Supplemental to: “An Adaptive Variational Finite Difference Framework for Efficient Symmetric Octree Viscosity"

1 DETAILS OF CONVERGENCE STUDY

As noted in the main paper, to evaluate the spatial accuracy of our octree discretization, we constructed analytical tests representing a single time step of the viscosity equations on closed box domains in 2D and 3D with a spatially varying viscosity function. This section describes those test cases and, briefly, their derivation.

1.1 2D Test Case

Consider the square domain $[0, \pi]^2$. Our methodology will be to choose an end-of-step velocity field at time $t = \Delta t$ that satisfies the no-slip boundary condition $\mathbf{u} = \mathbf{0}$ on the domain boundaries. There are many valid options, and we tested a few. The results in the paper are based on the following products of sinusoids,

$$u(x, y, t = \Delta t) = \sin(x)\sin(y),$$
$$v(x, y, t = \Delta t) = \sin(x)\sin(y).$$

Next, we choose a spatially varying function to determine the viscosity coefficient; again, many options are possible, provided the function remains non-negative over the domain. We choose

$$\mu(x) = \frac{x}{\pi} + \frac{1}{2}.$$

Finally, using analytical differentiation, we can take one step *backwards* in time to $t = 0$ following the viscosity equations, (semi-)discretized in time as in our method,

$$\mathbf{u}^* = \mathbf{u} - \Delta t \frac{1}{\rho} \nabla \cdot \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^\intercal). \tag{1}$$

This yields the initial conditions for our problem:

$$u(x, y, t = 0) = \sin(x)\sin(y) - \Delta t \left(\frac{2}{\pi}\cos(x)\sin(y) + (\cos(x + y) - 2\sin(x)\sin(y)) \left(\frac{x}{\pi} + \frac{1}{2} \right) \right),$$
$$v(x, y, t = 0) = \sin(x)\sin(y) - \Delta t \left((\cos(x)\cos(y) - 3\sin(x)\sin(y)) \left(\frac{x}{\pi} + \frac{1}{2} \right) + \frac{1}{\pi}\sin(x + y) \right).$$

We used a timestep of $\Delta t = 1$. The initial refinement pattern of the 2D domain is shown in Figure 1.

Author’s address:

2019. Manuscript submitted to ACM

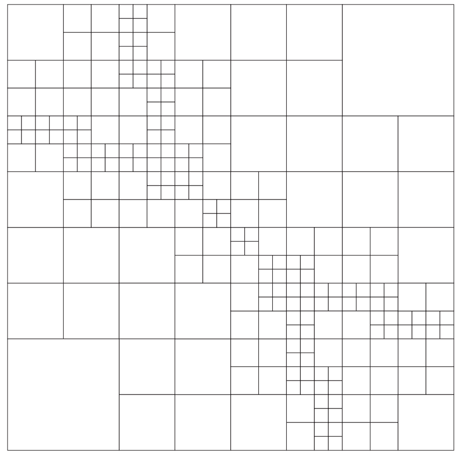

Fig. 1. The initial tree refinement pattern in 2D.

1.2 3D Test

Our 3D problem follows the same general approach, but over the domain $[0, \pi]^3$. The final velocity field is

$$u(x, y, z, t = \Delta t) = \sin(x)\sin(y)\sin(z),$$
$$v(x, y, z, t = \Delta t) = \sin(x)\sin(y)\sin(z),$$
$$w(x, y, z, t = \Delta t) = \sin(x)\sin(y)\sin(z).$$

