Supplementary Material for *Differentiable Curl-Noise: Boundary-Respecting Procedural Incompressible Flows Without Discontinuities*

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1 MOTIVATION: CONTINUITY OF ψ'

Proposition A: Let $\boldsymbol{x}$ be the query point and let $\hat{d}_{s_1}(\boldsymbol{x}), \hat{d}_{s_2}(\boldsymbol{x}), \cdots, \hat{d}_{s_n}(\boldsymbol{x})$ be the sorted LSE distance function defined in Section 4.2. Assume the user-defined potential $\psi(\boldsymbol{x})$ and $cp(\boldsymbol{x})$ are at least C^1 differentiable. If $d_0(\boldsymbol{x}) = \min\{\hat{d}_{s_2}(\boldsymbol{x}), d_0\}$, then

$$\text{Multiplicative Ramping:} \quad \psi'(\boldsymbol{x}) = \alpha(\boldsymbol{x})\psi(\boldsymbol{x}) + (1-\alpha(\boldsymbol{x}))\psi_g(\boldsymbol{x}), \tag{1}$$

and

$$\textit{Additive}\text{ Ramping:} \quad \psi'(\boldsymbol{x}) = \psi(\boldsymbol{x}) + (\psi_g(\boldsymbol{x}) - \psi(cp(\boldsymbol{x})))(1-\alpha(\boldsymbol{x})) \tag{2}$$

are both continuous, where

$$\alpha(\boldsymbol{x}) = ramp\left(\frac{\overline{d}^1(\boldsymbol{x})}{d_0(\boldsymbol{x})}\right), \quad \text{and} \quad ramp(r) = \begin{cases} 1 & \text{if } r \geq 1 \\ \frac{15}{8}r - \frac{10}{8}r^3 + \frac{3}{8}r^5 & \text{if } -1 < r < 1 \\ -1 & \text{if } r \leq -1 \end{cases}.$$

Proof. We first make the following five observations:
Observation 1: Directly following their definitions, we know $ramp(r) \in C^1$ and

$$d(\boldsymbol{x}) \leq \overline{d}^1(\boldsymbol{x}) = \frac{\sum_{i=1}^n \hat{d}_i(\boldsymbol{x}) \exp(-b\hat{d}_i(\boldsymbol{x}))}{\sum_{i=1}^n \exp(-b\hat{d}_i(\boldsymbol{x}))} = \frac{\sum_{i=1}^n \hat{d}_{s_i}(\boldsymbol{x}) \exp(-b\hat{d}_{s_i}(\boldsymbol{x}))}{\sum_{i=1}^n \exp(-b\hat{d}_{s_i}(\boldsymbol{x}))} \in C^1.$$

Observation 2: For all $\boldsymbol{x}$ located on the equidistant curve of the closest two obstacles (i.e., when $d(\boldsymbol{x}) = \hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$), we have $1 - \alpha(\boldsymbol{x}) = 0$.
Proof of observation 2: Since $\hat{d}_{s_1}(\boldsymbol{x}) \leq \hat{d}_{s_2}(\boldsymbol{x})$, there are only three possible cases regarding the order of $\hat{d}_{s_1}(\boldsymbol{x}), \hat{d}_{s_2}(\boldsymbol{x})$ and d_0. We analyze them one by one.
Case 1: If $d_0 \leq \hat{d}_{s_1}(\boldsymbol{x}) \leq \hat{d}_{s_2}(\boldsymbol{x})$, then $d_0(\boldsymbol{x}) = \min\{\hat{d}_{s_2}(\boldsymbol{x}), d_0\} = d_0$ and $\frac{\overline{d}^1(\boldsymbol{x})}{d_0(\boldsymbol{x})} \geq \frac{d(\boldsymbol{x})}{d_0} \geq 1$.
Case 2: If $\hat{d}_{s_1}(\boldsymbol{x}) \leq d_0 \leq \hat{d}_{s_2}(\boldsymbol{x})$, then $d_0(\boldsymbol{x}) = \min\{\hat{d}_{s_2}(\boldsymbol{x}), d_0\} = d_0$. When $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$, we must have $\hat{d}_{s_1}(\boldsymbol{x}) = d_0 = \hat{d}_{s_2}(\boldsymbol{x})$. So, $\frac{\overline{d}^1(\boldsymbol{x})}{d_0(\boldsymbol{x})} \geq \frac{d(\boldsymbol{x})}{d_0} = \frac{d(\boldsymbol{x})}{\hat{d}_{s_2}(\boldsymbol{x})} \geq 1$.
Case 3: If $\hat{d}_{s_1}(\boldsymbol{x}) \leq \hat{d}_{s_2}(\boldsymbol{x}) \leq d_0$, then $d_0(\boldsymbol{x}) = \min\{\hat{d}_{s_2}(\boldsymbol{x}), d_0\} = \hat{d}_{s_2}(\boldsymbol{x})$. When $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$, we

