PMATH 351 Lecture 5: January 13, 2010

Textbook on reserve in DC, call no 1359
Correction to question 2 on assignment 1: Let X and Y be sets, $X \neq \emptyset$ (insert)

Let X be a set, $\leq$ be a partial ordering on X. An element $a \in X$ in *maximal* if the only element $b \in X$ such that $a \leq b$ is $b = a$. Notation: $a < b$ means $a \leq b$ and $a \neq b$. So, $a \in X$ is maximal if there exists no $b \in X$, $a < b$. Notation: $a \geq b$ means $b \leq a$, and $a > b$ means $b < a$.

A subset C of X is *nested* if for any two elements $a, b \in C$, either $a \leq b$ or $b \leq a$. A nested subset is also known as a *chain*, or a *tower*.

An element $b \in X$ is an *upper bound* of $A \subset X$ if for each $a \in A$, $a \leq b$.

Zorn's Lemma: Let $(X, \leq)$ be a partially ordered set. Suppose that every chain C in X has an upper bound in X. Then there exists a maximal element in X.

Example: Let V be a vector space over a field F. Let $X = \{\, A \subset V : A \text{ is linearly independent} \,\}$. Let $\leq$ on X be set inclusion, i.e., $A_1 \leq A_2$ means $A_1 \subset A_2$.

If C is a chain in X, then $\bigcup C$(notation: $\bigcup_{A \in C} A) \in X$. [your assignment]. Clearly, for each $A \in C$, $A \subset \bigcup C$ (i.e., $A \leq \bigcup C$). Thus $\bigcup C$ is an upper bound of C.

Hence, the supposition of Zorn's Lemma is satisfied. Thus, by Zorn's Lemma, there exists, in X, a maximal B. That is:

(1) $B \in X$, i.e., B is linearly independent

(2) B is maximal in X, i.e., no linearly independent subset A (of V) is (strictly) larger than B.

Consider span(B), which is a subspace of V. If span$(B) \subsetneq V$, then we can take a $v_0 \in V$, $v_0 \notin$ span(B), and obtain a strictly larger linearly independent set $B \cup \{v_0\}$. That will contradict the maximality of B. This shows that, when B is maximal, span$(B) = V$.

B is thus a *basis* for V.

This example shows that, when we assume that axiom of choice *or* equivalently the Zorn's Lemma, it leads to the theorem: every vector space, over a field F, has a basis.

Example: Let us consider $X = \{\,]a, b[$[1] $: a, b \in \mathbb{R},\ a < b \,\}$. Let X be partially ordered by set inclusion. There is no *maximal* element, because for any $]a, b[\in X$, we see that $]a, b+1[$ is strictly larger.

The chain $C = \{\,]-n, n[: n \in \mathbb{N} = \{1, 2, \ldots\} \,\}$ has *no* upper bound in X.

PMATH 351 Lecture 6: January 15, 2010

Information Session on Grad Studies for 3rd and 4th year undergrads in the Faculty of Mathematics Thursday, January 21, 4:00 pm DC 1302
Refreshments will be served.

Topological Spaces

Let X be a set, $X \neq \emptyset$. A subset of $\mathcal{P}(X)$, $\mathcal{T}$, is called a *topology* on X if it is closed under taking finite intersection and arbitrary union. To be precise, we mean for any finite $\mathcal{A} \subset \mathcal{T}$, $\bigcap \mathcal{A} \in \mathcal{T}$ and for any $\mathcal{A} \subset \mathcal{T}$, $\bigcup \mathcal{A} \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space.

Example:

(1) $\mathcal{T} = \mathcal{P}(X)$ is a topology on X. This is called *the discrete topology* on X.

(2) $\mathcal{T} = \{\emptyset, X\}$ is called the *indiscrete topology* on X. Q: $\mathcal{T} = \emptyset$? No.

[1] open interval

(3) Let X be an infinite set. Let

$$\mathcal{T} = \left\{ \emptyset, X, A : X \setminus A^{2)} \text{ is finite} \right\}$$

Then $\mathcal{T}$ is a topology on X. This is called the co-finite topology *or* the topology of finite complements.

venn diagram of $A \cap B$ in X

$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Proposition: In a topological space $(X, \mathcal{T})$, $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
Proof: Let $\mathcal{A} = \emptyset$ ($\mathcal{A} \subset \mathcal{T}$), a finite set.

$$\begin{aligned} \bigcap \mathcal{A} &= \{\, x \in X : x \in A \text{ for } \textit{all } A \in \mathcal{A} \,\} \\ &= X \\ \bigcup \mathcal{A} &= \{\, x \in X : x \in A \text{ for } \textit{some } A \in \mathcal{A} \,\} \\ &= \emptyset \end{aligned}$$

$\mathcal{A} = \{A_1, A_2\}$
$\bigcap \mathcal{A} = A_1 \cap A_2$

(4) $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, X, \{a, b\}\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$

Proposition: Let $X \neq \emptyset$ and let $\{\, \mathcal{T}_i : i \in I \,\}$ be a family of topologies on X, say that $I \neq \emptyset$. Then $\bigcap_{i \in I} \mathcal{T}_i$ is a topology on X.

PMATH 351 Lecture 7: January 18, 2010

If $\{\, \mathcal{T}_i : i \in I \,\}$ is a non-empty family of topologies on X, then $\bigcap_{i \in I} \mathcal{T}_i$ is a top (on X)
Proof:

1. $\emptyset \in \mathcal{T}_i$ for each $i \in I$, as each $\mathcal{T}_i$ is a top. So $\emptyset \in \bigcap_{i \in I} \mathcal{T}_i$. Similarly, $X \in \bigcap_{i \in I} \mathcal{T}_i$.
2. We shall show that if A and B are in $\bigcap_{i \in I} \mathcal{T}_i$, then $A \cap B \in \bigcap_{i \in I} \mathcal{T}_i$. For each $i \in I$, $A \in \mathcal{T}_i$ and $B \in \mathcal{T}_i$ by definition of intersection. Since $\mathcal{T}_n$ is a topology, $A \cap B \in \mathcal{T}_i$. So $A \cap B \in \bigcap_{i \in I} \mathcal{T}_i$.
3. Let $A_j \in \bigcap_{i \in I} \mathcal{T}_i$ for each $j \in J$. Then, for each $i \in I$, $A_j \in \mathcal{T}_i$ for each $j \in J$. As $\mathcal{T}_i$ is a topology, $\bigcup_{j \in J} A_j \in \mathcal{T}_i$. As $i \in I$ is arbitrary, $\bigcup_{j \in J} A_j \in \bigcap_{i \in I} \mathcal{T}_i$. This shows that $\bigcap_{i \in I} \mathcal{T}_i$ is *closed* under arbitrary union.

Proposition: Let X be a non-empty set. Let $\mathcal{S}$ be any given family of subsets of X (i.e., $\mathcal{S} \subset \mathcal{P}(X)$). Then there exists a topology $\mathcal{T}_0$ on X such that (1) $\mathcal{T}_0 \supset \mathcal{S}$ (2) if $\mathcal{T}$ is a topology on X and $\mathcal{T} \supset \mathcal{S}$, then $\mathcal{T}_0 \subset \mathcal{T}$. So, $\mathcal{T}_0$ is the smallest topology on X which contains $\mathcal{S}$.

Proof: Consider $\mathcal{G} = \{\, \mathcal{T} : \mathcal{T} \text{ is a topology on } X,\ \mathcal{T} \supset \mathcal{S} \,\}$. Clearly, the discrete topology, $\mathcal{P}(X)$, contains $\mathcal{S}$ and so it is an element of $\mathcal{G}$. Thus $\mathcal{G} \neq \emptyset$.
Now $\mathcal{T}_0 \overset{\text{def}}{=} \bigcap \mathcal{G}$ is a topology on X by the previous theorem. Since each $\mathcal{T} \in \mathcal{G}$ clearly contains $\mathcal{T}_0$, this shows that (2) holds.

Definition: We call $\mathcal{T}_0$ the topology *generated* by $\mathcal{S}$.

Example: Let $X = \{a, b, c, d\}$. Let $\mathcal{S} = \{\{a\}, \{b\}, \{c, d\}\}$.
Then the topology generated by $\mathcal{S}$ is

$$\mathcal{T}_0 = \{\{a\}, \{b\}, \{c, d\}, \emptyset, X, \{a, b\}, \{a, c, d\}, \{b, c, d\}\}$$

Proposition: Let $\mathcal{S} \subset \mathcal{P}(X)$ be given. Let $\mathcal{B}$ be obtained from $\mathcal{S}$ by taking all possible finite intersections of members of $\mathcal{S}$. (Then $\mathcal{B}$ is closed under finite intersection.) Next, let $\mathcal{C}$ be obtained from $\mathcal{B}$ by taking all possible arbitrary union of members of $\mathcal{B}$. Then $\mathcal{C}$ is not just closed under arbitrary union, it is still closed under finite intersection. (Exercise.)
In particular, $\mathcal{C} = \mathcal{T}_0$.
Remark: By first taking arbitrary union of members of $\mathcal{S}$ then further by taking finite intersections, we don't always get $\mathcal{T}_0$.

2) complement of A in X

PMATH 351 Lecture 8: January 20, 2010

Grad Studies Info Session, tomorrow at 4, DC 1302
Midterm Exam Date: Mon Feb 22

Metric Spaces: An important class of topological spaces are the metric spaces.
Definition: Let X be a set. A function d which assigns to each pair of points of X a non-negative real number is called a metric on X if it satisfies

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) \leq d(x, z) + d(z, y)$ (the triangular inequality)

for all x, y, $z \in X$.

We refer to $d(x, y)$ as the *distance* between x and y.
Examples: Let X be any non-empty set. Let $d\colon X \times X \to \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

We call this the *discrete* metric on X.

Let X be $\mathbb{R}^n$, a real vector space. Let $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$, where $x = (x_i)_{i=1}^n$, $y = (y_i)_{i=1}^n$. It is called the Euclidean distance (the default).

Let (X, d) be a metric space $(X \neq \emptyset)$

$D(x, \epsilon) = \{\, y \in X : d(y, x) < \epsilon \,\}$, $\epsilon > 0$, is called a disc, or the ϵ-disc, about x.

A subset $A \subset X$ is called *open* if for *all* $a \in A$, there exists $\epsilon > 0$ so that $D(a, \epsilon) \subset A$.

Example: Let $X = \mathbb{R}^2$ with the default metric (distance function). Let $A = [0, 1] \times [0, 1]$. Then A is *not* open because $a = (0, 0)$ is a point which has no disc around it fully contained by A. figure: A with dashed circle around the origin

Let $B =]0, \infty[\, \times \mathbb{R}$ in $\mathbb{R}^2$. Then B is open.
For given $b = (b_1, b_2) \in B$, the disc $D(b, b_1)$ is contained in B. figure: $b \in B$

Let (X, d) be a metric space, $X \neq \emptyset$.
Let $\mathcal{T}$ be the set of all open subsets of X.
Proposition: $\mathcal{T}$ is a topology on X.
Proof:

(i) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$ because: The full X is *open* due to the observation that for each $x \in X$, $D(x, 1) \subset X$. So $X \in \mathcal{T}$. Clearly $\emptyset$ is open. So $\emptyset \in \mathcal{T}$.

(ii) Let A and $B \in \mathcal{T}$, and consider $A \cap B$. Let $x_0 \in A \cap B$ be given (arbitrarily). Then $x_0 \in A$ and $x_0 \in B$. Because A is open, there exists $\epsilon_1 > 0$ such that $D(x_0, \epsilon_1) \subset A$. Similarly, there exists $\epsilon_2 > 0$ such that $D(x_0, \epsilon_2) \subset B$. Then, for $\epsilon = \min(\epsilon_1, \epsilon_2) > 0$ figure: $A \cap B$

$$D(x_0, \epsilon) \begin{cases} \subset D(x_0, \epsilon_1) \subset A \\ \subset D(x_0, \epsilon_2) \subset B \end{cases}$$

and so $D(x_0, \epsilon) \subset A$ and B. So $D(x_0, \epsilon) \subset A \cap B$.