The viscosity function is

$$\mu(x, y) = \frac{x}{\pi} + y + 1.$$

The resulting initial velocity field is given by

$$\begin{aligned}
u(x, y, z, t = 0) = {} & \sin(x)\sin(y)\sin(z)(1 + 2\Delta t\mu(x, y)) - \Delta t(\sin(z)(\cos(x)\sin(y) + \sin(x)\cos(y) + \frac{2}{\pi}\cos(x)\sin(y)) + \\
& \mu(x, y)(\cos(x + y)\sin(z) + \cos(x + z)\sin(y))), \\
v(x, y, z, t = 0) = {} & \sin(x)\sin(y)\sin(z)(1 + 2\Delta t\mu(x, y)) - \Delta t(2\sin(x)\cos(y)\sin(z) + \frac{1}{\pi}\sin(x + y)\sin(z) + \\
& \mu(x, y)(\cos(x + y)\sin(z) + \sin(x)\cos(y + z))) \\
w(x, y, z, t = 0) = {} & \sin(x)\sin(y)\sin(z)(1 + 2\Delta t\mu(x, y)) - \Delta t(\sin(x)(\cos(y)\sin(z) + \sin(y)\cos(z)) + \frac{1}{\pi}\sin(x + z)\sin(y) + \\
& \mu(x, y)(\cos(x + z)\sin(y) + \sin(x)\cos(y + z)))
\end{aligned}$$

We used a timestep of $\Delta t = 1$. The initial refinement pattern of the domain (Figure 2) is generated from a union of two implicit spheres placed at $(0, 0, 0)$ and (π, π, π), with radii of $\frac{\sqrt{3}}{2}\pi$. Grid cells at the finest level are activated if their cell centers are within a distance $\frac{\Delta x}{2}$ of the surface of the spheres. We then proceed to grade the octree using our face-grading criteria. We chose this refinement pattern because it exercises all the relevant octree stencils.

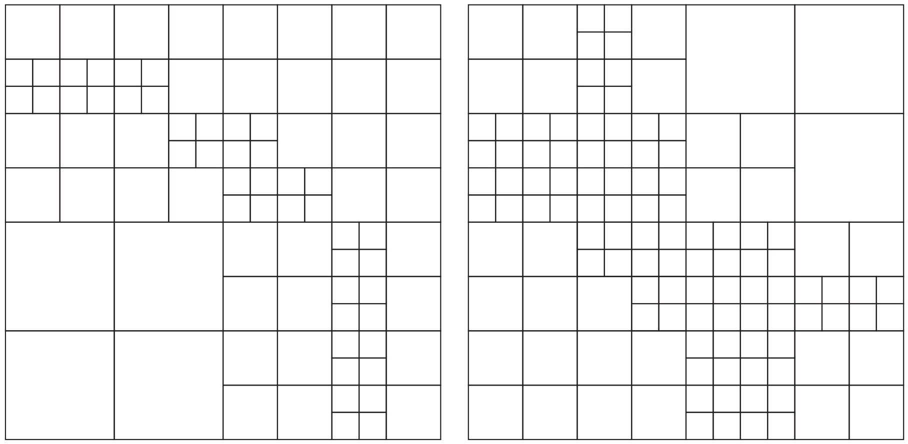

Fig. 2. Two slices of the initial tree refinement pattern in 3D. Both slices are on a yz-plane, at $x = 0$ and $x = \frac{\pi}{2}$, respectively.

Grid	$\|u - u^h\|_\infty$	Order	$\|v - v^h\|_\infty$	Order
32^2	1.8929 E -2		2.0159 E -2	
64^2	1.1169 E -2	0.76	1.1377 E -2	0.83
128^2	5.6016 E -3	1.00	5.4032 E -3	1.07
256^2	2.5636 E -3	1.13	2.5145 E -3	1.10
512^2	1.2273 E -3	1.06	1.1995 E -3	1.07
1024^2	6.2867 E -4	0.97	6.2013 E -4	0.95

Table 1. Convergence of 2D discretization in L_∞ on a quadtree grid, exhibiting approximately first order behavior.

Grid	$\|u - u^h\|_1$	Order	$\|v - v^h\|_1$	Order
32^2	4.2504 E -2		4.5429 E -2	
64^2	1.3526 E -2	1.65	1.3805 E -2	1.72
128^2	3.6910 E -3	1.87	3.7281 E -3	1.89
256^2	9.4728 E -4	1.96	9.5028 E -4	1.97
512^2	2.3910 E -4	1.99	2.3933 E -4	1.99
1024^2	6.0214 E -5	1.99	6.0514 E -5	1.98

Table 2. Convergence of 2D discretization in L_1 on a quadtree grid, exhibiting approximately second order behavior.

