An Overview of Fluid Animation

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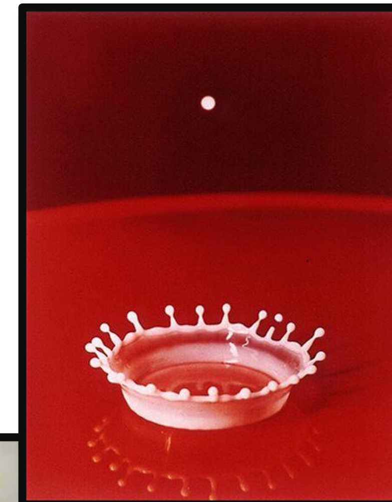
What distinguishes fluids?



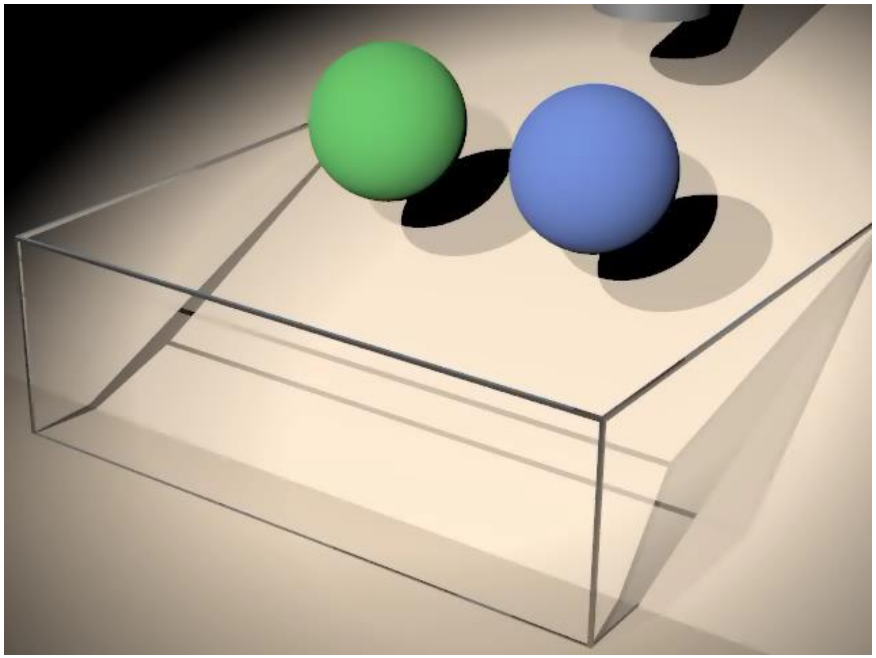


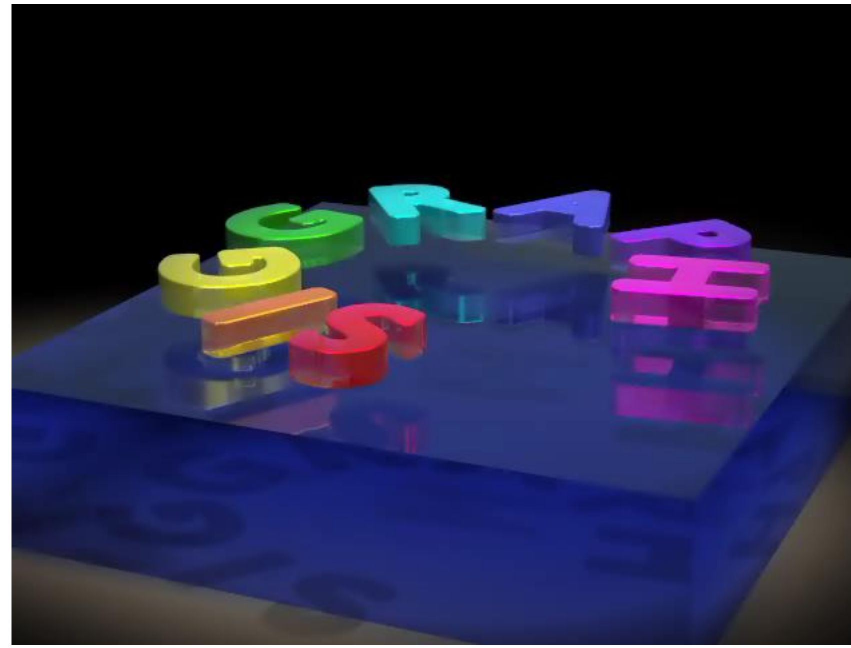


What distinguishes fluids?