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have $\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})} \geq \frac{d(\boldsymbol{x})}{\hat{d}_{s_2}(\boldsymbol{x})} = 1$.

Notice that, in all three cases, we showed that $\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})} \geq 1$ when $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$. So,

$$\alpha(\boldsymbol{x}) = ramp\left(\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})}\right) = 1 \implies 1 - \alpha(\boldsymbol{x}) = 0.$$

□

Observation 3: Eq. 1 and 2 are both of the general form

$$\psi'(\boldsymbol{x}) = \psi(\boldsymbol{x}) + (\psi_g(\boldsymbol{x}) - \psi(\boldsymbol{m}(\boldsymbol{x})))(1 - \alpha(\boldsymbol{x})). \tag{3}$$

We can rewrite Eq. 1 as

$$\psi'(\boldsymbol{x}) = \alpha(\boldsymbol{x})\psi(\boldsymbol{x}) + (1 - \alpha(\boldsymbol{x}))\psi_g(\boldsymbol{x}) = \psi(\boldsymbol{x}) + (\psi_g(\boldsymbol{x}) - \psi(\boldsymbol{x}))(1 - \alpha(\boldsymbol{x})),$$

where $\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{x}$. It is clear that $\boldsymbol{m}(\boldsymbol{x}) = cp(\boldsymbol{x})$ for Eq. 2.

Observation 4: $\hat{d}_{s_2}(\boldsymbol{x})$ is a strictly positive C^0 function. It is not differentiable over $\mathcal{D}$ wherever the second closest object to $\boldsymbol{x}$ changes. It is strictly positive since $\boldsymbol{x}$ is in the exterior of all objects. Therefore, $d_0(\boldsymbol{x})$ is a strictly positive C^0 function.

Observation 5: The user defined potential evaluated at the geometric center of the closest object to $\boldsymbol{x}$, i.e., $\psi_g(\boldsymbol{x})$, is a Heaviside step function. It is discontinuous wherever the closest object to $\boldsymbol{x}$ changes. That is, all discontinuities of $\psi_g(\boldsymbol{x})$ locate on the equidistant curve between the closest and second closest object to $\boldsymbol{x}$, where $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$. Moreover, as all the objects are defined in a finite and bounded domain $\mathcal{D}$, then ψ_g is bounded. Therefore, $\exists M \geq 0$ such that $|\psi_g| \leq M$.

Under the assumption that $\psi(\boldsymbol{x}), \boldsymbol{m}(\boldsymbol{x}) \in C^1$, and relying on the preceding Observations, it suffices to show that Eq. 3 is continuous at all $\boldsymbol{x}$ that satisfy $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$.

Take arbitrary $\epsilon > 0$. Set $\epsilon_0 = \frac{1}{3}\epsilon$. Let $\boldsymbol{x_0}$ be any query point such that $\hat{d}_{s_1}(\boldsymbol{x_0}) = \hat{d}_{s_2}(\boldsymbol{x_0})$.

