m}) = 2^{m-1}\det(\overline{L}_n)$. ∎

We can show that any vector in $\overline{L}_n$ including a nontrivial coefficient on $\boldsymbol{b}_{n+1}$ must be sufficiently large (have length at least α in the ℓ_1 norm).

LEMMA 2.4. *If* $\boldsymbol{x} = \sum_{i=1}^{n+1} e_i\boldsymbol{b}_i$ *then* $\|\boldsymbol{x}\|_1 \geq \alpha|e_{n+1}|$.

Proof. We have $\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n}|e_i|\log p_i + \bigl|\sum_{i=1}^{n} e_i\log p_i + e_{n+1}\alpha\bigr|$.

Without loss of generality suppose that $e_{n+1} > 0$ and for contradiction suppose $\|\boldsymbol{x}\|_1 < \alpha e_{n+1}$. Then

$$\sum_{i=1}^{n} e_i\log p_i + e_{n+1}\alpha \leq \Bigl|\sum_{i=1}^{n} e_i\log p_i + e_{n+1}\alpha\Bigr| < \alpha e_{n+1} - \sum_{i=1}^{n}|e_i|\log p_i$$

implies $\sum_{i=1}^{n}(e_i + |e_i|)\log p_i < 0$, and this is nonsensical since the left-hand side is nonnegative. ∎

For concreteness we will now take $\alpha := n^3$ as this will suffice for our ultimate aim.

2.5. Asymptotic formulae. Let $x := p_n$ and let $\pi(x)$ be the prime counting function, so that $n = \pi(x) - 1$. The prime number theorem [4] states that $\pi(x) \sim \operatorname{li}(x)$ where $\operatorname{li}(x)$ is the logarithmic integral $\int_0^x \frac{\mathrm{d}t}{\log t}$ with asymptotic expansion

$$\operatorname{li}(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + O\Bigl(\frac{x}{\log^4 x}\Bigr). \tag{2.1}$$

The following properties follow from the prime number theorem. For the convenience of the reader, proofs are given in the appendix.

LEMMA 2.5. $\sum_{i=1}^{n}\log p_i = n\log p_n - n - p_n/\log^2 p_n + O(p_n/\log^3 p_n)$.

LEMMA 2.6. $\sum_{i=1}^{n}\log\log p_i = n\log\log p_n - p_n/\log^2 p_n + O(p_n/\log^3 p_n)$.

These allow us to capture the growth rate of quantities like the volume of $\overline{L}_{n,m}$. Ultimately our choice of m will be approximately $n\log_2 n$, so this will be used in the next lemma.

LEMMA 2.7. *If* $m \sim n\log_2 n$ *then* $\det(\overline{L}_{n,m})^{1/(n+1)} = O(n^{1+\epsilon})$ *for all* $\epsilon > 0$.

Proof. By Lemmas 2.3 and 2.6,

$$\det(\overline{L}_{n,m}) = 2^{m-1}\alpha\exp\bigl(n\log\log p_n + o(n)\bigr).$$

Then $\sqrt[n+1]{2^{n\log_2 n + o(n\log_2 n) - 1}} = O(n^{1+\epsilon})$ for all $\epsilon > 0$, $\sqrt[n+1]{\alpha} \sim 1$ when $\alpha = n^3$, and

$$\sqrt[n+1]{\exp\bigl(n\log\log p_n + o(n)\bigr)} \sim \log n.$$ ∎

3. Exceptional *abc* triples. For our purposes the importance of the kernel sublattice is that it lets us show the existence of *abc* triples in which c is large relative to $\operatorname{rad}(abc)$. The following lemma shows how this may be done.

LEMMA 3.1. *For all* $m \lesssim n\log_2 n$ *and sufficiently large* n, *there exists an abc triple satisfying*

$$\frac{2^{m-1}}{\prod_{i=1}^{n} p_i}\operatorname{rad}(abc) \leq c \quad \textit{and} \quad 2\log c \leq \frac{n+1}{\delta}\Bigl(2^{m-1}n^3\prod_{i=1}^{n}\log p_i\Bigr)^{1/(n+1)}.$$

Proof. The constant δ has been chosen so that for all sufficiently large n there exists a nonzero $\boldsymbol{x} \in \overline{L}_{n,m}$ with

$$\|\boldsymbol{x}\|_1 \le \frac{n+1}{\delta} \det(\overline{L}_{n,m})^{1/(n+1)}. \tag{3.1}$$

Say $\boldsymbol{x} = \sum_{i=1}^{n+1} e_i \boldsymbol{b}_i$. For sufficiently large n we must have $e_{n+1} = 0$, since by Lemma 2.4 if $e_{n+1} \neq 0$ then $\|\boldsymbol{x}\|_1 \ge n^3$. This would contradict (3.1) since by Lemma 2.7 the right-hand side is $O(n^{2+\epsilon})$.

