Projected Walk on Spheres: A Monte Carlo Closest Point Method for Surface PDEs - Supplemental Note

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A GREEN'S FUNCTIONS AND THEIR DERIVATIVES

We list Green's functions on a ball with radius R in $\mathbb{R}^3$ and their derivatives for readers' convenience. As Sawhney and Crane [2020] summarized, when $\mathbf{x}$ is at the center of the ball, the Green's function for the Poisson equation is

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi}\frac{r-R}{rR}, \tag{11}$$

and the green's function for the screened Poisson equation is

$$G_\sigma(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi}\frac{\sinh((r-R)\sqrt{\sigma})}{r\sinh(R\sqrt{\sigma})}, \tag{12}$$

where $\mathbf{r} = \mathbf{y} - \mathbf{x}$ and $r = \|\mathbf{r}\|_2$.

The gradients of G and G_σ with respect to $\mathbf{x}$ when $\mathbf{x}$ is at the center of the ball are

$$\nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi}\left(\frac{1}{r^3} - \frac{1}{R^3}\right)\mathbf{r}, \tag{13}$$

and

$$\nabla_\mathbf{x} G_\sigma(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi}\left(\frac{\sqrt{\sigma}\cosh((R-r)\sqrt{\sigma})}{r\sinh(R\sqrt{\sigma})}\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{\sinh((R-r)\sqrt{\sigma})}{r\sinh(R\sqrt{\sigma}))}\left(\frac{1}{r^2} + \frac{\sqrt{\sigma}\cosh(R\sqrt{\sigma})}{R\sinh(R\sqrt{\sigma})}\right)\right). \tag{14}$$

We additionally derive $\nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y})$ in the general case when $\mathbf{x}$ is not at the center of the ball:

$$\nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi}\left(\frac{1}{r^3}\mathbf{r} + \frac{Ry}{q^3}\mathbf{q}\right), \tag{15}$$

where $\mathbf{q} = y\mathbf{x} - (R^2/y)\mathbf{y}$, $y = \|\mathbf{y}\|_2$, and $q = \|\mathbf{q}\|_2$. We also have $\nabla_\mathbf{z} G(\mathbf{x}, \mathbf{z}) = \nabla_\mathbf{z} G(\mathbf{z}, \mathbf{x})$ due to the symmetry of G. We use this expression for problems with a divergence of a vector field as their source term.

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B DIVERGENCE SOURCE TERM

For the solution estimator, when the source term $f = \nabla \cdot \mathbf{h}$, the volume term converts to

$$\begin{aligned} &\int_{B_r(\mathbf{x})} f(\mathbf{z})G(\mathbf{x}, \mathbf{z})\,\mathrm{d}\mathbf{z} \\ &= \int_{B_r(\mathbf{x})} (\nabla_\mathbf{z} \cdot \mathbf{h}(\mathbf{z}))\, G(\mathbf{x}, \mathbf{z})\,\mathrm{d}\mathbf{z}, \\ &= \int_{\partial B_r(\mathbf{x})} \mathbf{h}(\mathbf{z}) \cdot \mathbf{n}(\mathbf{z})\, G(\mathbf{x}, \mathbf{z})\,\mathrm{d}\mathbf{z} - \int_{B_r(\mathbf{x})} \mathbf{h}(\mathbf{z}) \cdot \nabla_\mathbf{z} G(\mathbf{x}, \mathbf{z})\,\mathrm{d}\mathbf{z}, \\ &= -\int_{B_r(\mathbf{x})} \mathbf{h}(\mathbf{z}) \cdot \nabla_\mathbf{z} G(\mathbf{x}, \mathbf{z})\,\mathrm{d}\mathbf{z}, \end{aligned} \tag{16}$$

and we evaluate the last integral instead, which does not require the explicit evaluation of the divergence of $\mathbf{h}$. We generate the samples to estimate the converted volume integral with $p(\mathbf{z}) \propto 1/\|\mathbf{x} - \mathbf{z}\|_2^2$, so the singularity of $\nabla_\mathbf{z} G$ cancels out.

C GRADIENT ESTIMATION

The gradient estimator replaces the integral equation for the first step of recursion with

$$\nabla u(\mathbf{x}) = \frac{1}{|B_r(\mathbf{x})|}\int_{\partial B_r(\mathbf{x})} u(\mathbf{y})\mathbf{n}(\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{B_r(\mathbf{x})} f(\mathbf{z})\nabla_\mathbf{x} G(\mathbf{x}, \mathbf{z})\,\mathrm{d}\mathbf{z}, \tag{17}$$

where $|B_r(\mathbf{x})|$ is the volume of the ball and $\mathbf{n}(\mathbf{y})$ is the outward unit normal of the ball at $\mathbf{y}$. For the screened Poisson equation, we multiply the first term by $c_{r,\sigma}$ and replace ∇G in the second term with ∇G_σ to get a similar integral equation. To evaluate the integrals, we uniformly sample a point on the sphere for the first term, and we generate the samples with $p(\mathbf{z}_i) \propto 1/\|\mathbf{x} - \mathbf{z}_i\|_2^2$ for the second term. The surface gradient of the solution to a surface PDE does not have a normal component, but the estimated solution may have a nonzero normal component before convergence. Thus, to improve the estimate, we set the normal component(s) of the estimated gradient to zero as a post-processing step.

D CONVERGENCE STUDY SETUP

We used the following problems to generate the error convergence plots in Fig. 4. Note that we finely discretized the surfaces we describe below to obtain the data we show in the figure.