Since $\psi(\boldsymbol{x}) \in C^1$, then $\forall \epsilon_0 > 0, \exists \delta_1 > 0$ such that

$$|\boldsymbol{x} - \boldsymbol{x_0}| < \delta_1 \implies |\psi(\boldsymbol{x}) - \psi(\boldsymbol{x_0})| < \epsilon_0. \tag{4}$$

By *Observation 1* and *Observation 4*, we know that $\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})}$ is continuous, and so is $ramp(\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})})$. By construction, $\alpha(\boldsymbol{x}) = ramp(\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})})$, and $1 - \alpha(\boldsymbol{x})$ are both continuous. Then $\forall \frac{\epsilon_0}{M} > 0, \exists \delta_2 > 0$ such that

$$|\boldsymbol{x} - \boldsymbol{x_0}| < \delta_2 \implies |(1 - \alpha(\boldsymbol{x})) - \underbrace{\cancelto{0}{(1 - \alpha(\boldsymbol{x_0}))}}_{\text{by } \textit{Observation 2}}| = |1 - \alpha(\boldsymbol{x})| < \frac{\epsilon_0}{M}. \tag{5}$$

By assumption, as $\psi(\boldsymbol{x}), \boldsymbol{m}(\boldsymbol{x}) \in C^1$, then $\psi(\boldsymbol{m}(\boldsymbol{x}))(1 - \alpha(\boldsymbol{x})) \in C^1$. Then $\forall \epsilon_0 > 0, \exists \delta_2 > 0$ such that

$$\begin{aligned} |\boldsymbol{x} - \boldsymbol{x_0}| < \delta_3 &\implies |\psi(\boldsymbol{m}(\boldsymbol{x}))(1 - \alpha(\boldsymbol{x})) - \psi(\boldsymbol{m}(\boldsymbol{x_0}))\underbrace{\cancelto{0}{(1 - \alpha(\boldsymbol{x_0}))}}_{\text{by } \textit{Observation 2}}| \\ &= |\psi(\boldsymbol{m}(\boldsymbol{x}))(1 - \alpha(\boldsymbol{x}))| < \epsilon_0. \end{aligned} \tag{6}$$

Take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, and assume $|\boldsymbol{x} - \boldsymbol{x_0}| < \delta$, then

$$
\begin{aligned}
&|\psi'(\boldsymbol{x}) - \psi'(\boldsymbol{x_0})| \\
&= |\psi(\boldsymbol{x}) + (\psi_g(\boldsymbol{x}) - \psi(\boldsymbol{m}(\boldsymbol{x})))(1-\alpha(\boldsymbol{x})) - \psi(\boldsymbol{x_0}) - (\psi_g(\boldsymbol{x_0}) - \psi(\boldsymbol{m}(\boldsymbol{x_0}))) \underbrace{(1-\alpha(\boldsymbol{x_0}))}_{\text{by } Observation\ 2 \ \to 0}| \\
&= |\psi(\boldsymbol{x}) + (\psi_g(\boldsymbol{x}) - \psi(\boldsymbol{m}(\boldsymbol{x})))(1-\alpha(\boldsymbol{x})) - \psi(\boldsymbol{x_0})| \\
&\leq |\psi(\boldsymbol{x}) - \psi(\boldsymbol{x_0})| + |\psi_g(\boldsymbol{x})||1-\alpha(\boldsymbol{x})| + |\psi(\boldsymbol{m}(\boldsymbol{x}))(1-\alpha(\boldsymbol{x})| \\
&\leq \epsilon_0 + M\frac{\epsilon_0}{M} + \epsilon_0 \qquad \text{by Eq. 4, 5, 6, and } Observation\ 5 \\
&= 3\epsilon_0 = \epsilon.
\end{aligned}
$$

Therefore, we showed that any $\psi'(\boldsymbol{x})$ having the form of Eq. 3 is continuous. Hence, by *Observation 3*, Eq. 1 and 2 are both continuous given that $\psi(\boldsymbol{x}), \boldsymbol{m}(\boldsymbol{x}) \in C^1$. □