(iii) Let $A_i \in \mathcal{T}$ for all $i \in I$. Without loss of generality, $I \neq \emptyset$, and consider $\bigcup_{i \in I} A_i$. Let $x_0 \in \bigcup_{i \in I} A_i$ be given. Then $x_0 \in A_{i_0}$ for some $i_0 \in I$. As A_{i_0} is open, there exists $\epsilon > 0$ such that $D(x_0, \epsilon) \subset A_{i_0}$. Then $D(x_0, \epsilon) \subset \bigcup_{i \in I} A_i$ follows. This proves that $\bigcup_{i \in I} A_i$ is open, hence in $\mathcal{T}$.

PMATH 351 Lecture 9: January 22, 2010

Chapter 2

Proposition: (2.1.2) Every ϵ-disc $D(x,\epsilon)$ is open.
Proof: Let $a \in D(x,\epsilon)$ be given. Let $a \in D(x,\epsilon)$ be given. Let $r = \epsilon - d(x,a)$. Then $r > 0$, because $a \in D(x,\epsilon)$, so $d(a,x) < \epsilon$.
Claim: $D(a,r) \subset D(x,\epsilon)$.
Proof: Let $y \in D(a,r)$ be given.
Then $d(y,a) < r$. Hence $d(y,x) \leq d(y,a) + d(a,x)$ (by the triangle inequality) $< r + d(a,x) = \epsilon$. So $d(y,x) < \epsilon$. This shows that $y \in D(x,\epsilon)$. As $a \in D(x,\epsilon)$ is arbitrarily given, this proves that $D(x,\epsilon)$ is open.

figure: $a,\, y \in D(x,r)$

Definition: Let $(X,\mathcal{T})$ be a topological space. Let $A \subset X$. $a \in A$ is called an *interior* point of A if there exists $G \in \mathcal{T}$ so that $a \in G \subset A$.

The set of all interior points of A is denoted $\text{int}(A)$.

figure: $a \in G \subset A$

A subset of X is called *open* if it is a member of the topology. Thus, $a \in \text{int}(A)$ if there exists open G so that $a \in G \subset A$.

Note: The finite intersection of open sets is open, and the (arbitrary) union of open sets is open. Also, X and $\emptyset$ are open.

Proposition: Let X be a topological space. (Implicitly there is a topology $\mathcal{S}$.) Let $A \subset X$. Then $\text{int}(A)$ is open.
Proof: Let $b \in \text{int}(A)$. Choose an open set G_b so that $b \in G_b \subset A$. Then $G_b \subset \text{int}(A)$. [**Proof:** Let $c \in G_b$. Then as $c \in G_b \subset A$, $c \in \text{int}(A)$.] Now $\text{int}(A) = \bigcup_{b \in \text{int}(A)} G_b$.

figure: $b \in G_b \subset A$

Being the union of open sets, $\text{int}(A)$ is open.

Proposition: If G is open and $G \subset A$, then $G \subset \text{int}(A)$. (seen from above) Thus $\text{int}(A)$ is the *largest* open subset of A.
Example: $X = \{a,b,c\}$, $\mathcal{T} = \{\emptyset, X, \{a\}\}$
$\text{int}(\{a,b\}) = \{a\}$. $\text{int}(\{a,b,c\}) = X$. $\text{int}(\emptyset) = \emptyset$, $\text{int}(\{b\}) = \emptyset$.

In a *discrete* topological space, $\text{int}(A) = A$, all A.
In an *indiscrete* topology space, $\text{int}(A) = \begin{cases} \emptyset & \text{if } A \neq \text{full } X \\ X & \text{if } A = X \end{cases}$

PMATH 351 Lecture 10: January 25, 2010

Example: Consider $\mathbb{R}$ under the usual metric (i.e., $d(x,y) = |x-y| = \sqrt{(x-y)^2}$). Let $A = (\mathbb{Q} \cap [0,1]) \cup [2,3]$. Then $\text{int}(A) =]2,3[$.
Consider the metric space A under the usual metric space $d(x,y) = |x-y|$.
Then $\text{int}(A) = A$.

figure: A on real line
figure: A not on real line

Definition: Let A be a subset of a topological space X. Then A is *closed* if $X \setminus A$ (notation A^c, the complement of A) is open.
Example: X, $\emptyset$ are closed.

Let $A \subset X$. A point $b \in X$ is called a *limit* point (or a *contact* point) of A if for every open set G, with $b \in G$, meets A (i.e., $G \cap A \neq \emptyset$).
If every open set G, with $b \in G$,
meets A at some point other than b itself, we say that b is an accumulation point of A.
The set of all limit points of A is called the *closure* of A, denoted $\text{cl}(A)$.

figure: b on boundary of A

Example: $X = \mathbb{R}$, usual metric. $A = \mathbb{Q} \cap [0,1] \cup [2,3]$. Then $\text{cl}(A) = [0,1] \cup [2,3]$.

Proposition: $\text{cl}(A)$ is a closed set in X. $\text{cl}(A) \supseteq A$ and is the *smallest* closed set which contains A.

Proposition: In a topological space X, for any subset $A \subset X$, $\text{int}(A)$ and $\text{cl}(A^c)$ are complementary

figures: $A \subset X$

sets, i.e., they form a partition, i.e.,

$$\text{int}(A)^c = \text{cl}(A^c).$$

PMATH 351 Lecture 11: January 27, 2010

Example: Consider $\mathbb{R}$ under the usual metric. Let $A = [0,1] \cup \{2\} \cup [3,4]$. Then 2 is a limit (contact) point of A. It is not an accumulation point of A. The open set $D(2,1/2)$ meets A at $\{2\}$.

Definition: Let X be a topological space. A set U is called a neighbourhood of $a \in X$ if U contains an open set G which has a as an element. Clearly, every open set which contains a is a neighbourhood of a.

figure: $a \in G$

$$\mathcal{U}(a) = \{\, U \subset X : U \text{ is a neighbourhood of } a \,\}$$

is called the *neighbourhood system at a*. Notice that $\mathcal{U}(a)$ is closed under finite intersection. Further, if $U \in \mathcal{U}(a)$ and $V \supset U$, then $V \in \mathcal{U}(a)$.

Definition: Let Δ be a set ($\neq \emptyset$) with a partial order $\leq$. Suppose that for any two elements a, $b \in \Delta$, there exists $c \in \Delta$ so that $a \leq c$ and $b \leq c$. We call such $(\Delta, \leq)$ a *directed* set.

Examples:

1. $\mathbb{N}$ under the usual ordering is a directed set.
2. Let X be a topological space, $a \in X$ be any point. Consider $\Delta = \mathcal{U}(a)$. Define on Δ the partial ordering $\leq$ by U, $V \in \mathcal{U}(a)$, $U \leq V$ if $V \subset U$. Then $(\mathcal{U}(a), \leq)$ is a directed set. In fact, if U and V are two neighbourhoods of a, then $U \cap V$ is a neighbourhood of a and is higher than both.

Definition: Let $(\Delta, \leq)$ be a directed set. Let X be a set. A function $\boldsymbol{x}\colon \Delta \to X$ is called a *net in X*. When $(\Delta, \leq)$ is $\mathbb{N}$ under the usual ordering, we call the net a *sequence in X*.

Definition: Let $(\Delta, \leq)$ be a directed set, X be a topological space. Let $\boldsymbol{x}$ be a net on Δ in X. The image of an element $\alpha \in \Delta$ under $\boldsymbol{x}$ will be denoted by $\boldsymbol{x}_\alpha$. The map $\boldsymbol{x}$ is sometimes recorded as $(\boldsymbol{x}_\alpha)_{\alpha \in \Delta}$.

Let $x_0 \in X$. We say that $\boldsymbol{x}$ *converges* to x_0 if for all $U \in \mathcal{U}(x_0)$, there exists $x \in \Delta$ such that $\boldsymbol{x}_\beta \in U$ for all $\alpha \leq \beta$.

Proposition: Let X be a topological space and $A \subset X$. Let $b \in X$. Then b is a limit point of A if and only if every neighbourhood $U \in \mathcal{U}(b)$ meets A if and only if there exists a net $\boldsymbol{x}\colon \Delta \to X$, with terms in A, so that $\boldsymbol{x}$ converges to b.

(Partial Proof). Suppose that b is a limit point of A. Consider $\Delta = \mathcal{U}(b)$, with the partial ordering $U \leq V$ if $V \subset U$. To each $U \in \mathcal{U}(b)$, choose $\boldsymbol{x}_u \in A \cap U$. [So, $\boldsymbol{x}$ is a choice function].

figure: b limit point of $A \subset X$

Then $\boldsymbol{x}$ is a net whose terms are in A. Moreover, we can check that indeed $\boldsymbol{x}$ converges to b.

PMATH 351 Lecture 12: January 29, 2010

Proposition: In a topological space X, a point b is a contact (limit) point of a set A if and only if there exists a net $\boldsymbol{x}\colon \Delta \to X$ with all terms in A which converges to b.

Proof: If b is a contact point of A, we constructed a net $\boldsymbol{x}\colon \mathcal{U}(b) \to A$ which converges to b. (Done)

Conversely, suppose that we have a net $\boldsymbol{x}\colon \Delta \to A$ which converges to b. We intend to show that b is a contact point of A.

Let $U \in \mathcal{U}(b)$ be given. Then, as $\boldsymbol{x}$ converges to b, there exists $\alpha \in \Delta$ such that $\boldsymbol{x}_\beta \in U$ for every $\alpha \leq \beta$. In particular, $\boldsymbol{x}_\alpha \in U$. As all terms of $\boldsymbol{x}$ are in A, we set $\boldsymbol{x}_\alpha \in A$. So $\boldsymbol{x}_\alpha \in A \cap U$. Thus $U \cap A \neq \emptyset$.

This proves that $b \in \text{cl}(A)$.

Example: Seen from the above is that if there exists a sequence $\boldsymbol{x}\colon \mathbb{N} \to A$ converging to b, then $b \in \text{cl}(A)$. Don't expect that the converse holds. Consider an uncountable infinite set X. On X we consider the co-countable topology

$$\mathcal{T} = \{\, A \subset X : A^c \text{ (i.e., } X \setminus A) \text{ is at most countable, or } A = \emptyset \,\}$$

Let $A = X \setminus \{x_0\}$, where $x_0 \in X$ is fixed. Is x_0 a limit (contact) point of A? Let $U \in \mathcal{U}(x_0)$ be given. There exists an open G such that $x_0 \in G \subset U$. Thus $G \in \mathcal{T}$.

Clearly $G \neq \emptyset$, so G^c is at most countable. If G does not meet A, then $G \subset A^c$, i.e., $G^c \supset A$. As G^c is at most countable, A is at most countable. This implies that $X = A \cup \{x_0\}$ is at most countable. This contradicts that X is more than countable. Then G must meet A. So will the larger U. This proves that x_0 is indeed a contact point of A. Does there exist a *sequence* $\boldsymbol{x} \colon \mathbb{N} \to A$ which converges to x_0? figure: $x_0 \in G \subset U \subset X$

Let $\boldsymbol{x} \colon \mathbb{N} \to A$ be arbitrarily given. Consider the neighbourhood $U = X \setminus \operatorname{range} \boldsymbol{x}$ of x_0. Notice that all terms of $\boldsymbol{x}$ are in A, no terms equal x_0. So $x_0 \in U$. Notice that U is open, because the range of $\boldsymbol{x}$ is at most countable. figure: $\boldsymbol{x}_i$s

As no term of $\boldsymbol{x}$ falls in the neighbourhood of x_0, $\boldsymbol{x}$ does not converge to x_0.

PMATH 351 Lecture 13: February 1, 2010

Let X and Y be topological spaces and $f \colon X \to Y$. Let $a \in X$. We say that f is *continuous* at a if for all $U \in \mathcal{U}(f(a))$ there exists a $V \in \mathcal{U}(a)$ such that $f(V) \subset U$. figure: $f \colon X \to Y$

If f is continuous at each $a \in X$ we say that f is continuous *on* X.

If X and Y are metric spaces under d and ρ respectively, then f is continuous at a if for all $D(f(a), \epsilon)$, there exists $D(a, \delta)$ such that $f(D(a, \delta)) \subset D(f(a), \epsilon)$, i.e., for all $\epsilon > 0$, there exists $\delta > 0$ such that for all x, $d(x, a) < \delta$ implies $\rho(f(x), f(a)) < \epsilon$.