- No “preferred” shape.
- Always flows when force is applied.
- Deforms to fit its container.
- Internal forces depend on *velocities*, not displacements (compare v.s., elastic objects)

Examples

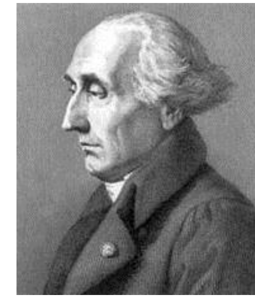




For further detail on today's material, see Robert Bridson's online fluid notes.
http://www.cs.ubc.ca/~rbridson/fluidsimulation/
(There's also a book.)

Basic Theory

Eulerian vs. Lagrangian

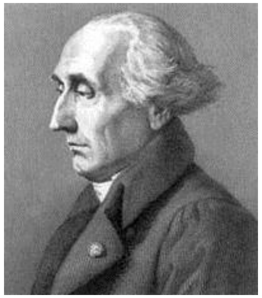


Lagrangian: Point of reference moves *with* the material.

Eulerian: Point of reference is *stationary.*

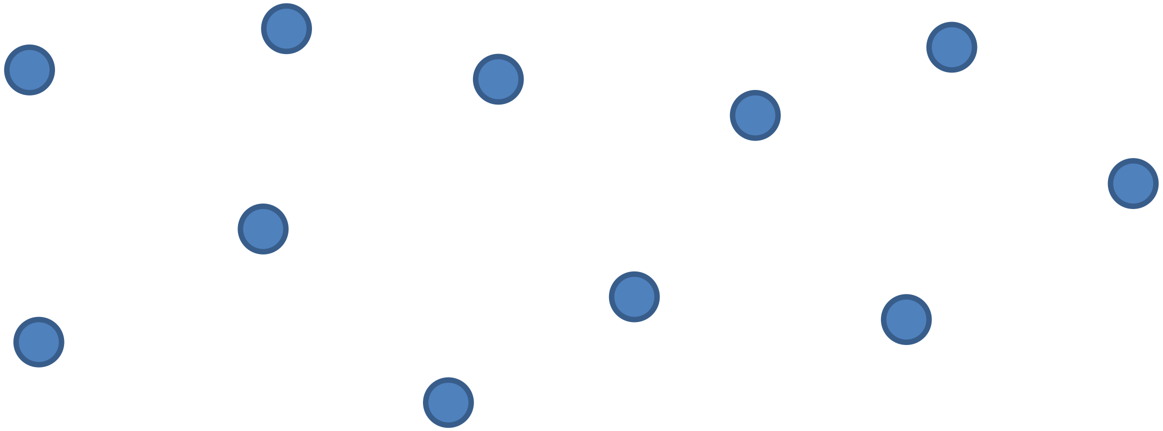
e.g. Weather balloon (Lagrangian) vs. weather station on the ground (Eulerian)

Eulerian vs. Lagrangian

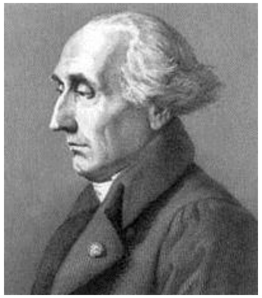


Consider an evolving scalar field (e.g., temperature).

Lagrangian view: Set of ***moving particles***, each with a temperature value.