2 ALGORITHM 1: DIFFERENTIABILITY OF ψ'

Proposition B: Let $\boldsymbol{x}$ be the query point and let $\hat{d}_{s_1}(\boldsymbol{x}), \hat{d}_{s_2}(\boldsymbol{x}), \cdots, \hat{d}_{s_n}(\boldsymbol{x})$ be the sorted LSE distance functions defined in Section 4.2. Assume the user-defined potential $\psi(\boldsymbol{x})$ and $cp(\boldsymbol{x})$ are at least C^1 differentiable. If $d_0(\boldsymbol{x})$ is approximated using the C^1 construction in Algorithm 1, then

$$\text{Eq. 1:} \quad \psi'(\boldsymbol{x}) = \alpha(\boldsymbol{x})\psi(\boldsymbol{x}) + (1-\alpha(\boldsymbol{x}))\psi_g(\boldsymbol{x}), \quad \text{and}$$

$$\text{Eq. 2:} \quad \psi'(\boldsymbol{x}) = \psi(\boldsymbol{x}) + (\psi_g(\boldsymbol{x}) - \psi(cp(\boldsymbol{x})))(1-\alpha(\boldsymbol{x}))$$

are both continuously differentiable, where

$$\alpha(\boldsymbol{x}) = ramp\left(\frac{\overline{d}^1(\boldsymbol{x})}{d_0(\boldsymbol{x})}\right), \quad \text{and} \quad ramp(r) = \begin{cases} 1 & \text{if } r \geq 1 \\ \frac{15}{8}r - \frac{10}{8}r^3 + \frac{3}{8}r^5 & \text{if } -1 < r < 1 \\ -1 & \text{if } r \leq -1 \end{cases}.$$

Proof. Notice that *Observations 1, 3, 5* in Proposition A directly hold true under this setup, and *Observation 4* is also true when a is large enough. Following Proposition A and based on the fact that $d_0(\boldsymbol{x})$ is now C^1, we know that $\psi'(\boldsymbol{x})$ can be continuous but not differentiable only when the closest object to $\boldsymbol{x}$ changes. For all other query points, $\psi'(\boldsymbol{x})$ is C^1 since $\psi_g(\boldsymbol{x})$ does not touch the discontinuity and thus all components are C^1. So, we only need to prove

$$\psi'(\boldsymbol{x}) = \psi(\boldsymbol{x}) + (\psi_g(\boldsymbol{x}) - \psi(\boldsymbol{m}(\boldsymbol{x})))(1-\alpha(\boldsymbol{x})),$$

is continuously differentiable, where we further assume $\boldsymbol{m}(\boldsymbol{x})$ is at least C^1. Therefore, it suffices to show that both $\frac{\partial \psi'(\boldsymbol{x})}{\partial x}$ and $\frac{\partial \psi'(\boldsymbol{x})}{\partial y}$ are continuous for all $\boldsymbol{x}$ such that $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$. In this proof, we will show that $\frac{\partial \psi'(\boldsymbol{x})}{\partial x}$ is continuous; the proof for differentiating y is similar.

Now, we want to show *Observation 2* in Proposition A is still true when $d_0(x)$ is calculated using Algorithm 1.

Observation 2: For all $\boldsymbol{x}$ located on the equidistant curve of the closest two obstacles, (i.e., when $d(\boldsymbol{x}) = \hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$), we have $1 - \alpha(\boldsymbol{x}) = 0$ and $\frac{\partial \alpha}{\partial x}(\boldsymbol{x}) = \frac{\partial \alpha}{\partial y}(\boldsymbol{x}) = 0$.

Proof of observation 2: From Section 4.2.1, we know that our differentiable $d_0(\boldsymbol{x})$ underestimates $\hat{d}_{s_2}(\boldsymbol{x})$ and overestimates $d(\boldsymbol{x}) = \hat{d}_{s_1}(\boldsymbol{x})$. Since $\hat{d}_{s_1}(\boldsymbol{x}) \leq \hat{d}_{s_2}(\boldsymbol{x})$, there are still three possible cases regarding the ordering of $\hat{d}_{s_1}(\boldsymbol{x}), \hat{d}_{s_2}(\boldsymbol{x})$ and d_0. We analyze them one by one.

Case 1: $d_0 \leq \hat{d}_{s_1}(\boldsymbol{x}) \leq \hat{d}_{s_2}(\boldsymbol{x})$. When $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$, we have $d_0 \leq \hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$. No matter which one of $\tilde{d}^2(\boldsymbol{x})$ or d_0 is smaller, we will always have $\overline{d}^1(\boldsymbol{x}) \geq d_0(\boldsymbol{x})$ since $\overline{d}^1(\boldsymbol{x}) \geq d(\boldsymbol{x}) = \hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x}) \geq \tilde{d}^2(\boldsymbol{x}) \geq d_0(\boldsymbol{x})$, and $d_0 \geq d_0(\boldsymbol{x})$ (LSE approximation of $\min\{\cdot\}$ is an underestimator).

