Supplemental to: "Variational Stokes: A Unified Pressure-Viscosity Solver for Accurate Viscous Liquids"

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Additional Key Words and Phrases: liquids, free surfaces, stokes, viscosity

1 DETAILS OF CONVERGENCE STUDY

We describe two test problems that we used to confirm the convergence of our method. We compute the L^∞ and L^1 errors, where our discrete L^1 norm is computed as $\|u_h\|_1 = \sum_i |u_i| h^d$ for a uniform grid spacing $h = \Delta x$ in d spatial dimensions

Free Surface Test (2D).. Our free surface Stokes test case is a fluid disk of radius $r = 0.75$ centred at the origin, with density $\rho = 1$ and viscosity $\mu = 0.1$, computed over a timestep $\Delta t = 1$. For simplicity of presentation, we describe the final velocity field in terms of a stream function, ψ, where the velocity field can be derived as $\boldsymbol{u}_{final} = \nabla \times \psi$. This also guarantees that the velocity field is divergence free. The stream function is:

$$\psi = \frac{128}{81} r^4 \cos(2\theta) \cos(\sqrt{3} \ln r) \left(15 - 30r + 16r^2\right) \tag{1}$$

This is a non-trivial velocity field designed to fulfill the free surface zero traction condition at $r = 0.75$, smoothly blended into a zero velocity at the origin ($r = 0$). The zero traction condition enforces a relationship between the surface pressure and the viscous stress resulting from this velocity field. To satisfy this condition, we use the following expression for pressure:

$$p = \frac{512\sqrt{3}}{81} r^2 \mu \sin(2\theta) \sin(\sqrt{3} \ln r)(15 - 30r + 16r^2) \tag{2}$$

The pressure in this expression will be non-zero at the interface; any method to solve this problem will need to correctly handle the coupling between pressure and viscous stresses. From this information, the expressions for the input velocity and final stresses can be derived using the equations of the PDE. We used a computer algebra system for this purpose. The convergence results are shown in Table 1a and the top half of Figure 1.

Solid Wall Test (2D).. Our Stokes solid boundary test case is an annulus centred at the origin with inner radius $r = 0.5$, outer radius $r = 1$, density $\rho = 1$, and viscosity $\mu = 0.1$, computed over a timestep $\Delta t = 1$. Inner and outer boundaries are static solids. We again use a stream function ψ to dictate our velocity field and ensure it is divergence free:

$$\psi = 256r^4 - 768r^3 + 832r^2 - 384r + 64 \tag{3}$$

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2017. Manuscript submitted to ACM

Grid	$\|u - u^h\|_\infty$	Order	$\|u - u^h\|_1$	Order
16^2	3.4171E-001		1.9727E-001	
32^2	7.4246E-002	2.20	4.3033E-002	2.20
64^2	2.6593E-002	1.48	1.2321E-002	1.80
128^2	9.2292E-003	1.53	3.3497E-003	1.88
256^2	6.7182E-003	0.46	1.5327E-003	1.13
512^2	3.0843E-003	1.12	5.9129E-004	1.37
1024^2	1.7877E-003	0.79	2.7579E-004	1.10

(a) Convergence of Stokes with free surface

Grid	$\|u - u^h\|_\infty$	Order	$\|u - u^h\|_1$	Order
16^2	4.4449E+000		3.2953E+000	
32^2	2.8765E+000	0.63	1.0500E+000	1.65
64^2	8.4194E-001	1.77	1.8614E-001	2.50
128^2	4.1100E-001	1.03	5.3975E-002	1.79
256^2	2.2147E-001	0.89	1.4992E-002	1.85
512^2	9.8967E-002	1.16	7.0519E-003	1.09
1024^2	5.3807E-002	0.88	3.6056E-003	0.97

(b) Convergence of Stokes with solid walls

Table 1. Convergence of Stokes in 2D

For pressure, we use:

$$p = r^2 \cos(\theta)\sin(\theta) \tag{4}$$

The equations of the PDE can be used to derive the input velocities and final stresses. The convergence results are shown in Table 1b and the bottom half of Figure 1.

2 EQUIVALENCE PROOFS FOR VARIATIONAL FORMS OF THE STOKES EQUATIONS

In the following two propositions we establish that the solution to our free surface and solid boundary variational Stokes problems solves the original (time-discretized) Stokes PDE:

$$\frac{\rho}{\Delta t}(\vec{u} - \vec{u}^*) = \nabla \cdot \tau - \nabla p \tag{5a}$$

$$\nabla \cdot \vec{u} = 0 \tag{5b}$$

$$\tau = \mu(\nabla \vec{u} + (\nabla \vec{u})^\top) \tag{5c}$$

subject to free surface or solid boundary conditions respectively. In the following proofs we assume all functions introduced are sufficiently smooth.