Theorem: The following statements are equivalent for a map $f \colon X \to Y$ on topological spaces.

(1) f is continuous on X

(2) $f^{-1}(G)$ is open in X for each open G in Y

(3) $f^{-1}(F)$ is closed in X for each closed F in Y

(4) $f(\operatorname{cl}(A)) \subset \operatorname{cl}(f(A))$ for all subsets $A \subset X$

Proof: [(1) $\Longrightarrow$ (2)] Assume (1). Let open G in Y be given. Consider $f^{-1}(G)$. Let $a \in f^{-1}(G)$. Then $f(a) \in G$ (by definition of pre-image). Now, $G \in \mathcal{U}(f(a))$ because G is open. Because f is continuous at a, there exists $U \in \mathcal{U}(a)$ such that $f(U) \subset G$. Without loss of generality, we may assume that U is open. [As there exists an *open neighbourhood* of a inside U.] As $f(U) \subset G$, $U \subset f^{-1}(G)$. Notice that $a \in U$. Then, it is clear that, figure: $f^{-1}(G)$

$$\bigcup \{\, U : U \text{ is open, } U \subset f^{-1}(G) \,\} = f^{-1}(G).$$

Being the union of open sets, $f^{-1}(G)$ is open.

[(2) $\Longrightarrow$ (3)] Assuming (2). Let $F \subset Y$ be a given closed set. Consider $f^{-1}(F)$. figure: $f^{-1}(F)$

Then F^c (i.e., $Y \setminus F$) is open in Y. By (2), $f^{-1}(F^c)$ is open in X.
As $f^{-1}(F^c) = [f^{-1}(F)]^c$, we see that $f^{-1}(F^c)$ is closed.

[(3) $\Longrightarrow$ (4)] Assume (3). Let $A \subset X$ be given. Consider $f^{-1}(\operatorname{cl}(A))$ figure: $\operatorname{cl}(A) \mapsto \operatorname{cl}(f(A))$
By (3), $f^{-1}(\operatorname{cl}(f(A)))$ is closed.
Notice that $\operatorname{cl}(f(A)) \supset f(A)$
so $f^{-1}(\operatorname{cl}(f(A))) \supset f^{-1}(f(A))$
so $f^{-1}(\operatorname{cl}(f(A))) \supset A$
So $\operatorname{cl}(A) \subset f^{-1}(\operatorname{cl}(f(A)))$ (by definition of closure). Therefore $f(\operatorname{cl}(A)) \subset f[f^{-1}(\operatorname{cl}(f(A)))] \subset \operatorname{cl}(f(A))$. We see (4).

PMATH 351 Lecture 14: February 3, 2010

To complete the proof of the equivalence of the four statements, we now show that

(4) $f(\mathrm{cl}(A)) \subset \mathrm{cl}(f(A))$

implies (2): f is continuous on X.
Proof: Let $a \in X$ be given.
Let $u \in \mathcal{U}(f(a))$ be given.
Without loss of generality, we may assume that u is open.
Then $F := u^c$ is closed and $f(a) \notin F$.
Consider $f^{-1}(u)$ which clearly contains a. We need only to show that $f^{-1}(u)$ is a neighbourhood of a.
Observe that $f^{-1}(u)^c = f^{-1}(F)$.
In particular $f[\underbrace{f^{-1}(u)^c}_{=A,\text{ say}}] \subset F$.
By assumption (4),

$$f(\mathrm{cl}[f^{-1}(u)^c]) \subset \mathrm{cl}(f(A))$$

Now, as $f(A) \subset F$ and F is closed, we have $\mathrm{cl}(f(A)) \subset F$.
Hence $f(\mathrm{cl}[A]) \subset F$.
So $\mathrm{cl}([A]) \subset f^{-1}(F) = A$ by definition of pre-image

$$\mathrm{cl}(A) \subset A$$

As $\mathrm{cl}(A) \supset A$ always, we get $\mathrm{cl}(A) = A$. So A is closed.
So $f^{-1}(u) = A^c$ is open.
So $f^{-1}(u)$ is a neighbourhood of a.

$f\colon X \to Y$
topological spaces
X and Y
figure: $a \mapsto f(a)$

Note:
$f(f^{-1}(F)) \subset F$.

Theorem: Let X be a set, Y be a topological space and let $f\colon X \to Y$ be a mapping.
Then the set

$$\mathcal{T} = \{\, f^{-1}(G) : G \text{ open in } Y \,\}$$

is a topology on X. Clearly, it is the smallest topology in X with which f is continuous.
Proof: [Checking that $\mathcal{T}$ is indeed a topology on X.]

(1) $\bigcap_{i\in I} f^{-1}(G_i)$ (where I is finite) $= f^{-1}(\bigcap_{i\in I} G_i)$, where $\bigcap_{i\in I} G_i$ is open. Then $\mathcal{T}$ is closed under finite intersection.

(2) Similarly $\mathcal{T}$ is closed under arbitrary union.

figure: step function

PMATH 351 Lecture 15: February 5, 2010

Definition: A mapping $f\colon X \to Y$ from topological space X to topological space Y is called a homeomorphism if it is bijective and both f and f^{-1} are continuous.

It follows that, for a homeomorphism f, a set $A \subset X$ is open *if and only if* $f(A) \subset Y$ is open:

(if) Suppose that $f(A)$ is open in Y. Then $A = f^{-1}(f(A))$ [because f is bijective] is open in X because f is continuous.

(only if) Suppose that A is open in X, then $f(A) = (f^{-1})^{-1}(A)$ is open because f^{-1} is continuous.

In short, the bijective f matches open sets of X to open sets of Y.

$f^{-1}(f(A)) \supset A$

$f\colon \mathbb{R} \to [0,\infty[$
$f(x) = x^2$
surjective
$A = [0,\infty[\subset \mathbb{R}$
$f(A) = [0,\infty[$
$f^{-1}(f(A)) =$
$f^{-1}([0,\infty[) = \mathbb{R}$

figure: $A \mapsto f(A)$

Definition: Topological spaces X and Y are homeomorphic if there exists a homeomorphism f from X to Y.

Example: Let $X = \{a, b, c\}$, $\mathcal{T} = \{X, \emptyset, \{a\}\}$. Let $Y = \{1, 2, 3\}$ and $\tilde{\mathcal{T}} = \{Y, \emptyset, \{3\}\}$. The spaces are homeomorphic. The map $f\colon X \to Y$ given by $f(a) = 3$, $f(b) = 1$, $f(c) = 2$ matches open sets.

Example: $[0, 1]$ and any closed interval $[a, b]$ ($a, b \in \mathbb{R}$, $a < b$), as metric spaces are homeomorphic. The map $f\colon [0, 1] \to [a, b]$, $f(t) = a + t(b - a)$, $t \in [0, 1]$ is a homeomorphism.

Definition: (Subspaces)
Let X be a topological space under a topology $\mathcal{T}$. Let $A \subset X$. Then $\mathcal{T}_A = \{\, G \cap A : G \in \mathcal{T} \,\}$ is a topology on A. With this topology, we call A a *subspace* of X.

Let (X, d) be a metric space. Let $A \subset X$. Then d_A defined by $d_A(a_1, a_2) = d(a_1, a_2)$ for all a_1, $a_2 \in A$ is also a metric. We call (A, d_A) a *subspace* of (X, d).

Question: Let (X, d) be a metric space. Let $A \subset X$. Then A has two topologies. First, A is a metric space under d_A, and so d_A induces a topology $\mathcal{T}_1$, say. Second, from d, we get a topology $\mathcal{T}$ on X, and that we get a topology $\mathcal{T}_A$ ($\mathcal{T}_2$) in A.

Are the two topologies the same? Answer: Yes.

Examples: $\mathbb{R}^2$ with the usual metric is a metric space. It is also a topological space.

e.g., the figures

$$A, B, C, D, \ldots, Z, \text{申},$$

are all (metric) and topological spaces.

Question: Are 8 and B homeomorphic? (Yes)

PMATH 351 Lecture 16: February 8, 2010

Definition: A topological space X is called Hausdorff if for each pair of *distinct* points x and y, there exist open neighbourhoods U and V of x and y, respectively such that $U \cap V = \emptyset$.

Proposition: Every metric space is Hausdorff.
Proof: Let (X, d) be a metric space, and $x \neq y$ in X be given. Then $d(x, y) > 0$ and so $r = \frac{1}{2} d(x, y) > 0$. The discs $D(x, r)$ and $D(y, r)$ are open and disjoint. If they were not disjoint, say that $z \in D(x, r) \cap D(y, r)$ exists, we would have $d(x, z) < r$, $d(z, y) < r$, resulting in $d(x, y) \leq d(x, z) + d(z, y) <^{3)} r + r = 2r = d(x, y)$, a contradiction.

figure: distinct disks with x, $y \in X$

A topological space X is said to be *metrizable* if there exists a metric d on X such that the topology induced by d agree with the topology on X.

A non-Hausdorff space is *not* metrizable, e.g., $X = \{a, b\}$, $\mathcal{T} = \{X, \emptyset, \{a\}\}$. Then $(X, \mathcal{T})$ is not metrizable.

Definition: A topological space X is *connected* if there exists no subset $A \subset X$ which is both open and closed, except $A = \emptyset$, and $A = X$.

Example: $[0, 1]$ is connected. (Try to prove it on your own.)
(Assuming that every non-empty subset of $\mathbb{R}$ which is bounded from above has a least upper bound in $\mathbb{R}$. Similarly, every non-empty subset of $\mathbb{R}$ which is bounded from below has a greatest lower bound in $\mathbb{R}$.)

Definition: A subset $I \subset \mathbb{R}$ is called an *interval* if whenever a, $b \in I$, so are all numbers $a \leq c \leq b$.
e.g., $I = [0, 1]$, $]0, 1[$, $]0, 1]$, $\mathbb{R}$, $\{1\}$, etc.

Example: A subset of $\mathbb{R}$ is connected if and only if it is an interval.
(Partial proof) If $A \subset \mathbb{R}$ and A is not an interval, we show that it is not connected:

There exist a, $b \in A$ and $a \leq c \leq b$ with $c \notin A$. Then $G_a = \{\, x : x \in A,\, x < c \,\}$ and $G_b = \{\, x : x \in A,\, c < x \,\}$. They are non-empty, and they are both open, partitioning A.
Notice that $G_a = \underbrace{A \cap \underbrace{]-\infty, c[}_{\text{open in } \mathbb{R}}}_{\text{open in the subspace } A}$

figure: hole at c

Similarly $G_b = A \cap \,]c, \infty[$ is open in space A

$$G_a \cup G_b = A.$$

3) strict

Hence G_a is both open and closed, and $G_a \neq A$, $\emptyset$. So A is not connected.

Proposition: The statements below are equivalent for a topological space X.

(1) The only subsets of X which are both open and closed are X and $\emptyset$.

(2) There is no (interesting) *partition* of X into two (disjoint) non-empty open sets.

Examples in $\mathbb{R}^2$
$A = \{ (\frac{1}{n}, y) : 0 \leq y \leq 1 \} \cup]0,1] \cup \{(0,1)\}$ figure: A
Then A is connected.

PMATH 351 Lecture 17: February 10, 2010

The *intermediate value theorem* in calculus states that a continuous function $f\colon [a,b] \to \mathbb{R}$ where $f(a) < 0$, $f(b) > 0$ must attain the value 0 at some point between a and b.

figure: root of f between a and b

The notion of a connected space is a characterization of such a property (intermediate value).

Theorem: A space X is connected if and only if for every continuous function $f\colon X \to \mathbb{R}$ satisfying $f(a) < 0$, $f(b) > 0$ for some a, $b \in X$, there exists a $c \in X$ so that $f(c) = 0$.
Proof:

Lemma: The continuous image of a connected space is connected. That is: if $f\colon X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Proof: Without loss of generality we may assume $f(X) = Y$. Suppose, to the contrary that Y is not connected, then we can partition Y into two disjoint non-empty open sets Y_1 and Y_2. Now $f^{-1}(Y_1)$ and $f^{-1}(Y_2)$ is a partition of X, where $f^{-1}(Y_1)$ and $f^{-1}(Y_2)$ are open due to the continuity of f, and both are non-empty (f surjective). This shows that X is not connected, a contradiction.