Therefore $\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})} \geq 1$.

Case 2: $\hat{d}_{s_1}(\boldsymbol{x}) \leq d_0 \leq \hat{d}_{s_2}(\boldsymbol{x})$. When $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$, we must have $\hat{d}_{s_1}(\boldsymbol{x}) = d_0 = \hat{d}_{s_2}(\boldsymbol{x})$. It follows that $\overline{d}^{1}(\boldsymbol{x}) \geq d(\boldsymbol{x}) = \hat{d}_{s_1}(\boldsymbol{x}) = d_0 = \hat{d}_{s_2}(\boldsymbol{x}) \geq \tilde{d}^2(\boldsymbol{x}) \geq d_0(\boldsymbol{x})$, which implies $\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})} \geq 1$.

Case 3: $\hat{d}_{s_1}(\boldsymbol{x}) \leq \hat{d}_{s_2}(\boldsymbol{x}) \leq d_0$. When $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$, we have $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x}) \leq d_0$. Then $\overline{d}^{1}(\boldsymbol{x}) \geq d(\boldsymbol{x}) = \hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x}) \geq \tilde{d}^2(\boldsymbol{x}) \geq d_0(\boldsymbol{x})$. Hence, $\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})} \geq 1$.

Notice that in all three cases, we showed that $\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})} \geq 1$, given that $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$. So,

$$\alpha(\boldsymbol{x}) = ramp\left(\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})}\right) = 1 \implies 1 - \alpha(\boldsymbol{x}) = 0, \quad \text{and} \tag{7}$$

$$\nabla\alpha(\boldsymbol{x}) = \underbrace{ramp'\left(\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})}\right)}_{=0} \cdot \nabla\left(\frac{\overline{d}^{1}(\boldsymbol{x})}{d_0(\boldsymbol{x})}\right) = 0 \implies \frac{\partial\alpha}{\partial x} = \frac{\partial\alpha}{\partial y} = 0 \tag{8}$$

□

We make one last observation based on the discussion in Section 4.2.1.

Observation 6: $\forall \boldsymbol{x} \in \mathcal{D}$, $\nabla\psi_g(\boldsymbol{x})(1 - \alpha(\boldsymbol{x})) = 0$.

Proof of Observation 6: In Section 4.2.1, we mentioned that the first order partial derivatives of $\psi_g(\boldsymbol{x})$ are linear combinations of Dirac delta functions, where the nonzero values occur for $\boldsymbol{x}$ on the equidistant curves between the two closest objects to $\boldsymbol{x}$. So, if $\boldsymbol{x}$ is not on any equidistant curve causing a change of the closest object, then $\nabla\psi_g(\boldsymbol{x}) = 0$. Otherwise, as we proved in *Observation 2* that $(1 - \alpha(\boldsymbol{x})) = 0$. Therefore, we will always have $\nabla\psi_g(\boldsymbol{x})(1 - \alpha(\boldsymbol{x})) = 0$. □

Let $\boldsymbol{x_0}$ be any query point such that $\hat{d}_{s_1}(\boldsymbol{x_0}) = \hat{d}_{s_2}(\boldsymbol{x_0})$.