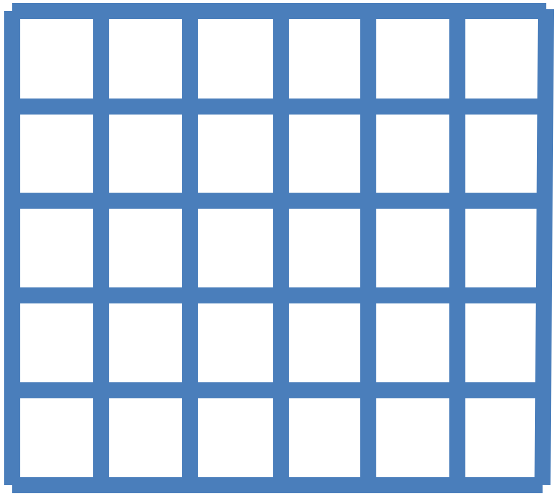


Eulerian vs. Lagrangian



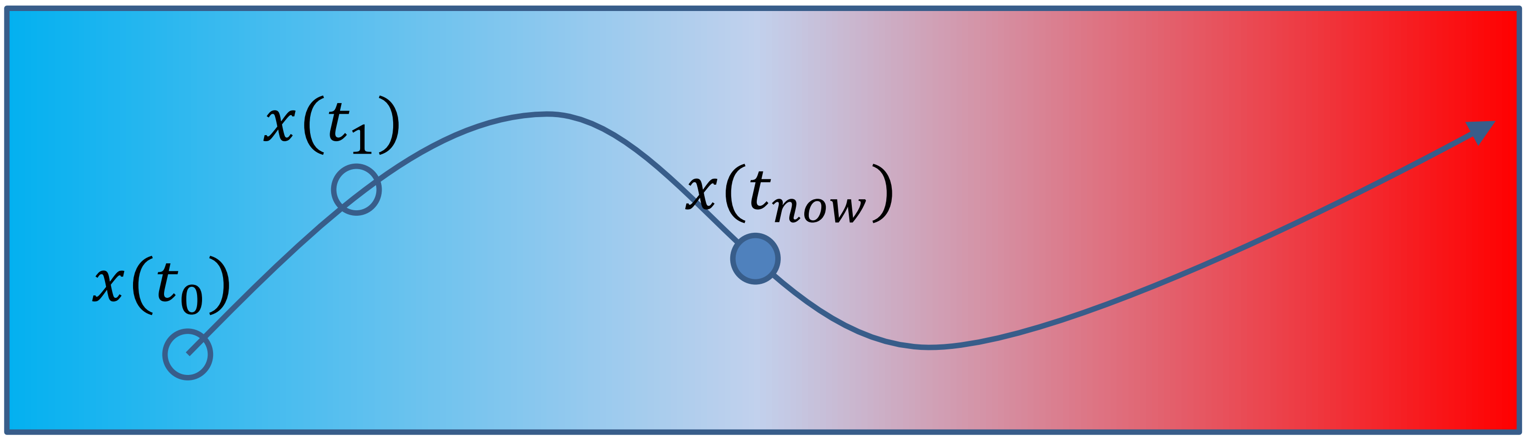
Consider an evolving scalar field (e.g., temperature).

Eulerian view: A fixed grid of temperature values, that temperature ***flows through.***



Relating Eulerian and Lagrangian

Consider the temperature $T(x,t)$ at a point following a given path, $x(t)$.



How can temperature measured at $x(t)$ change?

1. There is a hot/cold "source" at the current point.
2. Following the path, the point moves to a cooler/warmer location.

Time derivatives

Mathematically:

$$\frac{D}{Dt}T(x(t),t) = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\partial x}{\partial t}$$

Chain rule!

$$= \frac{\partial T}{\partial t} + \nabla T \cdot \frac{\partial x}{\partial t}$$

Definition of ∇

$$= \frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T$$

Choose $\frac{\partial x}{\partial t} = u$

Material Derivative

This is called the *material derivative*, and denoted $\frac{D}{Dt}$.

(AKA total derivative.)

Change at a point moving along the given path, x(t).

Change due to movement of the point.

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \boldsymbol{u}\nabla T$$

Change at the current (fixed) point.