(i) Suppose that X is connected. To show that the intermediate value property holds in X, let $f\colon X \to \mathbb{R}$ be a given continuous map, and suppose that there are points a and b such that $f(a) < 0$ and $f(b) > 0$.

By the Lemma, $f(X)$ is a connected space, and a subspace of $\mathbb{R}$ so $f(X)$ must be an interval. The interval has a negative value and a positive value. So the interval must contain all real numbers between them, In particular, 0 is there.

(ii) Suppose that X is not connected. Then there exists a partition of X into disjoint and non-empty open X_1, X_2. Let $f\colon X \to \mathbb{R}$ be defined by $f(x) = -1$ if $x \in X_1$ and $f(x) = +1$ if $x \in X_2$. Then f is continuous. There are only four possible images namely, X, X_1, X_2 or $\emptyset$. All are open. So f is continuous. The value 0 is not attained by f.

Proposition: Let X be a topological space. Let $\{ X_i : i \in I \}$ be a family of connected subsets of X. Suppose that $\bigcap_{i \in I} X_i \neq \emptyset$. Then $\bigcup_{i \in I} X_i$ is connected.
Proof: Exercise. [Sol: Lecture 34]

figures:
path $f\colon [a,b] \to X$
lines x_1, x_2, x_3
distinct lines in $\mathbb{R}^2$

PMATH 351 Lecture 18: February 12, 2010

Definition: A topological space X is *path connected* if for every two elements x, $y \in X$, there exists a (path) *continuous* map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

figure: $\gamma(t)$ from x to $y \in X$

Proposition: A path connected space is connected.
Proof: Fix an $x_0 \in X$. To each $x \in X$, fix a path γ_x in X *joining x to x_0*, i.e., $\gamma_x(0) = x$ and $\gamma_x(1) = x_0$. The family

figure: γ_x

$$\{ \gamma_x([0,1]) : x \in X \}$$

consists of connected subsets of X. The intersection is not empty (x_0 is there). So $\bigcup_{x \in X} \gamma_x([0,1])$ is connected by the previous theorem. But the union is equal to X.

The converse is not true. The example

figure: X

$$X = \{ (x,0) : x \in]0,1] \} \cup \{ \left(\tfrac{1}{n}, y\right) : y \in [0,1] \} \cup \{(0,1)\}$$

as a subspace of $\mathbb{R}^2$ is that of connected space which is not path connected. In fact $(0,1)$ and $(1,1)$ cannot be joined by a path in X.

Topological Vector Spaces: Let V be a real vector space. Suppose that $\mathcal{T}$ is a topology on V. We call V a topological vector space if the linear structure and the topological structure are compatible in the following sense:

(1) Vector addition: $\underbrace{V \times V}_{4)} \to V$ is closed and continuous

(2) Scalar multiplication: $\mathbb{R} \times V \to V$ is continuous
where the topology on $\mathbb{R} \times V$ is generated by $\{ G_1 \times G_2 : G_1 \text{ open in } \mathbb{R}, G_2 \text{ open in } V \}$

Examples: $\mathbb{R}^n$, $C[0,1]$ under the uniform metric defined by

$$d(f,g) = \sup\{ \min(|f(t) - g(t)|, 1) : t \in [0,1] \}$$

In a topological vector space over $\mathbb{R}$, a set A is *convex* if for all x, $y \in A$, the line segment joining x and y

$$\{ tx + (1-t)y : t \in [0,1] \}$$

is contained in A. figures: $A \subset V$

Proposition: A convex subset of a topological space is connected and in fact is path connected.

Remark: We have the theorem that $f\colon X \to Y$ is continuous if and only if $f^{-1}(G)$ is open for every open G. If $\mathcal{B}$ generates the topology on Y, then it is sufficient to observe that $f^{-1}(B)$ are open for each $B \in \mathcal{B}$. Example: $\mathbb{R}$ has the usual topology generated by

$$\mathcal{B} = \{]-\infty, a[,]a, \infty[: a \in \mathbb{Q} \}.$$

Thus $f\colon X \to \mathbb{R}$ is continuous if and only if $f^{-1}(]-\infty, a[)$ and $f^{-1}(]a, \infty[)$ are open (in X) for each rational a.

PMATH 351 Lecture 19: February 24, 2010

Compactness

Let X be a topological space. A family $\mathcal{C}$ of open sets is said to be an open *cover* of X if $\bigcup \mathcal{C} = X$.

If $\tilde{\mathcal{C}} \subset \mathcal{C}$ and $\bigcup \tilde{\mathcal{C}} = X$, we call $\tilde{\mathcal{C}}$ a *subcover* of $\mathcal{C}$.

The space X is called compact (cpct) is *every* open cover $\mathcal{C}$ of X has a *finite* subcover $\tilde{\mathcal{C}}$.

Example: $\mathbb{R}$ is *not* compact. The family $\{]-n, n[: n \in \mathbb{N} \}$ is an open cover of $\mathbb{R}$. Clearly it has no finite subcover.

A finite topological space X is compact. Here is the trivial argument: Let $X = \{x_1, x_2, \ldots, x_n\}$. Let $\mathcal{C}$ be any given open cover. Then $\bigcup \mathcal{C} = X$. So, for each $1 \le i \le n$, $x_i \in \bigcup \mathcal{C}$ and so there exists $G_i \in \mathcal{C}$ so that $x_i \in G_i$. Now $\tilde{\mathcal{C}} = \{ G_i : 1 \le i \le n \} \subset \mathcal{C}$. $\tilde{\mathcal{C}}$ is clearly a subcover of $\mathcal{C}$.

Let X be *any* set and consider the topology of finite complements. Then the space X is compact. Without loss of generality, X is infinite.

Proof: Let $\mathcal{C}$ be an open cover of X. Let $x_0 \in X$ be fixed. Then, as $\mathcal{C}$ covers X, there exists $G_0 \in \mathcal{C}$ so that $x_0 \in G_0$. Now, G_0 is open, therefore $X \setminus G_0$ is finite, say $X \setminus G_0 = \{x_1, x_2, \ldots, x_n\}$. To each x_i, there exists $G_i \in \mathcal{C}$ so that $x_i \in G_i$.

Now $\{G_0, G_1, G_2, \ldots, G_n\}$ is a finite subcover of $\mathcal{C}$.

Theorem: A subspace X of $\mathbb{R}^n$ is *compact* if and only if it is closed (in $\mathbb{R}^n$) and bounded.

---

4) where $V \times V$ has the topology generated by $\{ G_1 \times G_2 : G_1, G_2 \text{ open} \}$

Definition: $X \subset \mathbb{R}^n$ is bounded if there exists a (finite) radius r so that $X \subset D(0, r)$.

Definition: Sequential compactness. Let X be a topological space. If every sequence x_n in X has a convergent subsequence in X, we say X is *sequentially* compact.

Example: In $[0, 1]$, the sequence 0, 1, 0, 1, 0, 1, ..., is not convergent, but the sequence formed by the odd terms 0, 0, 0, ..., is convergent (illustrating the notion of convergent subsequence).

The full space $\mathbb{R}$ is *not* sequential compact.
Proof: The sequence $x_n = n$ is a sequence in $\mathbb{R}$ which has no convergent subsequence.

Theorem 3.1.3: (Bolzano–Weierstrass Theorem).
A (subset of a) metric space is *compact* if and only if it is *sequentially compact*. (Proof page 165).

Question on exam. Can we put a topology on P_2 so that P_2 is homeomorphic to $\mathbb{R}$?

Yes. P_2 can be matched with $\mathbb{R}^3$ by a bijective map. Also $|\mathbb{R}^3| = |\mathbb{R}|$. So $|P_2| = |\mathbb{R}|$. There is a bijection $f : P_2 \to \mathbb{R}$.

PMATH 351 Lecture 20: February 26, 2010

Theorem 3.1.3 (Bolzano–Weierstrass Theorem):
A subset A of a metric space M is compact if and only if it is sequentially compact.
Proof (page 165).

Lemma: A compact $A \subset M$ is closed in M.
Proof: Let A be compact. Let $x_0 \in M$, $x_0 \notin A$ be given.

figure: $A \subset M$

To each $a \in A$, because $a \neq x_0$, $r = d(a, x_0) > 0$ and $D(x_0, r/2)$ is disjoint from $D(a, r/2)$. Label them as U_a and V_a, and they are neighbourhoods (*open*) of a and x_0 respectively. Now $\{ U_a : a \in A \}$ is an *open cover* of A in the sense that $\bigcup_{a \in A} u_a \supset A$. Because A is compact, there exists finitely many $U_{a_1}, U_{a_2}, \ldots, U_{a_n}$ so that their union already contains A. Notice that $V_{a_1} \cap V_{a_2} \cap \cdots \cap V_{a_n} =: V_{x_0}$ is an open neighbourhood of x_0, and is disjoint from each U_{a_i} $(i = 1, \ldots, n)$. V_{x_0} does not meet $U_{a_1}, U_{a_2}, \ldots, U_{a_n}$ implies that V_{x_0} does not meet A.

figure: cover of A, $x_0 \notin A$

Hence x_0 is not a limit of A.

As $x_0 \notin A$ is arbitrary, this proves that A is closed.

Comment: The Lemma holds when M is any Hausdorff topological space.

Lemma 2: In a compact space, say X, a closed subset A is compact.
Proof: Let A be a closed set in X. Knowing X is compact, we wish to argue that A is compact.

figure: cover of $A \subset X$

Let $\mathcal{C}$ be a collection of open sets in X which covers A, i.e., $\bigcup \mathcal{C} \supset A$.

Now $\mathcal{C} \cup \{\underbrace{A^c}_{\text{open}}\}$ is an open cover of X. By compactness of X, a finite number of members of $\mathcal{C} \cup \{A^c\}$ covers X, say $\{u_1, u_2, \ldots, u_n, A^c\}$ covers X.
Then $\{u_1, u_2, \ldots, u_n\}$ covers A.
So A is compact.

Comment: In $\mathbb{R}^n$, a subset is compact if and only if it is closed and bounded (Heine–Borel Theorem).

With the Lemma above, if we can show that a *closed disk* (with finite radius) $\{ x \in \mathbb{R}^2 : d(x_0, 0) \leq r \}$ is compact, then it follows from the Lemma that every bounded closed set in $\mathbb{R}^n$ is compact.

PMATH 351 Lecture 21: March 1, 2010

New Midterm: Tuesday, 16 March, 2010 at 4:00–5:30 PM

Proof of the Bolzano–Weierstrass Theorem (page 165 text)
Let A be compact. Assume, to the contrary that A is not sequentially compact, that there exists a sequence $x_k \in A$ which has no convergent subsequence.

In particular, the sequence has *infinitely many distinct* points y_1, y_2, ..., y_n, ...

Claim: $\{y_1, y_2, \dots, y_n, \dots\}$ is closed.

Proof: Let $a \in A$, $a \notin \{y_1, \dots, y_n, \dots\}$. If a were a limit point of $\{y_1, \dots, y_n, \dots\}$ then every neighbourhood of a will meet this set. Hence, by picking elements in the intersection of $D(a, 1/n)$ with the set $\{y_1, \dots, y_n, \dots\}$, we get a convergent subsequence of x_k which converges to a. This would contradict that x_k has no convergent subsequence.

figure: x_{n_1}, x_{n_2} in neighbourhood of a, $n_2 > n_1$

Therefore $\{y_1, \dots, y_n, \dots\}$ is compact. ("closed subsets of a compact space A is compact").

Claim: Each element of $\{y_1, \dots, y_n, \dots\}$ is an *isolated point* of the set, i.e., to each y_i, there exists a positive δ such that $D(y_i, \delta)$ does not meet $\{y_1, \dots, y_n, \dots\}$ at any point other than y_i.

Consider the open cover of $\{y_1, \dots, y_n, \dots\}$

$$\mathcal{C} = \{ D(y_i, \delta_i) : i = 1, 2, \dots \}.$$

This $\mathcal{C}$ has no finite subcover. It contradicts the compactness of $\{y_1, \dots, y_n, \dots\}$. The above proves that compact A is sequentially compact.