Here, based on the general form (Eq. 3), we write down the partial derivative of $\psi'(\boldsymbol{x})$ with respect to x.

$$\frac{\partial\psi'}{\partial x}(\boldsymbol{x}) = \frac{\partial\psi}{\partial x}(\boldsymbol{x}) + \left(\frac{\partial\psi_g}{\partial x}(\boldsymbol{x}) + \frac{\partial\psi(\boldsymbol{m}(\boldsymbol{x}))}{\partial x}\right)(1 - \alpha(\boldsymbol{x})) - (\psi_g(\boldsymbol{x}) + \psi(\boldsymbol{m}(\boldsymbol{x})))\frac{\partial\alpha}{\partial x}(\boldsymbol{x}) \tag{9}$$

When $\boldsymbol{x} = \boldsymbol{x_0}$, then Eq. 9 is

$$\begin{aligned}\frac{\partial\psi'}{\partial x}(\boldsymbol{x_0}) &= \frac{\partial\psi}{\partial x}(\boldsymbol{x_0}) + \left(\frac{\partial\psi_g}{\partial x}(\boldsymbol{x_0}) + \frac{\partial\psi(\boldsymbol{m}(\boldsymbol{x_0}))}{\partial x}\right)(1 - \alpha(\boldsymbol{x_0})) - (\psi_g(\boldsymbol{x_0}) + \psi(\boldsymbol{m}(\boldsymbol{x_0})))\frac{\partial\alpha}{\partial x}(\boldsymbol{x_0}) \\ &= \frac{\partial\psi}{\partial x}(\boldsymbol{x_0}) \quad \text{By } \textit{Observation 2}, 1 - \alpha(\boldsymbol{x_0}) = 0 \text{ and } \frac{\partial\alpha}{\partial x}(\boldsymbol{x_0}) = 0.\end{aligned} \tag{10}$$

Take arbitrary $\epsilon > 0$. Set $\epsilon_0 = \frac{1}{4}\epsilon$.

Since $\psi(\boldsymbol{x}) \in C^1$, then $\forall\epsilon_0 > 0, \exists\delta_1 > 0$ such that

$$|\boldsymbol{x} - \boldsymbol{x_0}| < \delta_1 \implies |\frac{\partial\psi}{\partial x}(\boldsymbol{x}) - \frac{\partial\psi}{\partial x}(\boldsymbol{x_0})| < \epsilon_0. \tag{11}$$

Since $\alpha(\boldsymbol{x}) \in C^1$, then $\forall\frac{\epsilon_0}{M} > 0$, $\exists\delta_2 > 0$, where M is the upper bound of $\psi_g(\boldsymbol{x})$ in *Observation 5*, such that

$$|\boldsymbol{x} - \boldsymbol{x_0}| < \delta_2 \implies \left|\frac{\partial\alpha}{\partial x}(\boldsymbol{x}) - \underbrace{\cancelto{0}{\frac{\partial\alpha}{\partial x}(\boldsymbol{x_0})}}_{\text{by } \textit{Observation 2}}\right| = \left|\frac{\partial\alpha}{\partial x}(\boldsymbol{x})\right| < \frac{\epsilon_0}{M}. \tag{12}$$

By assumption, as $\psi(\boldsymbol{x}), \boldsymbol{m}(\boldsymbol{x}) \in C^1$, then $\frac{\partial \psi(\boldsymbol{m}(\boldsymbol{x}))}{\partial x}(1-\alpha(\boldsymbol{x}))$ is continuous. Then $\forall \epsilon_0 > 0, \exists \delta_3 > 0$ such that

$$|\boldsymbol{x} - \boldsymbol{x_0}| < \delta_3 \implies \left| \frac{\partial \psi(\boldsymbol{m}(\boldsymbol{x}))}{\partial x}(1-\alpha(\boldsymbol{x})) - \frac{\partial \psi(\boldsymbol{m}(\boldsymbol{x_0}))}{\partial x} \underbrace{\cancelto{0}{(1-\alpha(\boldsymbol{x_0}))}}_{\text{by } \textit{Observation 2}} \right|$$

$$= \left| \frac{\partial \psi(\boldsymbol{m}(\boldsymbol{x}))}{\partial x}(1-\alpha(\boldsymbol{x})) \right| < \epsilon_0. \tag{13}$$