2.1 Free Surface

Definition 2.1. Let D be the rate of deformation operator defined as

$$D(\vec{u}) = \frac{\nabla \vec{v} + (\nabla \vec{v})^\top}{2}.$$

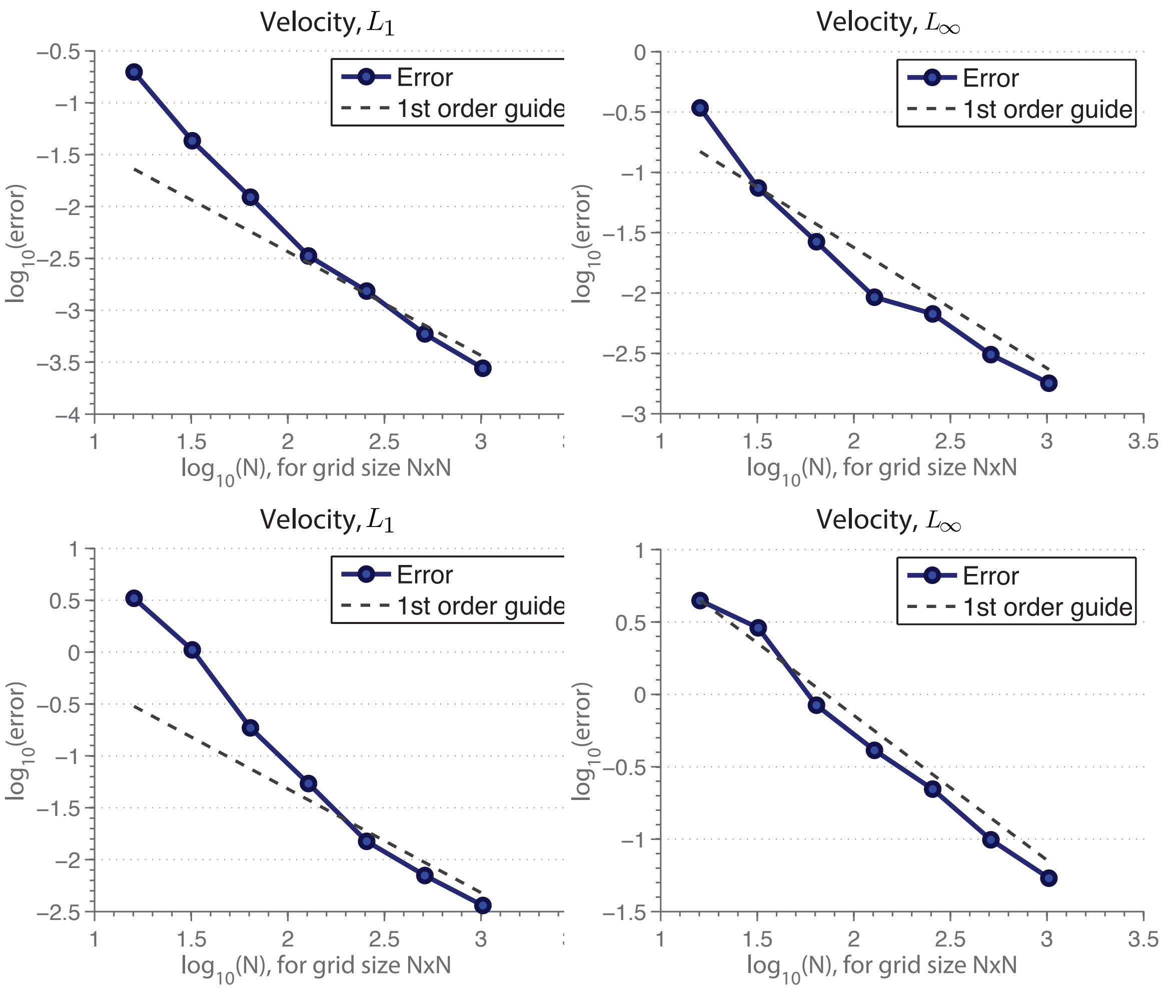

Fig. 1. Convergence graphs for a single step of a time-dependent 2D Stokes problem in an irregular domain with (top pair) free surface boundaries and (bottom pair) solid wall boundaries.