Advection

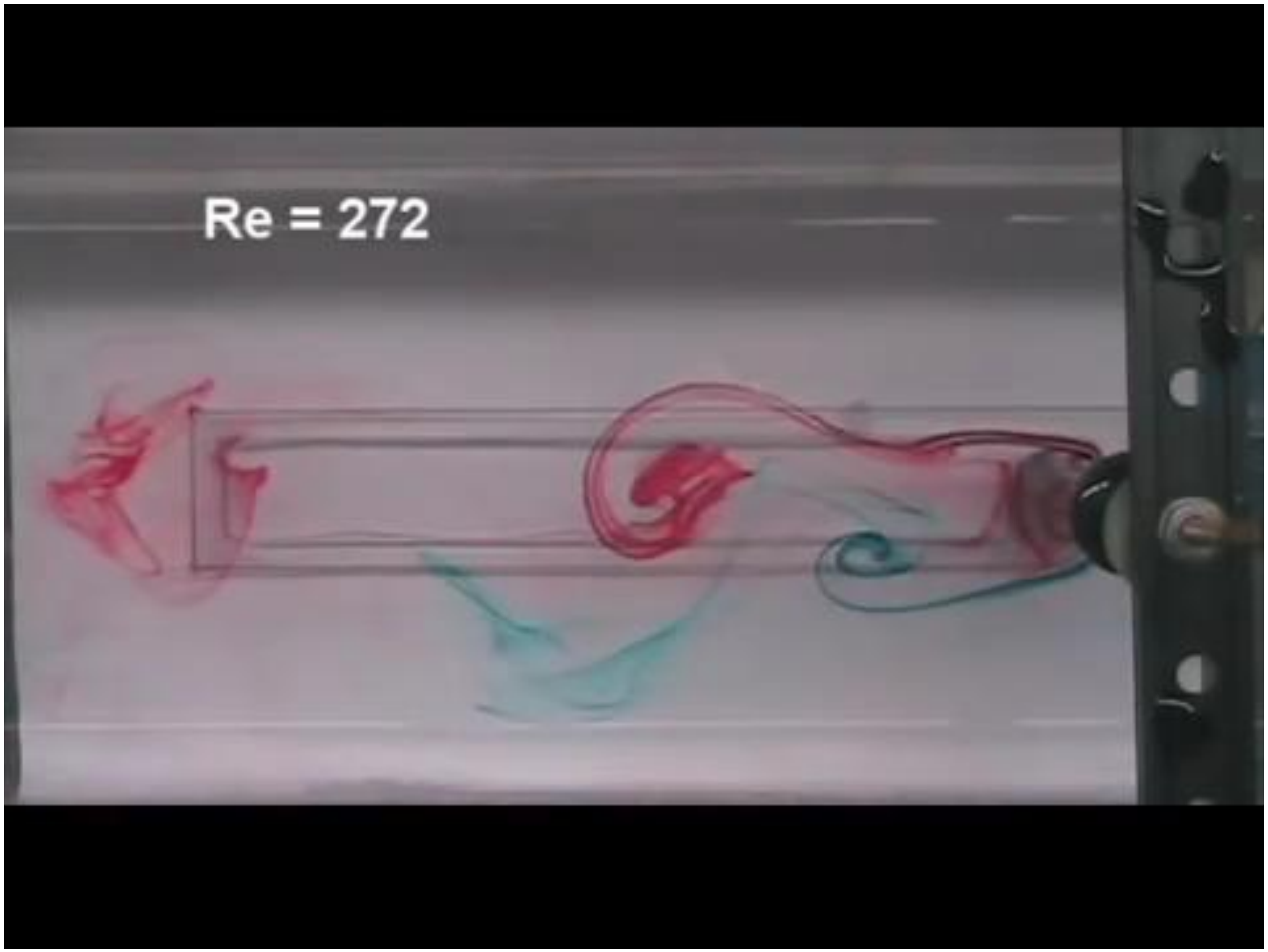
To track a quantity T moving (passively) through a velocity field:

$$\frac{DT}{Dt} = 0 \qquad \text{or equivalently} \qquad \frac{\partial T}{\partial t} + \boldsymbol{u}\nabla T = 0$$

This is the *advection equation*.

Think of colored dye or massless particles drifting around in fluid.

Advection

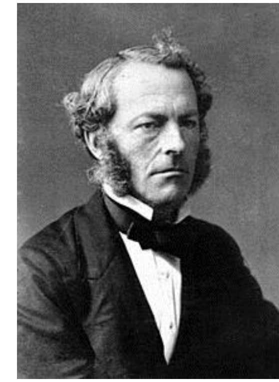


Equations of Motion

For general materials, we have Newton's second law: F = ma.

The *Navier-Stokes equations* are essentially the same equation, specialized to fluids.