For a similar reason, $\psi(\boldsymbol{m}(\boldsymbol{x}))\frac{\partial \alpha}{\partial x}(\boldsymbol{x})$ is also continuous. So, $\forall \epsilon_0 > 0, \exists \delta_4 > 0$ such that

$$|\boldsymbol{x} - \boldsymbol{x_0}| < \delta_3 \implies \left| \psi(\boldsymbol{m}(\boldsymbol{x}))\frac{\partial \alpha}{\partial x}(\boldsymbol{x}) - \psi(\boldsymbol{m}(\boldsymbol{x_0})) \underbrace{\cancelto{0}{\frac{\partial \alpha}{\partial x}(\boldsymbol{x_0})}}_{\text{by } \textit{Observation 2}} \right|$$

$$= \left| \psi(\boldsymbol{m}(\boldsymbol{x}))\frac{\partial \alpha}{\partial x}(\boldsymbol{x}) \right| < \epsilon_0. \tag{14}$$

Take $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, and assume $|\boldsymbol{x} - \boldsymbol{x_0}| < \delta$, then

$$\begin{aligned}
&\left| \frac{\partial \psi'}{\partial x}(\boldsymbol{x}) - \frac{\partial \psi'}{\partial x}(\boldsymbol{x_0}) \right| \\
&= \left| \frac{\partial \psi}{\partial x}(\boldsymbol{x}) + \left(\frac{\partial \psi_g(\boldsymbol{x})}{\partial x} + \frac{\partial \psi(\boldsymbol{m}(\boldsymbol{x}))}{\partial x} \right)(1-\alpha(\boldsymbol{x})) - (\psi_g(\boldsymbol{x}) + \psi(\boldsymbol{m}(\boldsymbol{x})))\frac{\partial \alpha}{\partial x}(\boldsymbol{x}) - \frac{\partial \psi}{\partial x}(\boldsymbol{x_0}) \right| \\
&\qquad \text{Sub in Eq. 9 and Eq. 10} \\
&\le \left| \frac{\partial \psi}{\partial x}(\boldsymbol{x}) - \frac{\partial \psi}{\partial x}(\boldsymbol{x_0}) \right| + \underbrace{\cancelto{0}{\left| \frac{\partial \psi_g(\boldsymbol{x})}{\partial x}(1-\alpha(\boldsymbol{x})) \right|}}_{\text{by } \textit{Observation 6}} + \left| \frac{\partial \psi(\boldsymbol{m}(\boldsymbol{x}))}{\partial x}(1-\alpha(\boldsymbol{x})) \right| + \\
&\qquad \left| (\psi_g(\boldsymbol{x}) + \psi(\boldsymbol{m}(\boldsymbol{x})))\frac{\partial \alpha}{\partial x}(\boldsymbol{x}) \right| \\
&\le \left| \frac{\partial \psi}{\partial x}(\boldsymbol{x}) - \frac{\partial \psi}{\partial x}(\boldsymbol{x_0}) \right| + \left| \frac{\partial \psi(\boldsymbol{m}(\boldsymbol{x}))}{\partial x}(1-\alpha(\boldsymbol{x})) \right| + |\psi_g(\boldsymbol{x})| \left| \frac{\partial \alpha}{\partial x}(\boldsymbol{x}) \right| + \left| \psi(\boldsymbol{m}(\boldsymbol{x}))\frac{\partial \alpha}{\partial x}(\boldsymbol{x}) \right| \\
&\le \epsilon_0 + \epsilon_0 + M\frac{\epsilon_0}{M} + \epsilon_0 \quad \text{By Eq. 11, 12, 13, 14 and } \textit{Observation 5} \\
&= 4\epsilon_0 = \epsilon.
\end{aligned}$$

Hence, we showed that $\frac{\partial \psi'}{\partial x}$ is continuous even when the closest object to $\boldsymbol{x}$ changes. A similar conclusion can be drawn for $\frac{\partial \psi'}{\partial y}$. Therefore, we know that $\psi'(\boldsymbol{x}) \in C^1$ for all $\boldsymbol{x}$ such that $\hat{d}_{s_1}(\boldsymbol{x}) = \hat{d}_{s_2}(\boldsymbol{x})$. As for all other query points, $\psi'(\boldsymbol{x})$ is already C^1, we conclude that $\psi'(\boldsymbol{x}) \in C^1, \forall \boldsymbol{x} \in \mathcal{D}$.