2 DETAILED NUMERICAL RESULTS

Beginning from the initial refinement pattern, we recursively subdivided every cell, evaluating the L_∞ and L_1 velocity errors at each step. The 2D results are presented in Table 2, and 3D results for sloped and enhanced gradients in Tables 5 and 6, respectively; the corresponding convergence plots are given in the main paper. The grid size listed in these tables corresponds to the finest virtual grid size of the tree structure (i.e. the effective grid resolution).

Grid	$\|u - u^h\|_\infty$	Order	$\|v - v^h\|_\infty$	Order	$\|w - w^h\|_\infty$	Order
16^3	4.0994 E-2		3.6799 E-2		4.1257 E-2	
32^3	2.2438 E-2	0.87	2.2182 E-2	0.73	2.2709 E-2	0.86
64^3	1.2464 E-2	0.85	1.2338 E-2	0.85	1.2538 E-2	0.86
128^3	6.6379 E-3	0.91	6.5528 E-3	0.91	6.6554 E-3	0.91
256^3	3.4793 E-3	0.93	3.4317 E-3	0.93	3.4818 E-3	0.93
512^3	1.8087 E-3	0.94	1.7859 E-3	0.94	1.8082 E-3	0.95

Table 3. Convergence of 3D octree discretization in L_∞ with *sloped* gradients, exhibiting approximately first order behavior.

Grid	$\|u - u^h\|_\infty$	Order	$\|v - v^h\|_\infty$	Order	$\|w - w^h\|_\infty$	Order
16^3	3.4294 E-2		2.8760 E -2		3.4065 E -2	
32^3	2.5290 E-2	0.44	2.6265 E -2	0.13	2.4756 E -2	0.46
64^3	1.0322 E-2	1.30	1.0698 E -2	1.30	1.0301 E -2	1.27
128^3	4.1865 E-3	1.30	4.3085 E -3	1.31	4.2229 E -3	1.29
256^3	1.8039 E-3	1.21	1.8412 E -3	1.23	1.8227 E -3	1.21
512^3	8.2871 E-4	1.13	8.3664 E -4	1.14	8.3273 E -4	1.13

Table 4. Convergence of 3D octree discretization in L_∞ with *enhanced gradients*, exhibiting approximately first order behavior.

Grid	$\|u - u^h\|_1$	Order	$\|v - v^h\|_1$	Order	$\|w - w^h\|_1$	Order
16^3	2.2728 E -1		2.2728 E -1		2.2846 E -1	
32^3	8.3192 E -2	1.45	8.4643 E -2	1.43	8.3151 E -2	1.46
64^3	3.4742 E -2	1.26	3.5813 E -2	1.24	3.4676 E -2	1.26
128^3	1.7170 E -2	1.02	1.7768 E -2	1.01	1.7126 E -2	1.02
256^3	8.6849 E -3	0.98	8.9916 E -3	.98	8.6589 E -3	.98
512^3	4.1313 E -3	1.07	4.2207 E -3	1.09	4.1245 E -3	1.07

Table 5. Convergence of 3D octree discretization in L_1 with *sloped* gradients, exhibiting approximately first order behavior.

Grid	$\|u - u^h\|_1$	Order	$\|v - v^h\|_1$	Order	$\|w - w^h\|_1$	Order
16^3	2.0364 E-1		2.0128 E -1		2.0504 E -1	
32^3	5.2047 E-2	1.97	5.1960 E -2	1.95	5.2332 E -2	1.97
64^3	1.2568 E-2	2.05	1.2622 E -2	2.04	1.2617 E -2	2.05
128^3	3.0287 E-3	2.05	3.0522 E -3	2.05	3.0355 E -3	2.06
256^3	7.1526 E-4	2.08	7.2298 E -4	2.08	7.1639 E -4	2.08
512^3	1.4614 E-4	2.29	1.4797 E-4	2.29	1.4630 E -4	2.29

Table 6. Convergence of 3D octree discretization in L_1 with *enhanced gradients*, exhibiting approximately *second* order behavior.