(a). The helix curve we use has three turns, has a radius of 1, and the endpoints have a height difference of 2. We solve the Laplace

equation defined along the curve length ϕ as

$$\begin{aligned} \frac{\partial^2 u_{\mathcal{S}}}{\partial \phi^2} &= 0, \\ u_{\mathcal{S}}(0) &= 0, \\ u_{\mathcal{S}}(\psi) &= 1, \end{aligned} \tag{18}$$

where the boundary conditions are specified at the two ends of the curve, $\phi = 0$ and $\phi = \psi$. The analytical solution is $u_{\mathcal{S}}(\phi) = \phi/\psi$.

(b) to (d). The problem we solve is defined along the curve length ϕ as

$$\begin{aligned} \frac{\partial^2 u_{\mathcal{S}}}{\partial \phi^2} &= 0.02, \\ u_{\mathcal{S}}(0) &= 0, \\ u_{\mathcal{S}}(\psi) &= 1, \end{aligned} \tag{19}$$

where the boundary conditions are specified at the two ends of the curve, $\phi = 0$ and $\phi = \psi$, similar to (a). The analytical solution is $u_{\mathcal{S}}(\phi) = 0.01\phi^2 + \frac{1-0.01\psi^2}{\psi}\phi$. The helix curve in (b) is identical to the one in (a). The z-order curve in (c) and (d) is defined using 8 points, $(\pm 1.0, \pm 1.0, \pm 1.0)$.

(e). This scene is one of the scenes in the grid-based CPM paper by King et al. [2023]. On a unit circle, we have a two-sided Dirichlet boundary. In polar coordinates, the problem we solve in terms of the angle θ is

$$\begin{aligned} \frac{\partial^2 u_{\mathcal{S}}}{\partial \theta^2} &= -2\cos(\theta - \theta_c), \\ u_{\mathcal{S}}(\theta_c^-) &= 2, \\ u_{\mathcal{S}}(\theta_c^+) &= 22, \end{aligned} \tag{20}$$

where $\theta_c = 1.022\pi$ is the position of the Dirichlet boundary. The analytical solution to this problem is $u_{\mathcal{S}}(\theta) = 2\cos(\theta - \theta_c) + \frac{10}{\pi}(\theta - \theta_c)$.

(f). The surface we used is a torus with a major radius $R = 3$ and a minor radius $r = 1$. The Dirichlet boundary curve is a torus knot expressed as a parametric curve

$$x_1(s) = v(s)\cos(as), \quad x_2(s) = v(s)\sin(as), \quad x_3(s) = \sin(bs), \tag{21}$$

where $v(s) = R + \cos(bs), a = 3, b = 7$, and $s \in [0, 2\pi]$. We solve the Laplace equation on the torus with boundary condition $\sin(s)$ along the curve. We used the grid-based CPM implementation of King et al. [2023] with a grid spacing of 0.02 to generate a reference solution and measured the error of PWoS against it.

(g) and (h). The surface we used for these setups is the one given by Dziuk [1988] and later used in multiple CPM works [Chen and Macdonald 2015; King et al. 2023]. The surface is expressed as $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^3 | (x_1 - x_3)^2 + x_2^2 + x_3^2 = 1\}$. The problem we solve is

$$\Delta_{\mathcal{S}} u_{\mathcal{S}}(\mathbf{x}) - u_{\mathcal{S}}(\mathbf{x}) = -f_{\mathcal{S}}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{S}, \tag{22}$$

where $f_{\mathcal{S}}$ has an analytical, yet complex, expression we can derive as in the work by Chen and Macdonald [2015], so the solution of the problem becomes $u_{\mathcal{S}}(\mathbf{x}) = x_1 x_2$. For (g), we use a unit circle on the $x_1 x_2$-plane with the analytical solution specified on it as the boundary value as the Dirichlet boundary. For (h), we did not use any boundary to show the algorithm's convergence for the screened Poisson equation without any boundaries.

(i) to (p). The scenes consider the unit sphere with a spherical harmonic function as the analytical solution as is done in the study of mesh Laplacians [Bunge and Botsch 2023]. The sphere mesh is punched inward at $x_3 = 0.25$ for (m) to (p) to test the algorithm on a geometry with sharp corners. Given a spherical harmonic $Y_2^3(\mathbf{x}) = \frac{1}{4}\sqrt{\frac{105}{\pi}}(x_1^2 - x_2^2)^2 x_3$ with eigenvalue -12 as the solution, we solve the Poisson equation

$$\Delta_{\mathcal{S}} u_{\mathcal{S}}(\mathbf{x}) = -12 Y_2^3(\mathbf{x}), \quad \mathbf{x} \in \mathcal{S}, \tag{23}$$

for (i), (j), (m), and (n) and the screened Poisson equation

$$\Delta_{\mathcal{S}} u_{\mathcal{S}}(\mathbf{x}) - u_{\mathcal{S}}(\mathbf{x}) = -13 Y_2^3(\mathbf{x}), \quad \mathbf{x} \in \mathcal{S}, \tag{24}$$

for (k), (l), (o), and (p). For (i), (k), (m), and (o), we use the unit circle on the $x_1 x_2$-plane as the Dirichlet boundary, and for (j) and (n), we use the unit semicircle where $x_2 > 0$ as the Dirichlet boundary. We observe the expected convergence behavior with all of the cases in (i) to (p) and suspect that it has something to do with the fact that the source term is a constant multiple of the solution.

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