3 EXTEND TO 3D: A RAY MARCHING ALGORITHM

CLAIM: Take arbitrary $i \in [n]$. Consider the i^{th} object, O_i, defined in the domain of interest $\mathcal{D}$. Let F_i be the set of primitives composing O_i. If $\frac{\partial^2 w_\ell}{\partial \phi_\ell^2} = 0, \forall \ell \in F_i$, then $\nabla d(x) \in C^1$.

PROOF. We need $\nabla d(\boldsymbol{x})$ to be differentiable if we want the updating step of the Ray Marching algorithm to be differentiable. However, the design of weight function by **?** only guarantees $\nabla \hat{d}_i(\boldsymbol{x}) \in C^0$, and thus $\nabla d(\boldsymbol{x}) \in C^0$ (by the construction of our Equation 7 in the paper). Hence, the key step is to design a continuously differentiable $\nabla \hat{d}_i(\boldsymbol{x})$. To make sure $\nabla \hat{d}_i(\boldsymbol{x}) \in C^1$, we need to confine the second order behaviour of $w_\ell(\boldsymbol{x})$. We first reiterate the expression for $\nabla\hat{d}_i(\boldsymbol{x})$ in our notation (Equation 4 in ?):

$$\nabla\hat{d}_i(\boldsymbol{x}) = \frac{\sum_{f_\ell \in F_i} w_\ell(\boldsymbol{x}) \exp(-ad_\ell^*)\nabla d_\ell^* - \frac{1}{a}\exp(-ad_\ell^*)\nabla w_\ell(\boldsymbol{x})}{\sum_{f_\ell \in F_i} w_\ell(\boldsymbol{x}) \exp(-ad_\ell^*)}.$$

Following the nature of the SC Exterior Mapping, we are only querying the points in the domain $\mathcal{D}$ that is also in the exterior of the objects. Then, $\nabla d_\ell^* \in C^1$ holds for all such query points. Therefore, taking a closer look at the expression for $\nabla\hat{d}_i(\boldsymbol{x})$, we realized that each component of $\nabla\hat{d}_i(\boldsymbol{x})$ is already C^1, except for $\nabla w_\ell(\boldsymbol{x})$. Let us write down $\nabla^2 w_\ell(\boldsymbol{x})$ explicitly:

$$\begin{aligned}
\nabla^2 w_\ell(\boldsymbol{x}) &= \frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial \pi_\ell}{\partial \boldsymbol{x}}\frac{\partial \phi_\ell}{\partial \pi_\ell}\frac{\partial w_\ell}{\partial \phi_\ell}\right) \\
&= \frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial \pi_\ell}{\partial \boldsymbol{x}}\right)\frac{\partial \phi_\ell}{\partial \pi_\ell}\frac{\partial w_\ell}{\partial \phi_\ell} + \frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial \phi_\ell}{\partial \pi_\ell}\right)\frac{\partial \pi_\ell}{\partial \boldsymbol{x}}\frac{\partial w_\ell}{\partial \phi_\ell} + \frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial w_\ell}{\partial \phi_\ell}\right)\frac{\partial \pi_\ell}{\partial \boldsymbol{x}}\frac{\partial \phi_\ell}{\partial \pi_\ell} \\
&= \frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial w_\ell}{\partial \phi_\ell}\right)\frac{\partial \pi_\ell}{\partial \boldsymbol{x}}\frac{\partial \phi_\ell}{\partial \pi_\ell} \quad \left(\text{since } \frac{\partial w_\ell}{\partial \phi_\ell} = 0 \text{ by ?}\right) \\
&= \frac{\partial^2 w_\ell}{\partial \phi_\ell^2}\left(\frac{\partial \pi_\ell}{\partial \boldsymbol{x}}\frac{\partial \phi_\ell}{\partial \pi_\ell}\right)^2.
\end{aligned}$$

Set $\frac{\partial^2 w_\ell}{\partial \phi_\ell^2} = 0$ so that the continuity of $\nabla^2 w_\ell$ is not spoiled by the discontinuity of $\frac{\partial \pi_\ell}{\partial \boldsymbol{x}}$. Tracing back, we have $\nabla\hat{d}_i(\boldsymbol{x}) \in C^1$ and eventually $\nabla d(\boldsymbol{x}) \in C^1$ as desired.