In the following proposition, we will use the fact that D is a linear operator and the integration-by-parts formula

$$\iiint_\Omega D(\vec{v}) : \sigma \, dV = \iint_{\partial\Omega} \vec{v} \cdot \sigma \hat{n} \, dA - \iiint_\Omega \vec{v} \cdot \nabla \cdot \sigma \, dV, \tag{6}$$

for any vector field $\vec{v}$ and symmetric rank-two tensor field σ, where $\hat{n}$ is the outward normal to $\partial\Omega$. We will also use the standard vector integration-by-parts formula

$$\iiint_\Omega q \nabla \cdot \vec{v} \, dV = \iint_{\partial\Omega} \vec{v} \cdot (q \boldsymbol{I} \hat{n}) \, dA - \iiint_\Omega \vec{v} \cdot \nabla q \, dV, \tag{7}$$

for any vector field $\vec{v}$ and scalar field q, where $\boldsymbol{I}$ is the identity tensor.

Proposition 2.2. *Let Ω_L be our liquid domain with a free surface boundary $\partial\Omega_L$. Consider the objective function of our variational Stokes formulation with natural free surface boundary conditions:*

$$F(p,\tau,\vec{u}) = \iiint_{\Omega_L} \frac{\rho}{2}\|\vec{u}-\vec{u}^*\|^2 - \Delta t p\nabla\cdot\vec{u} + \Delta t\tau : D(\vec{u}) - \frac{\Delta t}{4\mu}\|\tau\|_F^2. \tag{8}$$

Solving the minimization problem

$$\max_{p,\tau}\min_{\vec{u}} F(p,\tau,\vec{u}) \tag{9}$$

provides the solution to the original Stokes PDE (5) *on Ω_L, subject to the free surface boundary condition:*

$$(-p\boldsymbol{I}+\tau)\hat{n} = 0 \quad \text{on } \partial\Omega_L \tag{10}$$

where $\hat{n}$ is the outward normal to the boundary $\partial\Omega_L$.

Proof. Suppose p, τ, and $\vec{u}$ are the values of the pressure, shear stress, and velocity fields respectively at the stationary point of the objective function F. Let ϵ be a scalar, and define a scalar function $g(\epsilon)$ such that

$$g(\epsilon) = F(p+\epsilon q, \tau+\epsilon\sigma, \vec{u}+\epsilon\vec{v}), \tag{11}$$

where q is an arbitrary scalar function, σ is an arbitrary symmetric rank-two tensor function, and $\vec{v}$ is an arbitrary vector function. It is straightforward to show that g is quadratic in ϵ. Indeed rearranging the expression (11), we get

$$\begin{aligned}
g(\epsilon) = &\iiint_{\Omega_L} \frac{\rho}{2}\|\vec{u}-\vec{u}^*\|^2 - \Delta t\left(p\nabla\cdot\vec{u} - \tau : D(\vec{u}) + \frac{1}{4\mu}\|\tau\|^2\right) dV \\
&+ \epsilon \iiint_{\Omega_L} \rho(\vec{u}-\vec{u}^*)\cdot\vec{v} - \Delta t\left(q\nabla\cdot\vec{u} + p\nabla\cdot\vec{v} - \tau : D(\vec{v}) - \sigma : D(\vec{u}) + \frac{1}{2\mu}\tau : \sigma\right) dV \\
&+ \epsilon^2 \iiint_{\Omega_L} \frac{\rho}{2}\|\vec{v}\|^2 - \Delta t\left(q\nabla\cdot\vec{v} - \sigma : D(\vec{v}) + \frac{1}{4\mu}\|\sigma\|^2\right) dV.
\end{aligned}$$

Since we chose p, $\vec{u}$ and τ to be optimal, it must be that $g'(0) = 0$. Thus the linear term above must be identically zero:

$$0 = \iiint_{\Omega_L} \rho(\vec{u}-\vec{u}^*)\cdot\vec{v} - \Delta t p\nabla\cdot\vec{v} + \Delta t\tau : D(\vec{v})\, dV \tag{12a}$$

$$- \Delta t \iiint_{\Omega_L} q\nabla\cdot\vec{u}\, dV \tag{12b}$$

$$+ \Delta t \iiint_{\Omega_L} \left(D(\vec{u}) - \frac{1}{2\mu}\tau\right) : \sigma\, dV. \tag{12c}$$