Next, assume that A is sequentially compact. Let $\mathcal{C}$ be a given open cover of A.

Claim: There exists $r > 0$ such that for each $y \in A$, $D(y, r) \subset U$ for some $U \in \mathcal{C}$.
... Read the book.

PMATH 351 Lecture 22: March 3, 2010

Theorem: (4.2.2) Let $f \colon X \to Y$ be continuous where X and Y are topological spaces. If X is compact, then $f(X)$ is compact.
Proof: Let $\{ G_i : i \in I \}$ be an open cover of $f(X)$. Then $\{ f^{-1}(G_i) : i \in I \}$ is an open cover of X. Each $f^{-1}(G_i)$ is open because f is continuous and G_i is open.

$$\bigcup_{i \in I} f^{-1}(G_i) = f^{-1}\Big(\bigcup_{i \in I} G_i\Big) \supset f^{-1}(f(X)) \supset X.$$

As X is compact, there exists i_1, i_2, ..., $i_N \in I$ such that $\{f^{-1}(G_{i_1}), f^{-1}(G_{i_2}), \dots, f^{-1}(G_{i_N})\}$ covers X. Then $\{G_{i_1}, G_{i_2}, \dots, G_{i_N}\}$ covers $f(X)$. This proves that $f(X)$ is compact.

Comment: In calculus, we have the theorem: a continuous function (into $\mathbb{R}$) on $[a, b]$ attains maximum and minimum.
Proof: $[a, b]$ is compact. Therefore $f[a, b]$ is compact ($\subset \mathbb{R}$). So $f[a, b]$ is closed and bounded (clearly non-empty, as $a \leq b$ is understood). It contains a maximum and minimum. (sup and inf exist for bounded non-empty sets in $\mathbb{R}$, and they are limit points).

Example: The continuous map $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = x$, attains no max/min *on* $\mathbb{R}$ which is *not* compact. The continuous map $f \colon]0, 1[\to \mathbb{R}$, $f(x) = \frac{1}{x}$ attains no maximum and minimum $]0, 1[$. Note $f(]0, 1[) =]1, \infty[$.

Example: Show that the figures (in $\mathbb{R}^2$)

0 and 8 are not homeomorphic

Proof: If any point is removed from the first figure, what is left is a connected space. However, the removal of the point A gives 8 which is not connected. Hence they are not homeomorphic.

figure: 8 with centre point missing

Theorem: A bijective f from a compact space X to a Hausdorff space which is continuous is a homeomorphism. (That is, the inverse map is continuous).
Proof: Let $f \colon X \to Y$ be continuous, bijective, X is compact, Y is Hausdorff.

To show that $f^{-1}\colon Y \to X$ is continuous, let $F \subset X$ be a given closed set.
Consider $(f^{-1})^{-1}(F) = f(F)$. Because X is compact, F closed, F is compact. As f is continuous, $f(F)$ is compact. Being in a Hausdorff space Y, $f(F)$ is closed in Y. Thus $(f^{-1})^{-1}(F)$ is closed in Y.

figure: $f\colon X \to Y$ and its inverse

This proves that f^{-1} is *continuous.*

Corollary: Continuous and *injective* images of the circle $\{ (x,y) : x^2 + y^2 = 1 \}$ in $\mathbb{R}^2$ are homeomorphic.

figures: homeomorphic to a circle

PMATH 351 Lecture 23: March 5, 2010

Midterm on March 16, Tuesday, 4:00–5:30, MC 4042

§4.6 Uniform Continuity
Let X and Y be metric spaces under metrics d and ρ, respectively. A map $f\colon X \to Y$ is said to be *uniformly* continuous on X if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $(d(x_1,x_2) < \delta \implies \rho(f(x_1), f(x_2)) < \epsilon)$.
Clearly, uniform continuity of f on X implies continuity on X.

Example: Let $X =]0,1[$, $Y = \mathbb{R}$. Let $f(x) = \frac{1}{x}$. Then f is continuous on X, but *not* uniformly continuous.

Proposition: *If X is compact*, then continuous $f\colon X \to Y$ is uniformly continuous.
Proof: Assume that $f\colon X \to Y$ is continuous, and that X is compact. Let $\epsilon > 0$ be given.

To each $x \in X$, there exists a $\delta_x > 0$ such that $\rho(f(x), f(x_2)) < \epsilon/2$ for all $d(x,x_2) < \delta_x$. [continuity of f at x]
Now the family $\{ D(x, \delta_x/2) : x \in X \}$ is an open cover of X. By compactness of X, there exists $a_1, a_2, \ldots, a_n \in X$ so that $\{ D(a_i, \delta_{a_i}/2) : i = 1, \ldots, n \}$ covers X. Let $\delta = \min_{i=1,\ldots,n}(\delta_{a_i}/2)$. Then $\delta > 0$.

Let x_1, $x_2 \in X$ be given with $d(x_1,x_2) < \delta$.

Because the discs $D(a_i, \delta_{a_i}/2)$ cover X, there exists i so that $x_1 \in D(a_i, \delta_{a_i}/2)$. So, $d(x_1, a_i) < \delta_{a_i}/2$.

$$d(x_2, a_i) \le d(x_1, a_i) + d(x_1, x_2) < \delta_{a_i}/2 + \delta < \delta_{a_i}/2 + \delta_{a_i}/2 = \delta_{a_i}$$

So $\rho(f(x_2), f(a_i)) < \epsilon/2$. Also, $\rho(f(x_1), f(a_i)) < \epsilon/2$. Hence

$$\rho(f(x_1), f(x_2)) \le \rho(f(x_2), f(a_i)) + \rho(f(x_1), f(a_i)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves the uniform continuity of f.

Complete metric spaces
Definition: Let X be a metric space with metric d.
A sequence x_k in X is called *Cauchy* if

$$\lim_{k,l \to \infty} d(x_k, x_l) = 0, \text{ i.e., } \forall \epsilon > 0, \exists N \text{ such that } (k, l \ge N \implies d(x_k, x_l) < \epsilon).$$

Clearly, if x_k is a convergent sequence in X, then it is Cauchy.

The converse is not true in general.
Example: Consider $]0,1](= X)$. The sequence $\frac{1}{k}$ $(k \in \mathbb{N})$ is Cauchy. It does not converge to a point in $]0,1]$.
Definition: A metric space (X,d) is *complete* if every Cauchy sequence converges (to a point of X).
Proposition: $\mathbb{R}^n$, $\mathbb{C}^n$ are complete metric spaces.
Proposition: A subspace A of a complete metric space X is complete if and only if A is closed in X.
Proposition: Compact metric spaces are complete.

Read Theorem 3.1.5

PMATH 351 Lecture 24: March 8, 2010

Definition: (3.1.4). A metric space is *totally bounded* if for all $\epsilon > 0$, there exist finitely many $x_1, \ldots, x_n$ in the space so that $\{ D(x_i, \epsilon) : i = 1, \ldots, n \}$ covers the space.

Example: The square $[0,1] \times [0,1]$ in $\mathbb{R}^2$ is totally bounded.

figure: a square is totally bounded

Theorem: (3.1.5). A metric space (X, d) is compact if and only if it is complete and totally bounded. (A generalization of the Heine–Borel Theorem for subspaces of $\mathbb{R}^n$).

Proof: (Page 166). To see the converse we suppose that (X, d) is complete and totally bounded, and proceed to argue that X is sequentially compact.

compactness implies sequentially complete and totally bounded

Let y_k be a sequence in X.

Without loss of generality, we may assume that all terms of y_k are distinct. Consider $\epsilon = 1$. There are a finite number of discs $D(x_1, 1)$, $D(x_2, 1)$, $\ldots$, $D(x_k, 1)$ which covers X. There must be one disc, say $D(x_1, 1)$, which holds infinitely many y_k terms.

Extract a subsequence

$$y_{11}, y_{12}, y_{13}, \ldots, y_{1j}, \ldots$$

of $y_1, y_2, \ldots, y_k, \ldots$ with all terms in $D(x_1, 1)$.
Next, repeat the argument using $\epsilon = 1/2$, and claim that there exists a disc $D(x_2, 1/2)$ and a subsequence

$$y_{21}, y_{22}, y_{23}, \ldots$$

of the previous $y_{11}, y_{12}, \ldots$ so that all terms are in $D(x_2, 1/2)$

figure: finite cover of discs of radius 1/2

$\vdots$

By induction, get sequence

$$y_{l1}, y_{l2}, \ldots,$$

which is a subsequence of $y_{l-1,1}, y_{l-1,2}, \ldots$ so that all terms are in $D(x_l, 1/l)$.
Consider the diagonal sequence

$$y_{11}, y_{22}, y_{33}, \ldots, y_{nn}, \ldots$$

It is Cauchy. As X is complete, it converges to a point of X.

Don't expect the statement: A metric space (X, d) is compact if and only if it is complete and bounded.

Example: $\mathbb{R}^2$ is complete, but not compact. However, $(\mathbb{R}^2, \rho = \min(d^{5)}, 1))$ has the same topology as $(\mathbb{R}^2, d)$.

$\mathbb{R}^2$ is bounded by $D_\rho(\mathbf{0}, 2)$

PMATH 351 Lecture 25: March 10, 2010

The Banach Fix Point Theorem (or the Contraction mapping theorem): Let (X, d) be a metric space. A mapping $T \colon X \to X$ is *contractive* if there exists a constant $k < 1$ such that $d(T(x), T(y)) \leq kd(x, y)$ for *all* $x, y \in X$. (Clearly, contractive maps are uniformly continuous.) If (X, d) is complete. Then every contractive map T has a unique fixed point $x_0 \in X$ (i.e., $T(x_0) = x_0$).

Proof: Uniqueness first. Suppose x_0 and $\tilde{x}_0$ are both fixed points of T. Consider $d(T(x_0), T(\tilde{x}_0)) \leq kd(x_0, \tilde{x}_0)$ we get $d(x_0, \tilde{x}_0) \leq kd(x_0, \tilde{x}_0)$.
With $k < 1$, we get $d(x_0, \tilde{x}_0) = 0$. Hence $x_0 = \tilde{x}_0$.

(Existence).
Let $x_1 \in X$ be a fixed element in X and consider $x_2 = T(x_1)$, $x_3 = T(x_2)$, $\ldots$, $x_k = T(x_{k-1}) = T^{(k-1)}(x_1)$, $\ldots$
Claim: The sequence x_k converges to a fixed point of T.

figure: $x_1 \to x_2 \to x_3 \to \cdots$

5) Euclidean

Proof:

$$\begin{aligned}
d(x_2,x_3) &= d(T(x_1),T(x_2)) \le k d(x_1,x_2)\\
d(x_3,x_4) &= d(T(x_2),T(x_3)) \le k d(x_2,x_3) \le k^2 d(x_1,x_2)\\
&\vdots\\
d(x_n,x_{n+1}) &\le k^{n-1} d(x_1,x_2)\\
d(x_n,x_{n+j}) &\le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+j-1},x_{n+j})\\
&\le \left[k^{n-1} + k^n + \cdots + k^{n+j-2}\right] d(x_1,x_2)\\
&\le \left[k^{n-1} + k^n + \cdots\right] d(x_1,x_2) = \frac{k^{n-1}}{1-k} d(x_1,x_2)
\end{aligned}$$

by $0 \le k < 1$ $\sum_{n=0}^{\infty} k^m = \frac{1}{1-k}$

The RHS tends to 0 as $n \to \infty$. So the sequence is Cauchy. The space X is complete, so there exists $x_0 \in X$ such that $x_n \to x_0$.

Since T is continuous,

$$T(x_0) = T\Big(\lim_{n\to\infty} x_n\Big) \overset{6)}{=} \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x_0$$

Application

$T(\mathrm{cl}(A)) \subseteq \mathrm{cl}(T(A))$

Show that there exists a *continuous* function $f_0\colon [0,1] \to \mathbb{R}$ satisfying the integral equation

$$f_0(x) = e^x + \int_0^x \frac{(\sin t)^3}{2} f_0(t)\,\mathrm{d}t \qquad \text{for all } x \in [0,1].$$

Such a f_0 is unique.