Let $\prod_{i=1}^{n} p_i^{e_i} = p/q$ be expressed in lowest terms. By construction of the kernel sublattice, we have that $p/q \equiv 1 \pmod{2^m}$. Let $c \coloneqq h(p/q) = \max\{p, q\}$, $b \coloneqq \min\{p, q\}$, and $a \coloneqq c - b$, so that a, b, c form an abc triple. Furthermore, we see that

$$c \equiv b \pmod{2^m}$$

so that $c = b + k2^m$ for some positive integer $k \le c/2^m$. Note a is divisible by 2 and any other prime that divides it also divides k, so that $\operatorname{rad}(a) \le 2k \le c/2^{m-1}$. Furthermore, by construction of b and c, $\operatorname{rad}(bc) \le \prod_{i=1}^{n} p_i$ and the first bound follows. The second bound follows from (3.1) and Lemmas 2.1 and 2.3. ∎

3.1. Optimal choice of m. The first bound in Lemma 3.1 allows us to show the existence of infinitely many abc triples whose ratio of c to $\operatorname{rad}(abc)$ grows arbitrarily large. Using the second bound, we can even show that this ratio grows faster than a function of c. It is not immediately clear how to choose m optimally, i.e., to maximize the ratio $c/\operatorname{rad}(abc)$.

For convenience, let R denote the right-hand side of the second inequality in Lemma 3.1. Then $2^{m-1} = (\frac{\delta R}{n+1})^{n+1}/(n^3 \prod_{i=1}^{n} \log p_i)$, so the bounds of Lemma 3.1 can be rewritten in terms of R:

$$\frac{(\delta R/(n+1))^{n+1}}{n^3 \prod_{i=1}^{n} p_i \log p_i} \operatorname{rad}(abc) \le c \quad \text{and} \quad 2 \log c \le R. \tag{3.2}$$

The question now becomes how to choose R in terms of n so that $c/\operatorname{rad}(abc)$ is maximized.

Taking the logarithm of the first inequality in (3.2) gives

$$(n+1)\log\Big(\frac{\delta R}{n+1}\Big) - 3\log n - \sum_{i=1}^{n} \log p_i - \sum_{i=1}^{n} \log\log p_i + \log\operatorname{rad}(abc) \le \log c.$$

Using the asymptotic formulae in Lemmas 2.5 and 2.6 with $\log(n+1) = \log n + o(1)$, this becomes

$$n\log\Big(\frac{e\delta R}{np_n \log p_n}\Big) + \frac{2p_n}{\log^2 p_n} + O\Big(\frac{p_n}{\log^3 p_n}\Big) + \log\operatorname{rad}(abc) \le \log c. \tag{3.3}$$

By the prime number theorem $n \sim \operatorname{li}(p_n)$ and (2.1) the leftmost term becomes

$$n\log\bigg(\frac{e\delta R}{p_n^2\big(1 + 1/\log p_n + O(1/\log^2 p_n)\big)}\bigg),$$

and with $\log(1 + 1/x) = 1/x + O(1/x^2)$ as $x \to \infty$, this becomes

$$n\log\Big(\frac{e\delta R}{p_n^2}\Big) - \frac{n}{\log p_n} + O\Big(\frac{n}{\log^2 p_n}\Big).$$

Using (2.1) again on the last two terms and putting this back into (3.3), we get

$$n\log\Big(\frac{e\delta R}{p_n^2}\Big)+\frac{p_n}{\log^2 p_n}+O\Big(\frac{p_n}{\log^3 p_n}\Big)+\log\operatorname{rad}(abc)\le\log c, \tag{3.4}$$

and our goal becomes to choose R as a function of n to maximize $n\log(e\delta R/p_n^2)$. Choosing R as asymptotically slow-growing as possible in terms of n will maximize this in terms of R. We must take $R>p_n^2/(e\delta)$ for the logarithm to be positive, so we take $R\coloneqq kp_n^2$ for some constant k. Note that with this choice $m\sim n\log_2 n$, so Lemma 3.1 applies. We have that $n\log(e\delta R/p_n^2)$ simplifies to

$$n\log(e\delta k)\sim\frac{p_n}{\log p_n}\log(e\delta k)=\frac{\sqrt{R/k}}{\log\sqrt{R/k}}\log(e\delta k)\sim\frac{2\sqrt{R/k}}{\log R}\log(e\delta k).$$

For fixed R this is maximized when $k\coloneqq e/\delta$. Using $R=ep_n^2/\delta$ in our previous result (3.4),

$$2n+\frac{p_n}{\log^2 p_n}+O\Big(\frac{p_n}{\log^3 p_n}\Big)+\log\operatorname{rad}(abc)\le\log c.$$

By the prime number theorem and (2.1) again,

$$\frac{2p_n}{\log p_n}+\frac{3p_n}{\log^2 p_n}+O\Big(\frac{p_n}{\log^3 p_n}\Big)+\log\operatorname{rad}(abc)\le\log c.$$