By applying the integration-by-parts formulas (7) and (6) to the second and third terms of (12a) respectively, we can rewrite the integral in (12a) as

$$\iiint_{\Omega_L} \left(\rho(\vec{u}-\vec{u}^*) + \Delta t\nabla p - \Delta t\nabla\cdot\tau\right)\cdot\vec{v}\, dV + \Delta t \iint_{\partial\Omega_L} (-p\boldsymbol{I}+\tau)\,\hat{n}\cdot\vec{v}\, dA \tag{12a$'$}$$

Observe that q, $\vec{v}$ and σ were chosen arbitrarily. This lets us recover the original Stokes PDE from (12). Indeed, setting $q = 0$ and $\sigma = \mathbf{0}$ in (12), but leaving $\vec{v}$ to be arbitrary requires (12a$'$) to be zero, which gives

$$\rho(\vec{u}-\vec{u}^*) + \Delta t\nabla p - \Delta t\nabla\cdot\tau = 0 \quad \text{on } \Omega_L, \text{ and} \tag{13}$$

$$(-p\boldsymbol{I}+\tau)\hat{n} = 0 \quad \text{on } \partial\Omega_L. \tag{14}$$

These are precisely the conditions (5a) and (10) respectively. Similarly, setting $\vec{v} = \vec{0}$ and $\sigma = \mathbf{0}$, but leaving q to be arbitrary, leaves us with (12b), which gives the divergence free condition (5b). Finally setting $q = 0$ and $\vec{v} = \vec{0}$, with σ arbitrary leaves (12c), which yields

$$D(\vec{u}) - \frac{1}{2\mu}\tau = 0. \tag{15}$$

This is exactly (5c) when rearranged. Thus we have recovered the three equations comprising the PDE form of the Stokes problem as well as the free surface boundary condition.

□

Although the proof for the solid boundary formulation can be done in the same way, we choose to write it in more detail to reveal the structure of the optimization problem. By considering each variable of the objective function separately and looking at the signs of the quadratic terms, we can see why the optimization indeed approaches a stationary point as we prescribe it.

2.2 Solid Boundary

Proposition 2.3. *Let Ω_F be the fluid domain with a solid boundary $\partial\Omega_F$. Consider the objective function of our variational Stokes formulation with natural solid boundary conditions:*

$$F(p, \tau, \vec{u}) = \iiint_{\Omega_F} \frac{\rho}{2}\|\vec{u} - \vec{u}^*\|^2 + \Delta t \vec{u} \cdot (\nabla p - \nabla \cdot \tau) - \frac{\Delta t}{4\mu}\|\tau\|_F^2 \, dV, \tag{16}$$

Solving the minimization problem

$$\max_{p,\tau} \min_{\vec{u}} F(p, \tau, \vec{u}) \tag{17}$$

provides the solution to the original Stokes PDE (5) *on Ω_F, subject to the solid boundary condition:*

$$\vec{u} = \vec{0} \quad \text{on } \partial\Omega_F. \tag{18}$$

Proof. Let p and τ be fixed and let $\vec{v}$ be an arbitrary vector function. Assume that $\vec{u}$ minimizes F. Then for some scalar ϵ, define

$$g(\epsilon) := F(p, \tau, \vec{u} + \epsilon\vec{v}).$$

It is straightforward to check that g is a quadratic function:

$$g(\epsilon) = \left(\iiint_{\Omega_F} \frac{\rho}{2}\|\vec{u} - \vec{u}^*\|^2 + \Delta t \vec{u} \cdot (\nabla p - \nabla \cdot \tau) - \frac{\Delta t}{4\mu}\|\tau\|_F^2 \, dV\right) \tag{19}$$

$$+ \epsilon \left(\iiint_{\Omega_F} \rho(\vec{u} - \vec{u}^*) \cdot \vec{v} + \Delta t(\nabla p - \nabla \cdot \tau) \cdot \vec{v} \, dV\right) \tag{20}$$

$$+ \epsilon^2 \left(\iiint_{\Omega_F} \frac{\rho}{2}\|\vec{v}\|^2 \, dV\right). \tag{21}$$

Since $\vec{u}$ is optimal, $g'(0) = 0$. Therefore the linear term above must be identically zero:

$$0 = g'(0) = \iiint_{\Omega_F} \rho(\vec{u} - \vec{u}^*) \cdot \vec{v} + \Delta t(\nabla p - \nabla \cdot \tau) \cdot \vec{v} \, dV. \tag{22}$$