Proof: Background: Consider $C([0,1],\mathbb{R}) = \{\, f\colon [0,1] \to \mathbb{R} : f \text{ continuous} \,\}$. It is a vector space over $\mathbb{R}$. Equip the space with a norm:

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)| = \max_{x\in[0,1]} |f(x)|$$

The norm induces a metric

$$d(f,g) = \|f-g\|_\infty$$

Fact: $(C[0,1], d)$ is complete.

Consider $T\colon C[0,1] \to C[0,1]$ defined by

$$T(f) = e^x + \int_0^x \frac{(\sin t)^3}{2} f(t)\,\mathrm{d}t \qquad x \in [0,1].$$

Then the f_0 we are looking for is a fixed point of T. T is contractive:

Proof:

$$\begin{aligned}
|T(f)(x) - T(g)(x)| &= \left| \cancel{e^x} + \int_0^x \frac{(\sin t)^3}{2} f(t)\,\mathrm{d}t - \left(\cancel{e^x} + \int_0^x \frac{(\sin t)^3}{2} g(t)\,\mathrm{d}t\right)\right|\\
&= \left|\int_0^x \frac{(\sin t)^3}{2}(f(t)-g(t))\,\mathrm{d}t\right|\\
&\le \int_0^x \left|\frac{\sin(t)^3}{2}|f(t)-g(t)|\right|\,\mathrm{d}t\\
&\le \tfrac{1}{2}\int_0^x |f(t)-g(t)|\,\mathrm{d}t \le \tfrac{1}{2}\int_0^1 |f(t)-g(t)|\,\mathrm{d}t \le \tfrac{1}{2}\|f-g\|_\infty\\
\sup_{x\in[0,1]} |T(f)(x) - T(g)(x)| &\le \tfrac{1}{2}\|f-g\|_\infty\\
\|T(f)-T(g)\|_\infty &\le \tfrac{1}{2}^{7)} \|f-g\|_\infty
\end{aligned}$$

6) continuity

7) $k = \frac{1}{2}$

PMATH 351 Lecture 26: March 12, 2010

§5.5

A (real) vector space X is *normed* if there is a map $\|\cdot\|\colon X \to \mathbb{R}$ (called norm) satisfying

(1) $\|x+y\| \leq \|x\| + \|y\|$ for all x, $y \in X$

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$

(3) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.

The norm induces a metric on X by

$$d(x,y) = \|x-y\|$$

and is therefore a metric space as well as a topological space. If X is *complete*, we call X a Banach space.

Examples: $(\mathbb{R}^n, \|\cdot\|_p)$ where $\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$

The usual Euclidean norm is using $p = 2$.

$$(\mathbb{R}^n, \|\cdot\|_2),\ (\mathbb{R}^n, \|\cdot\|_1),\ (\mathbb{R}^n, \|\cdot\|_\infty)$$
$$\text{where } \|x\|_\infty \stackrel{\text{def}}{=} \sup_{i \leq n} |x_i|$$

are examples of Banach spaces.

Definition: Let X be a topological space. A sequence $f_n\colon X \to \mathbb{R}$ is said to converge *pointwise* (on X) if for each fixed $x \in X$, the sequence $f_n(x)$ in $\mathbb{R}$ is convergent.

When f_n is pointwise convergent,
$f(x) = \lim_{n\to\infty} f_n(x)$, $f\colon X \to \mathbb{R}$, is called *the pointwise* limit of f_n. We write "$f_n \to f$ pointwise".

Thus it means that for each $x \in X$ and $\epsilon > 0$, there exists N such that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$.

If N exists and is independent of x, we say that $f_n \to f$ *uniformly* on X.

In fact, the above can be formulated for any set X. Consider $C(X,\mathbb{R})$ the vector space of all *continuous* functions on X, and confine ourself further, to $C_b(X,\mathbb{R})$, the space of *bounded* continuous functions.

Theorem: Let X be a topological space. Let f_n be a sequence in $C(X,\mathbb{R})$. If f_n tends to $f\colon X \to \mathbb{R}$ *uniformly* on X, then $f \in C(X,\mathbb{R})$. (Proof: Exercise)

Definition: On $C_b(X,\mathbb{R})$, we define $\|\cdot\|_\infty$ by

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \qquad \text{(a finite number because } f \text{ is bounded)}$$

Claim that $\|\cdot\|_\infty$ is a *norm* on $C_b(X,\mathbb{R})$ under which the space $C_b(X,\mathbb{R})$ is a Banach space. Observe that, if X is compact, then

$$C(X,\mathbb{R}) = C_b(X,\mathbb{R}).$$

We can observe that

$$f_n \to f \text{ uniformly on } X$$
$$\text{if and only if } (f_n - f) \to 0 \text{ uniformly on } X$$
$$\text{and } g_n \to 0 \text{ uniformly on } X$$
$$\text{if and only if } \|g_n\|_\infty \to 0 \text{ (in } \mathbb{R})$$

Note: When X is finite with n elements, using the discrete topology, $C(X,\mathbb{R})$ is essentially the same as $\mathbb{R}^n$.

PMATH 351 Lecture 27: March 15, 2010

The Arzela–Ascoli Theorem (Page 299, §5.6)

Let $A \subset M^{8)}$ be compact and $\mathcal{B} \subset C^{9)}(A, N^{10)})$

Definition: $\mathcal{B}$ is called *equicontinuous* on A if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x,y) < \delta \implies \rho(f(x), f(y)) < \epsilon, \text{ all } f \in \mathcal{B}.$$

Note: δ does not depend on $f \in \mathcal{B}$.
$\mathcal{B}$ is bounded means that $\{\, \|f\|_\infty : f \in \mathcal{B} \,\}$ is bounded set, i.e., $\sup_{x \in A} |f(x)| < b$, finite b, for all $f \in \mathcal{B}$.
$\mathcal{B}$ is *pointwise compact* if $\{\, f(x) : f \in \mathcal{B} \,\}$ is compact for each fixed $x \in A$.

Theorem: $\mathcal{B}$ is compact if and only if $\mathcal{B}$ is closed, equicontinuous and pointwise compact.
Proof: Suppose that $\mathcal{B}$ is closed, equicontinuous and pointwise compact. We wish to show that $\mathcal{B}$ is compact.

Since A is compact, for each $\delta > 0$, there exists a finite set $C_\delta = \{y_1, \ldots, y_k\}$ such that each $x \in A$ is within δ of some $y_i \in C_\delta$. [total boundedness of compact A]

Thus $C_{1/n}$ is a finite set for each $n \in \mathbb{N}$ and $C = \bigcup_{n \in \mathbb{N}} C_{1/n}$ is a countable set (and is dense in A).

Let f_n be a given sequence of functions in $\mathcal{B}$. Let $C = \{x_1, x_2, \ldots\}$ be a listing of elements of the countable C.

The sequence $\{\, f_n(x_1) : n \in \mathbb{N} \,\}$ is a sequence in $\{\, f(x_1) : f \in \mathcal{B} \,\}$ which is compact by *pointwise* compactness of $\mathcal{B}$. By the Bolzano–Weierstrass theorem, $f_n(x_1)$ has a convergent subsequence, say $f_{11}(x_1)$, $f_{12}(x_1)$, $f_{13}(x_1)$, …
Repeat this idea to the sequence f_{1k} $(k = 1, 2, \ldots)$
at x_2, we get a (second) subsequence of f_{1k} $(k = 1, \ldots)$

$$f_{21}(x_2),\ f_{22}(x_2),\ f_{23}(x_2),\ \ldots$$

which is convergent. Note: $f_{21}(x_1)$, $f_{22}(x_1)$, …, is also convergent.
Repeating the above,
we set

$$f_{31}(x_3),\ f_{32}(x_3),\ f_{33}(x_3),\ f_{34}(x_3),\ \ldots \qquad \text{convergent.}$$

⋮

Consider the diagonal sequence f_{nn} which is a subsequence of all previous ones, and will therefore have the property that

$$f_{nn}(x_j) \qquad (n = 1, \ldots) \text{ is convergent for each } j$$

Let $g_n = f_{nn}$, a subsequence of f_n. It converges at each $x_j \in C$. Let $\epsilon > 0$ be given, and let $\delta > 0$ be found, according to equicontinuity of $\mathcal{B}$. Let $C_\delta = \{y_1, y_2, \ldots, y_k\}$ be the finite set consisting of points of C. [use n with $\frac{1}{n} < \delta$]

There exists N_0 such that $m, n \geq N_0$

$$\rho(g_m(y_i), g_n(y_i)) < \epsilon \text{ for each } 1 \leq i \leq k.$$

Therefore

$$\begin{aligned} \rho(g_n(x), g_m(x)) &\leq \rho(g_n(x), g_n(y_j)) + \rho(g_n(y_j), g_m(y_j)) + \rho(g_m(y_j), g_m(x)) \\ &\leq \epsilon + \epsilon + \epsilon = 3c \end{aligned}$$

8) metric space
9) all continuous maps from A to N
10) metric space

for all n, $m \geq N_0$.

This shows that g_n is uniformly Cauchy, i.e., Cauchy in norm $\|\cdot\|_\infty$. The space $C(A, N)$ is complete, so g_n is convergent in $C(A, N)$. $\mathcal{B}$ is closed, it converges in $\mathcal{B}$.

PMATH 351 Lecture 28: March 17, 2010

Note, the proof of the Arzela–Ascoli Theorem has these lines

$$\begin{array}{cccccl} f_1, & \dots, & f_n, & \dots & & \\ \boxed{f_{11}}, & f_{12}, & f_{13}, & f_{14}, & \dots, & \text{converging at } x_1 \\ f_{21}, & \boxed{f_{22}}, & f_{23}, & f_{24}, & \dots, & \text{converging at } x_2 \\ \vdots & & & & & \\ f_{m1}, & f_{m2}, & \dots, & \boxed{f_{mm}}, & & \text{converging at } x_m \\ \vdots & & & & & \end{array}$$

Let $g_n = f_{nn}$.
Claim: g_n is a subsequence of all f_{m1}, f_{m2}, ...
(From text page 300)
The claim should be modified as g_n, *starting with the mth term*, is a subsequence of f_{m1}, f_{m2}, f_{m3}, ...

Example: Consider the sequence of functions $f_n\colon [0,1] \to \mathbb{R}$ belonging to $C([0,1], \mathbb{R})$ given by

$$f_n(t) = \begin{cases} 0 & 1 \geq t \geq \frac{1}{n} \\ 1 - nt & 0 \leq t \leq \frac{1}{n} \end{cases}$$

figure of $f_n(t)$

$$\|f_n\|_\infty = 1 \text{ for each } n$$

For each fixed t, the sequence

$$\begin{cases} f_n(t) \to 0 & \text{if } 0 < t \\ f_n(0) \to 1 & \text{if } 0 = t \end{cases}$$

That is, f_n tends to the function $\phi\colon [0,1] \to \mathbb{R}$

$$\begin{cases} \phi(t) = 0 & \text{if } t > 0 \textit{ pointwise} \\ \phi(t) = 1 & \text{otherwise} \end{cases}$$

Is $\phi \in C([0,1], \mathbb{R})$? No.

Does f_n converge to some function in the $C([0,1], \mathbb{R})$ under $\|\cdot\|_\infty$?
i.e., Does f_n tends to some f_n in $C([0,1], \mathbb{R})$ uniformly?
No (uniform convergence implies pointwise convergent.)

Does f_n has a convergent subsequence in $C([0,1], \mathbb{R})$ under $\|\cdot\|_\infty$?
No.
Let $\mathcal{B} = \{\, f_n : n \in \mathbb{N} \,\} \subset C([0,1], \mathbb{R})$.

$\mathcal{B}$ is not sequentially compact. It is not compact (we are dealing with metric spaces).

Some conditions of the A–A theorem must fail.
$\mathcal{B}$ is clearly bounded, as $\|f_n\|_\infty = 1$. Exercise: Is $\mathcal{B}$ weakly compact? Is $\mathcal{B}$ equicontinuous?

Approximating continuous functions.