Rewriting in terms of R,

$$\frac{2\sqrt{\delta R/e}}{\log\sqrt{\delta R/e}}+\frac{3\sqrt{\delta R/e}}{\log^2\sqrt{\delta R/e}}+O\Big(\frac{\sqrt{R}}{\log^3 R}\Big)+\log\operatorname{rad}(abc)\le\log c.$$

Simplifying,

$$\frac{4\sqrt{\delta R/e}}{\log(\delta R/e)}+\frac{12\sqrt{\delta R/e}}{\log^2(\delta R/e)}+O\Big(\frac{\sqrt{R}}{\log^3 R}\Big)+\log\operatorname{rad}(abc)\le\log c.$$

Using $1/(x+y)=1/x-y/x^2+O(x^{-3})$ as $x\to\infty$ this gives

$$\frac{4\sqrt{\delta R/e}}{\log(R/2)}+\frac{(12-4\log(2\delta/e))\sqrt{\delta R/e}}{\log^2 R}+O\Big(\frac{\sqrt{R}}{\log^3 R}\Big)+\log\operatorname{rad}(abc)\le\log c.$$

Using that $2\delta<e^4$ the second term on the left is positive, and so for sufficiently large R the middle two terms are necessarily positive. Therefore for sufficiently large R this can be simplified to

$$\frac{4\sqrt{\delta R/e}}{\log(R/2)}+\log\operatorname{rad}(abc)\le\log c.$$

Using that $2\log c\le R$ from (3.2) and the increasing monotonicity of $\sqrt{R}/\log(R/2)$ for sufficiently large R, we finally achieve that

$$\frac{4\sqrt{2(\delta/e)\log c}}{\log\log c}+\log\operatorname{rad}(abc)\le\log c.$$

Taking the exponential, this proves the following theorem.

THEOREM 3.2. *There are infinitely many abc triples satisfying*

$$\exp\left(\frac{4\sqrt{2(\delta/e)\log c}}{\log\log c}\right)\operatorname{rad}(abc) \leq c.$$

Using Rankin's result on δ, the constant in the exponent becomes approximately 6.563.

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Appendix.

LEMMA 2.5. $\sum_{i=1}^{n} \log p_i = n\log p_n - n - p_n/\log^2 p_n + O(p_n/\log^3 p_n)$.

Proof. Let $x \coloneqq p_n$, so by the prime number theorem $n \sim \operatorname{li}(x)$ and rearranging the asymptotic expansion of the logarithmic integral (2.1) gives

$$\begin{aligned} x &= n\log x - \frac{x}{\log x} - \frac{2x}{\log^2 x} + O\Big(\frac{x}{\log^3 x}\Big) \\ &= n\log x - n - \frac{x}{\log^2 x} + O\Big(\frac{x}{\log^3 x}\Big). \end{aligned}$$

An equivalent form of the prime number theorem gives $x \sim \sum_{p\leq x} \log p$, so the left-hand side may be replaced by $\sum_{i=1}^{n} \log p_i$ from which the result follows. ∎

LEMMA 2.6. $\sum_{i=1}^{n} \log\log p_i = n\log\log p_n - p_n/\log^2 p_n + O(p_n/\log^3 p_n)$.

Proof. By Abel's summation formula with $f(k) \coloneqq \log\log k$ and

$$a_k \coloneqq \begin{cases} 1 & \text{if } k \text{ is an odd prime} \\ 0 & \text{otherwise} \end{cases}$$

for k up to $x \coloneqq p_n$, we have

$$\sum_{i=1}^{n} \log\log p_i = n \log\log x - \int_2^x \frac{\pi(t)-1}{t\log t}\,\mathrm{d}t.$$

We have $\pi(t) - 1 = t/\log t + O(t/\log^2 t)$ by the prime number theorem, so that

$$\int_2^x \frac{\pi(t)-1}{t\log t}\,\mathrm{d}t = \int_2^x \frac{\mathrm{d}t}{\log^2 t} + O\Big(\int_2^x \frac{\mathrm{d}t}{\log^3 t}\Big).$$

The first integral on the right works out to

$$\int_2^x \frac{\mathrm{d}t}{\log^2 t} = \mathrm{li}(x) - \frac{x}{\log x} + O(1) = \frac{x}{\log^2 x} + O\Big(\frac{x}{\log^3 x}\Big)$$

by the asymptotic expansion of the logarithmic integral. The second integral on the right can split in two (around $\sqrt{x}$) and then estimated by

$$\int_2^{\sqrt{x}} \frac{\mathrm{d}t}{\log^3 t} + \int_{\sqrt{x}}^x \frac{\mathrm{d}t}{\log^3 t} \le \frac{\sqrt{x}}{\log^3 2} + \frac{x-\sqrt{x}}{\log^3\sqrt{x}} = O\Big(\frac{x}{\log^3 x}\Big).$$

Putting everything together gives

$$\sum_{i=1}^{n} \log\log p_i = n\log\log x - \frac{x}{\log^2 x} + O\Big(\frac{x}{\log^3 x}\Big).$$

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