Recall that $\vec{v}$ is chosen arbitrarily on Ω_F, which yields:

$$\rho(\vec{u} - \vec{u}^*) + \Delta t(\nabla p - \nabla \cdot \tau) = 0. \tag{23}$$

This is precisely the condition in (5a), and it can be rewritten in terms of $\vec{u}$ as

$$\vec{u} = \vec{u}^* - \frac{\Delta t}{\rho}(\nabla p - \nabla \cdot \tau) \tag{24}$$

Now assume that p is further chosen to be optimal in (17), and define a new function

$$h(\epsilon) := F(p + \epsilon q, \tau, \vec{u}).$$

for some arbitrary scalar function q. We learn that h is also a quadratic function by substituting in (24) for $\vec{u}$:

$$\begin{aligned}
h(\epsilon) &= \iiint_{\Omega_F} \frac{\rho}{2}\left\|\frac{\Delta t}{\rho}(\nabla(p+\epsilon q) - \nabla \cdot \tau)\right\|^2 + \Delta t\left(\vec{u}^* - \frac{\Delta t}{\rho}(\nabla(p+\epsilon q) - \nabla\cdot\tau)\right)\cdot(\nabla(p+\epsilon q) - \nabla\cdot\tau) - \frac{\Delta t}{4\mu}\|\tau\|_F^2\, dV \\
&= \iiint_{\Omega_F} \frac{\Delta t^2}{2\rho}\|\nabla(p+\epsilon q) - \nabla\cdot\tau\|^2 + \Delta t\vec{u}^*\cdot(\nabla(p+\epsilon q) - \nabla\cdot\tau) - \frac{\Delta t^2}{\rho}\|\nabla(p+\epsilon q) - \nabla\cdot\tau\|^2 - \frac{\Delta t}{4\mu}\|\tau\|_F^2\, dV \\
&= \iiint_{\Omega_F} \Delta t\vec{u}^*\cdot(\nabla(p+\epsilon q) - \nabla\cdot\tau) - \frac{\Delta t^2}{2\rho}\|\nabla(p+\epsilon q) - \nabla\cdot\tau\|^2 - \frac{\Delta t}{4\mu}\|\tau\|_F^2\, dV \\
&= \iiint_{\Omega_F} \Delta t\vec{u}^*\cdot(\nabla p - \nabla\cdot\tau) + \epsilon\Delta t\vec{u}^*\cdot\nabla q - \frac{\Delta t^2}{2\rho}\left(\|\nabla p - \nabla\cdot\tau\|^2 - 2\epsilon\nabla q\cdot(\nabla p - \nabla\cdot\tau) - \epsilon^2\|\nabla q\|^2\right) - \frac{\Delta t}{4\mu}\|\tau\|_F^2\, dV \\
&= \left(\iiint_{\Omega_F} \Delta t\vec{u}^*\cdot(\nabla p - \nabla\cdot\tau) - \frac{\Delta t^2}{2\rho}\|\nabla p - \nabla\cdot\tau\|^2 - \frac{\Delta t}{4\mu}\|\tau\|_F^2\, dV\right) \\
&\quad + \epsilon\left(\iiint_{\Omega_F} \Delta t\vec{u}^*\cdot\nabla q - \frac{\Delta t^2}{\rho}\nabla q\cdot(\nabla p - \nabla\cdot\tau)\, dV\right) \\
&\quad - \epsilon^2\left(\iiint_{\Omega_F} \frac{\Delta t^2}{2\rho}\|\nabla q\|^2\, dV\right).
\end{aligned} \tag{25}$$

As before, since p was chosen to be optimal, $h'(0) = 0$, and so

$$0 = h'(0) = \iiint_{\Omega_F}\left(\vec{u}^* - \frac{\Delta t}{\rho}(\nabla p - \nabla\cdot\tau)\right)\cdot\nabla q\, dV = \iiint_{\Omega_F}\vec{u}\cdot\nabla q\, dV.$$

In order to express this in terms of q and not its derivative, we apply integration-by-parts to get

$$0 = \iint_{\partial\Omega_F} q\vec{u}\cdot\hat{n}\, dA - \iiint_{\Omega_F} q\nabla\cdot\vec{u}\, dV. \tag{26}$$

Since q was chosen arbitrarily, we get two conditions:

$$\nabla\cdot\vec{u} = 0 \quad \text{on } \Omega_F \tag{27}$$

$$\vec{u}\cdot\hat{n} = 0 \quad \text{on } \partial\Omega_F, \tag{28}$$

where the first corresponds to (5b). Note that the second condition doesn't quite satisfy our no-slip boundary condition (18).