The e^x can be approximated by finite polynomials on $[a, b]$ in the sense that for all $\epsilon > 0$, there exists polynomial p so that $|f(x) - p(x)| \leq \epsilon$ for all $x \in [a, b]$

$$\text{i.e., } \|f-p\|_\infty < \epsilon \text{ in } C([a,b],\mathbb{R}).$$

(Taylor series)

Question: Can a continuous function $f\colon [a,b] \to \mathbb{R}$ be approximated by a polynomial?
Theorem: (Weierstrass Approximation Theorem): Every $f \in C([a,b],\mathbb{R})$ can be approximated by a polynomial $p \in C([a,b],\mathbb{R})$.

Rephrased: The set of polynomials is dense in $C([a,b],\mathbb{R})$.
See Theorem 5.8.1 (page 305).
Indeed the Bernstein polynomials

$$p_n(x) = \sum_{r=0}^{n} \binom{n}{r} f\left(\frac{r}{n}\right) x^r (1-x)^{n-r}$$

is a sequence of polynomials approximating a continuous $f\colon [0,1] \to \mathbb{R}$
i.e., $\|p_n - f\|_\infty \to 0$ as $n \to \infty$.

PMATH 351 Lecture 29: March 19, 2010

Theorem: (Weierstrass Approximation Theorem)
f is a continuous function from $[a,b]$ to $\mathbb{R}$.
Then there exists a (finite) polynomial p such that after $\epsilon > 0$ is specified, $\|f-p\|_\infty < \epsilon$.
Proof: Without loss of generality, $[a,b]=[0,1]$, and may assume $f(0)=f(1)=0$. Extend f to $\mathbb{R}$ by $f(t)=0$ for $t \notin [0,1]$. Then f is uniformly continuous on $\mathbb{R}$.

Let $Q_n(x) = C_n(1-x^2)^n$ on $[-1,1]$ where $C_n = 1/\int_{-1}^{1}(1-x^2)^n \,\mathrm{d}x$. With that normalization constant, $\int_{-1}^{1} Q_n(x)\,\mathrm{d}x = 1$.

figure of $Q_n(x)$

Observation 1: $F(x) = (1-x^2)^n - (1-nx^2) \geq 0$ on $[0,1]$
Proof: $F(0)=0$, $F'(x) = -2nx(1-x^2)^{n-1} + 2nx$
$= 2nx(1-(1-x^2)^{n-1}) \geq 0$ on $[0,1]$

Observation 2: $\int_{-1}^{1}(1-x^2)^n\,\mathrm{d}x = 2\int_0^1 (1-x^2)^n\,\mathrm{d}x \geq 2\int_0^{1/\sqrt{n}}(1-x^2)^n\,\mathrm{d}x$
$\geq 2\int_0^{1/\sqrt{n}}(1-nx^2)\,\mathrm{d}x = \frac{4}{3\sqrt{n}} \geq \frac{1}{\sqrt{n}}$
i.e., $C_n \leq \sqrt{n}$.

Let $1 > \delta > 0$ be fixed.
Then $Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$ for $x \in [-1,-\delta] \cup [\delta,1]$

Let $P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)\,\mathrm{d}t$
$= \int_{-x}^{1-x} f(x+t)Q_n(t)\,\mathrm{d}t$ (if $t < -x$, then $x+t<0$, then $f(x+t)=0$)
$= \int_0^1 f(t)Q_n(t-x)\,\mathrm{d}t$ $\left[\begin{smallmatrix} x+t=s \\ t=s-x \\ \mathrm{d}t=\mathrm{d}s \end{smallmatrix}\right]$

Observation 3: $P_n(x)$ is a polynomial in x.

Proof:

$$\begin{aligned}\frac{\mathrm{d}^{2n+1}}{\mathrm{d}x}P_n(x) &= \frac{\mathrm{d}^{2n+1}}{\mathrm{d}x}\int_0^1 f(t)Q_n(t-x)\,\mathrm{d}t \\ &= \int_0^1 f(t)\frac{\mathrm{d}^{2n+1}}{\mathrm{d}x}Q_n(t-x)\,\mathrm{d}t \\ &= \int_0^1 f(t)0\,\mathrm{d}t = 0.\end{aligned}$$

Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ so that if $|x-y| < 2\delta$, then $|f(x)-f(y)| < \epsilon/2$.

Since $Q_n(t) \geq 0$, we get
Theorem: (Weierstrass Approximation Theorem)

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} [f(x+t) - f(x)] Q_n(t)\,\mathrm{d}t \right| \qquad \text{(note: } \int_{-1}^{1} Q_n = 1)$$

$$= \left| \int_{-1}^{-\delta} [f(x+t) - f(x)] Q_n(t)\,\mathrm{d}t + \int_{-\delta}^{\delta} [f(x+t) - f(x)] Q_n(t)\,\mathrm{d}t + \int_{\delta}^{1} [f(x+t) - f(x)] Q_n(t)\,\mathrm{d}t \right|$$

$$\leq \left| \int_{-1}^{-\delta} [f(x+t) - f(x)] Q_n(t)\,\mathrm{d}t \right| + \left| \int_{-\delta}^{\delta} [f(x+t) - f(x)] Q_n(t)\,\mathrm{d}t \right| + \left| \int_{\delta}^{1} [f(x+t) - f(x)] Q_n(t)\,\mathrm{d}t \right|$$

$$\leq 2M \int_{-1}^{\delta} Q_n(t)\,\mathrm{d}t + \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\,\mathrm{d}t + 2M \int_{\delta}^{1} Q_n(t)\,\mathrm{d}t$$

where $M = \|f\|_\infty$

$$\leq 4M\sqrt{n}(1 - \delta^{2\,11)})^n + \frac{\epsilon}{2}.$$

The first term tends to 0 as $n \to \infty$.
Large N, we get [12] $\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon$ and such $\|P_N - f\|_\infty \leq \epsilon$.

PMATH 351 Lecture 30: March 22, 2010

The Stone Weierstrass Theorem (generalisation of Weierstrass approximation theorem)

Let A be a compact metric space, $\mathcal{B} \subset C(A, \mathbb{R})$
Assuming that $\mathcal{B}$ satisfies:

i) $\mathcal{B}$ is an algebra, i.e., $f,\, g \in \mathcal{B} \implies f + g \in \mathcal{B},\ fg^{13)} \in \mathcal{B}$
$\qquad \implies \lambda f \in \mathcal{B}^{14)},\ \lambda \in \mathbb{R}$, multiplicative

ii) constant function $1 \in \mathcal{B}$

iii) $\mathcal{B}$ separates points of A

then the closure of $\mathcal{B}$, denoted $\overline{\mathcal{B}}$, equals $C(A, \mathbb{R})$

Example: $A = [a, b]$, $\mathcal{B} = \{\, p(x) : p \text{ is a polynomial on } [a, b] \,\}$
i, ii, iii) obvious, (iii) take the identity.
Every continuous function in $[a, b]$ can be approximated by a polynomial
Proof: By the Weierstrass approximation theorem, for every n, exists p_n such that

$$||t| - p_n(t)| < 1/n \text{ for } -n \leq t \leq n$$

Thus $||f(x)| - p_n(f(x))| < 1/n$ for $-n \leq f(x) \leq n$ (n be large enough since A is compact).

This shows that $\overline{\mathcal{B}}$ is closed under taking absolute value, i.e., $f \in \overline{\mathcal{B}}$ implies $|f| \in \overline{\mathcal{B}}$.
First $\mathcal{B}$ is an algebra, is $\overline{\mathcal{B}}$ also an algebra? Yes, since

$$\left.\begin{array}{l} f \in \overline{\mathcal{B}} \implies \exists \text{ an approx} \implies |f - f_n| < \epsilon \\ g \in \overline{\mathcal{B}} \implies \exists \text{ an approx} \implies |g - g_n| < \epsilon \end{array}\right\} f + g \in \overline{\mathcal{B}}$$

Check $+$ is a continuous function on $C(A, \mathbb{R}) \times C(A, \mathbb{R})$ to $C(A, \mathbb{R})$

Similarly, x is also continuous, $f \in \overline{\mathcal{B}},\ g \in \overline{\mathcal{B}} \implies fg \in \overline{\mathcal{B}}$
$\rightsquigarrow \overline{\mathcal{B}}$ is an algebra

---

[11] arrow to below
[12] arrow from above
[13] pointwise
[14] with $f + g \in \mathcal{B}$, vector space + linear algebra $fg \in \mathcal{B}$

If $f \in \overline{\mathcal{B}}$, so is $p_n(f)$ (because $\overline{\mathcal{B}}$ is an algebra)
Also, $p_n(f)(x) = p_n(f(x))$ and $\big||f(x)| - \underbrace{p_n(f(x))}_{\in \overline{\mathcal{B}}}\big| < 1/n$ means that $|f(x)|$ can be approximated by an element of $\overline{\mathcal{B}}$, then $|f(x)| \in \overline{\mathcal{B}}$ since $\overline{\mathcal{B}}$ is closed and $|f(x)|$ is a limit point of $\overline{\mathcal{B}}$.

Aside: A is compact, f is bounded on A, there exists large enough n such that $-n \leq f(x) \leq n$

$$\begin{aligned}
\text{Define } f \vee g &= \max(f, g) \text{ pointwise} \\
f \wedge g &= \min(f, g) \text{ pointwise} \\
\text{and observe that } f \vee g &= \frac{f+g}{2} + \frac{|f-g|}{2} \\
f \wedge g &= \frac{f+g}{2} - \frac{|f-g|}{2}
\end{aligned}$$

We see that $\overline{\mathcal{B}}$ is closed under maximum and minimum.

figure: distance between $\frac{a+b}{2}$ and b on real line

Let $h \in C(A, \mathbb{R})$ and $x_1 \neq x_2 \in A$, then by (iii), there exists $g \in \mathcal{B}$ such that $g(x_1) \neq g(x_2)$. By choosing α, $\beta \in \mathbb{R}$ correctly, we can have

$$\begin{aligned}
\alpha g + \beta \text{ achieving } (\alpha g + \beta)(x_1) &= h(x_1) \\
(\alpha g + \beta)(x_2) &= h(x_2)
\end{aligned}$$

Call such $\alpha g + \beta$ by the name: $f_{x_1 x_2}$ — That is $f_{x_1 x_2} \in \mathcal{B}$ and

$$\begin{aligned}
f_{x_1 x_2} &= h(x_1) \\
f_{x_1 x_2} &= h(x_2)
\end{aligned}$$

— textbook 5.8.2

$$\begin{aligned}
f_{yx}(y) = h(y) &\implies f_{yx}(y) > h(y) - \epsilon \\
\text{for } z \in U \subset \mathcal{U}(y) &\implies f_{yx}(z) > h(z) - \epsilon \text{ by continuity of } h
\end{aligned}$$

should be also by continuity of f_{yx}

Is the metric used?

PMATH 351 Lecture 31: March 24, 2010

$f \in \overline{\mathcal{B}}$
$\implies p(f) \in \overline{\mathcal{B}}$
$f^2 + 2f + 10 f_{x_1 x_2} = h(x_1)$, $f_{x_1 x_2}(x_2) = h(x_2)$
f_{xy}
Let $\epsilon > 0$ and $x \in A$. For $y \in A$, $\exists$ neighborhood $\mathcal{U}(y)$ of y such that

$$f_{yx}(z) > h(z) - \epsilon \text{ for all } z \in \mathcal{U}(y)$$

(simply because h is continuous)

$$\begin{aligned}
f_{yx}(y) &= h(y) \\
f_{yx}(y^{15)}) &> h(y^{16)}) - \epsilon \\
f_{yx}(z) &> h(z) - \epsilon
\end{aligned}$$

Baire's Category Theorem
Reference on page 175, chapter 3, Exercise 33. Let M be a metric space. A set $S \subset M$ is called *nowhere* dense (in M) if for every [nonempty] open U, we have $\mathrm{cl}(S) \cap U \neq U$, or equivalently

$$\mathrm{int}(\mathrm{cl}(S)) =^{17)} \emptyset$$

15) $z \in \mathcal{U}(y)$
16) z
17) (typo $\neq$ in text)

Show that $\mathbb{R}^n$ cannot be written as a countable union of nowhere dense sets.