It remains to show that (5c) and (18) hold. We proceed as before: assume that τ maximizes (17), and take an arbitrary symmetric rank-two tensor function σ (not to be confused with Cauchy stress). Then we define yet another function

$$f(\epsilon) := F(p, \tau + \epsilon\sigma, \vec{u}),$$

which is also quadratic. We can follow steps similar to (25), to show that f is quadratic in ϵ:

$$f(\epsilon) = \left(\iiint_{\Omega_F} \Delta t \vec{u}^* \cdot (\nabla p - \nabla \cdot \tau) - \frac{\Delta t^2}{2\rho} \|\nabla p - \nabla \cdot \tau\|^2 - \frac{\Delta t}{4\mu} \|\tau\|_F^2 \, dV \right) \tag{29}$$

$$+ \epsilon \left(\iiint_{\Omega_F} \frac{\Delta t^2}{\rho} (\nabla p - \nabla \cdot \tau) \cdot \nabla \cdot \sigma - \Delta t \vec{u}^* \cdot \nabla \cdot \sigma - \frac{\Delta t}{2\mu} \tau : \sigma \, dV \right) \tag{30}$$

$$+ \epsilon^2 \left(\iiint_{\Omega_F} \frac{\Delta t^2}{2\mu} \|\nabla \cdot \sigma\|_F^2 + \frac{\Delta t}{4\mu} \|\sigma\|_F^2 \, dV \right). \tag{31}$$

As before, since τ is optimal, $f'(0) = 0$, and so

$$0 = f'(0) = \iiint_{\Omega_F} \frac{\Delta t}{\rho} (\nabla p - \nabla \cdot \tau) \cdot \nabla \cdot \sigma - \vec{u}^* \cdot \nabla \cdot \sigma - \frac{1}{2\mu} \tau : \sigma \, dV \tag{32}$$

$$= - \iiint_{\Omega_F} \vec{u} \cdot \nabla \cdot \sigma + \frac{1}{2\mu} \tau : \sigma \, dV. \tag{33}$$

Following integration-by-parts, we get

$$0 = \iiint_{\Omega_F} \frac{1}{2} \left(\nabla \vec{u} + (\nabla \vec{u})^\top - \frac{1}{\mu} \tau \right) : \sigma \, dV - \iint_{\partial \Omega_F} \vec{u} \cdot \sigma \hat{n} \, dA, \tag{34}$$

by symmetry of σ. Since σ is chosen arbitrarily on $\partial \Omega_F$ and Ω_F, we get

$$\nabla \vec{u} + (\nabla \vec{u})^\top - \frac{1}{\mu} \tau = 0 \quad \text{on } \Omega_F \tag{35}$$

$$\vec{u} = \vec{0} \quad \text{on } \partial \Omega_F, \tag{36}$$

which are exactly the constraints (5c) and (18) respectively. □

3 SCENE CONFIGURATIONS

In the following table we show the configuration parameters for various animation results.

Name	Frames	Particle Count	Particle Separation	Grid Scale	Viscosity (Pa s)	Surface Tension (N/m)
Coiling at different viscosities	600	3620 to 907654	0.001	1.5	25 and 100	0
Piling Armadillos	720	823580 per armadillo	0.0015	2	200	0
Colliding Characters	600	422603	0.0025	2	100	0
Collapsing Liquid Squab	960	1148019	0.00025	1.5	0.9	0.5
Molasses Coil	300	4785 to 1008594	0.001	1.5	35	0.5
Ball Through Goop	320	395089	0.002	1.5	3	0
Conveyor Belt	600	8836 to 2690823	0.000263	1.5	50	0.2
Variable Viscosity Block	300	805128	0.0013	1.5	0.0256 - 162	0
Caramel on Wafer	600	13157 to 2175060	0.00025	2	10	0
Surface Tension Comparisons	120	58713	0.001	1.5	50	20
Method Comparisons	600	5045 to 1889959	0.001	1.5	50	0

Table 2. Scene configurations for each animation in the video. Particle separation is a Houdini parameter indicating distance between particles. Grid scale is another Houdini parameter computed as $\Delta x/$(particle separation), which gives an approximate indication of the number of particles per cell along an axis.