Definition: A set $A \subset M$ is of *first* category (in M) if it is the union of countably many nowhere dense sets. Else A is of second category.

The exercise above can be phrased as: $\mathbb{R}^n$ is of 2nd category.

Theorem: (Baires) Every complete metric space M is of 2nd category (in M).

Examples: Let the metric space M be $\mathbb{R}$. Is $\mathbb{N} \subseteq \mathbb{R}$ of 1st category or 2nd category? Answer: 1st. $\mathbb{N}$ is of first category in $\mathbb{R}$.

Baire's Theorem gives:

$$\mathbb{N} \text{ is of 2nd category in } \mathbb{N}$$

In $\mathbb{N}$,

$$\text{cl}(\{2\}) = \{2\}$$
$$\text{int}(\text{cl}\{2\}) = \text{int}(\{2\}) = \{2\} \neq \emptyset$$

PMATH 351 Lecture 32: March 26, 2010

Baire Category Theorem. A complete metric space X is of 2nd category, i.e., it is not the union of countably many nowhere dense sets.

Proof: Let S_n be a sequence of nowhere dense sets, i.e., $\overline{S_n}$ has empty interior for each n. Let $U_n = X \setminus \overline{S_n}$[18]. Then each U_n is open and dense. In particular, every non-empty open set in X meets U_n.

We shall show that $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$.

Let $x_1 \in U_1$ be fixed. Let r_1 be a positive radius so that

$$D_1 = D(x_1, r_1) \subset U_1.$$

Since U_2 is dense, there exists a point x_2 of U_2 which is in D_1. Since U_2 is open, there exists a small enough radius r_2 so that $D_2 = D(x_2, r_2) \subset U_2$. We may assume that r_2 is small enough that $r_2 < \frac{1}{2}r_1$, and smaller than $r_1 - d(x_1, x_2)$ [note: $x_2 \in D_1$].

Then $\overline{D_2} \subset D_1$. By induction, we get a sequence of discs D_n with centres x_n and radii r_n so that

$$\overline{D_n} \subset D_{n-1},\ D_n \subset U_n,\ r_n < \tfrac{1}{2}r_{n-1}.$$

In particular $r_n \to 0$ as $n \to \infty$.
Note: $n, m \geq N \implies x_n, x_m \in D_N \implies d(x_n, x_m) < 2r_N$. This sequence x_n is Cauchy and therefore converges to an x in the complete space X.

$x_n \in D_N$ for all $n \geq N \implies x \in \overline{D_N} \subset D_{N-1}$.
Thus $x \in D_k$ for every k.
So $x \in \bigcap_{k=1}^{\infty} D_k$. So $x \in \bigcap_{n=1}^{\infty} U_k$ as each $D_k \subset U_k$.

Now $x \in \bigcap_{n=1}^{\infty} U_n \implies x \notin \left[X \setminus \bigcap_{n=1}^{\infty} U_n\right] \implies x \notin \bigcup_{n=1}^{\infty}(X \setminus U_n)$
$\implies x \notin \bigcup_{n=1}^{\infty} \overline{S_n} \implies x \notin \bigcup_{n=1}^{\infty} S_n$.
Hence $\bigcup_{n=1}^{\infty} S_n \neq X$.

Corollary: (The uniform boundedness principle). Let $\mathcal{B}$ be a family of real valued continuous functions on a complete metric space M (i.e., $\mathcal{B} \subset C(X, \mathbb{R})$).

Suppose that for $x \in M$, there is a bound b_x such that $|f(x)| \leq b_x$ *for all* $f \in \mathcal{B}$. [*pointwise boundedness*[19] of the family $\mathcal{B}$] Then there exists an *open set* $G \subset X$, $G \neq \emptyset$, and a constant b such that

$$|f(x)| \leq b \text{ for all } f \in \mathcal{B} \text{ and all } x \in G.$$

PMATH 351 Lecture 33: March 29, 2010

[18] closure
[19] in X

The uniform boundedness principle.

Let $\mathcal{B}$ be a family of continuous functions on a complete metric space M, and suppose that for each $x \in M$, there exists a constant b_x such that $|f(x)| \leq b_x$ for all $f \in \mathcal{B}$ [pointwise boundedness]. Then there is a non-empty open set (say a disc) G such and a constant b such that

$$|f(x)| \leq b \text{ for all } x \in G \text{ and } f \in \mathcal{B}$$

[uniform boundedness of $\mathcal{B}$ on G.]
Proof: For each $n \in \mathbb{N}$, let

$$F_n = \{\, x \in M : |f(x)| \leq n \text{ for all } f \in \mathcal{B} \,\}$$

Then each F_n is a closed set in M, because

$$F_n = \bigcap_{f \in \mathcal{B}} \{\, x \in M : f(x) \in [-n, n] \,\} = \bigcap_{f \in \mathcal{B}} f^{-1}([-n, n])$$

For each $x \in M$, there exists $n \in \mathbb{N}$ such that

$$x \in F_n \qquad \text{(by pointwise boundedness and take } n \geq b_x)$$

Therefore $\bigcup_{n=1}^{\infty} F_n = M$.
Baire's Theorem asserts that M is *not* of 1st category as M is complete. So, at least some F_{n_0} which is *not* nowhere dense. So $(\overline{F_{n_0}})^\circ \neq \emptyset$. As F_{n_0} is closed, $\overline{F_{n_0}} = F_{n_0}$. So ${F_{n_0}}^\circ \neq \emptyset$. Take $G = {F_{n_0}}^\circ$. $\circ$: interior
Thus $x \in G \implies x \in F_{n_0} \implies |f(x)| \leq n_0$ for all $f \in \mathcal{B}$. So $|f(x)| \leq n_0$ for all $f \in \mathcal{B}$ and $x \in G$. Take $b = n_0$.

Space-filling curves (paths). figure: Hilbert curve

Proposition: There exists a continuous (path) $f \colon [0,1] \to [0,1] \times [0,1]$ which is surjective.

$$\begin{aligned} |f_1(t) - f_2(t)| &\leq \sqrt{2}\delta \text{ for all } t \\ \|f_1 - f_2\|_\infty &\leq \sqrt{2}\delta,\ f_1,\ f_2 \in C([0,1], \mathbb{R}^2) \\ \|f_3 - f_2\|_\infty &\leq \sqrt{2}(\tfrac{\delta}{2}) \\ \text{etc } \|f_{n+1} - f_n\|_\infty &\leq \sqrt{2}(\tfrac{\delta}{2^{n-1}}) \text{ inductively} \end{aligned}$$

We get from the above that f_n is a Cauchy sequence in the complete space $C([0,1], \mathbb{R}^2)$. It converges to an $f \in C([0,1], \mathbb{R}^2)$.

Question: Is f injective? No.

Is $\left\{\, x \in \mathbb{R} : \underbrace{\sin(x) + \cos(e^x) + \sqrt{2}x^7}_{f(x)} < 10 \,\right\}$ open?
$= f^{-1}(]-\infty, 10[)$

PMATH 351 Lecture 34: March 31, 2010

Example: If X is a topological space and A, $B \subset X$ are connected subsets, $A \cap B \neq \emptyset$, then $A \cup B$ is connected.
Proof: (Version 1). Suppose that U and V are open, disjoint sets partitioning $A \cup B$. We intend to show that one of them is empty.

Since A is connected, figure: U and V partition $A \cup B$
[$U_A = U \cap A$ is open in A, $V_A = V \cap A$ is open in A, and U_A and V_A partition A]

$U \cap A$ *or* $V \cap A$ must be empty. Hence either $A \subset U$ or $A \subset V$, without loss of generality, say $A \subset U$.

Similarly, either $B \subset U$ *or* $B \subset V$.

Case 1: Suppose that $B \subset U$.
Hence $A \cup B \subset U$.
Then, as $A \cup B = U$[20] and V[21].
So $V = \emptyset$.

figure: A, $B \subset U$

Case 2: Suppose that $B \subset V$. As U and V are disjoint, A and B must be disjoint. A contradiction to $A \cap B \neq \emptyset$.

figure: $A \subset U$, $B \subset V$

Version 2: We show $A \cup B$ has the IVP. Let $f\colon A \cup B \to \mathbb{R}$ be continuous and that $f(x_1) > 0$ and $f(x_2) < 0$ for given x_1, $x_2 \in A \cup B$. Let $x_0 \in A \cap B$ be fixed (exists by assumption).
Case 1: $f(x_0) = 0$. (Done)
Case 2: Suppose that $f(x_0) < 0$.
Subcase: If x_1 and x_2 are both from A, by the continuity of $f|_A\colon A \to \mathbb{R}$ and the connectedness of A, there exists $c \in A$ where $f(c) = 0$.
Subcase: If x_1 and x_2 are both from B, similarly, we get that there exists $c \in B$ where $f(c) = 0$.
Subcase: If $x_1 \in A$, $x_2 \in B$, then by continuity of $f|_A\colon A \to \mathbb{R}$ and connectedness of A, and $f(x_1) > 0$, $f(x_0) < 0$, there exists $c \in A$ with $f(c) = 0$.

figure: x_1, $x_2 \in A \cup B$

figure: connected sets which are not path connected sets

(M, d) a metric space
$d\colon M \times M \to \mathbb{R}$
ρ metric on $M \times M$ may be defined by $\rho((x_1, x_2), (y_1, y_2)) = \max(d(x_1, y_1), d(x_2, y_2))$

$$\begin{aligned} D(x_0, r) &= \{\, x \in M : d(x_0, x) < r \,\} \\ &= \{\, x \in M : \underbrace{d(x_0, x)}_{f(x)} \in \,]{-\infty}, r[\, \} \\ &= f^{-1}(]{-\infty}, r[) \end{aligned}$$

Therefore $D(x_0, r)$ is open.

$\{\, x \in M : 1 < d(x_0, x) < 2 \,\} = f^{-1}(]1, 2[)$

PMATH 351 Lecture 35: April 5, 2010

Exercise 1. Let T_1 and $T_2\colon \mathbb{R}^n \to \mathbb{R}^n$ be two contractions. Let a_1 and a_2 be the unique fixed points of T_1 and T_2 respectively. Show that there exists $c < 1$ such that

$$\|a_1 - a_2\| \leq \tfrac{1}{1-c} \Big(\sup_{x \in \mathbb{R}^n} \|T_1(x) - T_2(x)\| \Big).$$

Exercise 2. Let (M, d) be a metric space with a countable dense set. (We call M *separable.*) Show that for every subset $A \subset M$, there exists a countable (at most countable) subset of A which is dense in A.

$C(X$[22]$, \mathbb{R})$

$\overline{\overline{A}} = \overline{A}$

A sequence of functions $f_n\colon X \to (M, d)$
is *pointwise* Cauchy if for each $x \in X$, $f_n(x)$ (a sequence in X) is Cauchy, i.e., $\forall \epsilon > 0$, $\exists N$ such that $d(f_n(x), f_m(x)) < \epsilon$[23] for n, $m \geq N$.
It is *uniformly* Cauchy if for for all $\epsilon > 0$, $\exists N$ such that $d(f_n(x), f_m(x)) < \epsilon$ *for all* $x \in X$.

Example: Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous. Show $(f \vee g)(x) = \max(f(x), g(x))$ is a continuous function.
Proof: Use $f \vee g = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$
or **Proposition:** a function $\phi\colon \mathbb{R} \to \mathbb{R}$ is continuous if and only if

$$\phi^{-1}(]{-\infty}, a[) \text{ and } \phi^{-1}(]a, \infty[)$$

[20] disjoint
[21] disjoint
[22] compact
[23] not $|f_n(x) - f_m(x)| < \epsilon$

are open for each $a \in \mathbb{R}$.
$(f \vee g)^{-1}(]-\infty, a[)$
$= \{ x \in \mathbb{R} : (f \vee g)(x) < a \} = \{ x \in \mathbb{R} : f(x) < a \textit{ and } g(x) < x \}$
$= \{ x \in \mathbb{R} : f(x) < a \} \cap \{ x \in \mathbb{R} : g(x) < a \}$
$= f^{-1}(]-\infty, a[)$[24] $\cap\, g^{-1}(]-\infty, a[)$[25]

[24] open by continuity of f
[25] open by continuity of g