Navier-Stokes



Density × Acceleration = Sum of Forces

$$\rho \frac{D\boldsymbol{u}}{Dt} = \sum_i \boldsymbol{F}_i$$

Expanding the material derivative…

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\rho(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \sum_i \boldsymbol{F}_i$$

What are the forces on a fluid?

Primarily for now:

- Pressure
- Viscosity
- Simple "external" forces
  - (e.g. gravity, buoyancy, user forces)

Also:

- Surface tension
- Coriolis
- Possibilities for more exotic fluid types:
  - Elasticity (e.g. silly putty)
  - Shear thickening / thinning (e.g. "oobleck", ketchup, paints)
  - Electromagnetic forces: magnetohydrodynamics, ferrofluids, etc.
- Sky's the limit…

Exotic Fluids - Oobleck

Exotic Fluids - Ferrofluid

In full...

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\rho(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \sum_i \boldsymbol{F}_i$$

Change in velocity at a ***fixed point***

Advection (of *velocity*)

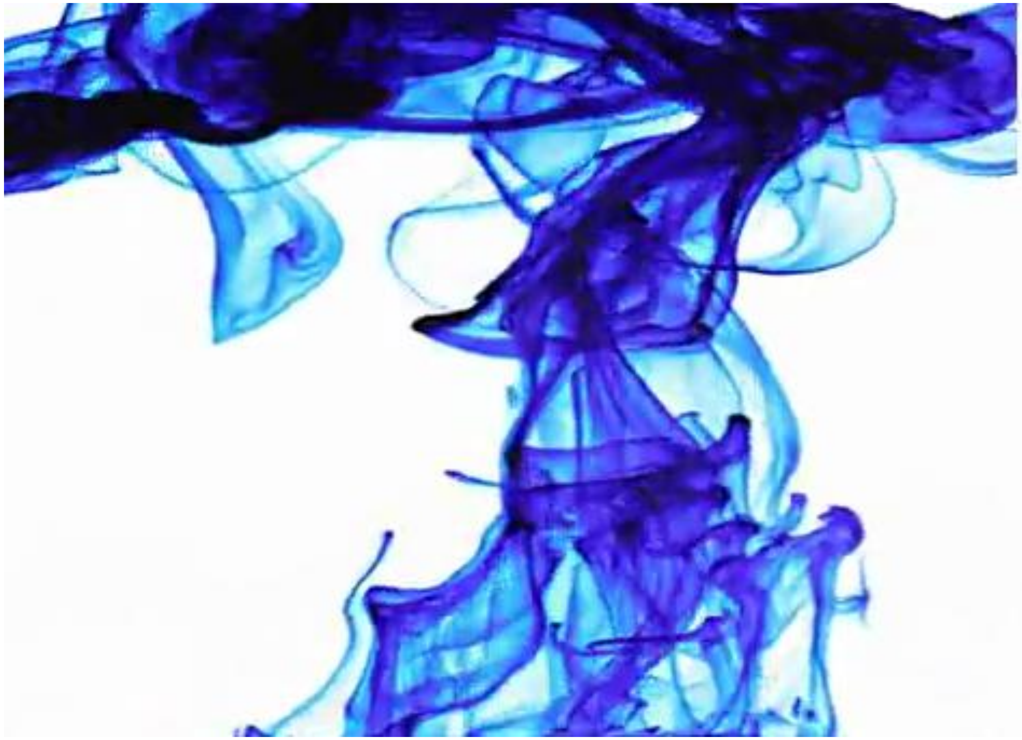
Forces (pressure, viscosity gravity,...)

Operator splitting

Break the full, nonlinear equation into sub-steps:

1. Advection: $\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\rho(\boldsymbol{u} \cdot \nabla \boldsymbol{u})$
2. Pressure: $\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F}_{pressure}$
3. Viscosity: $\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F}_{viscosity}$
4. External: $\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F}_{other}$

1. Advection



Advection

Earlier, we considered advection of a passive scalar quantity, T, by velocity $\boldsymbol{u}$.

$$\frac{\partial T}{\partial t} = -\boldsymbol{u} \cdot \nabla T$$

In Navier-Stokes we saw:

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{u} \cdot \nabla \boldsymbol{u}$$

Velocity $\boldsymbol{u}$ is advected *by itself!*

Advection

That is, (u, v, w) components of velocity $\boldsymbol{u}$ are advected as separate scalars.

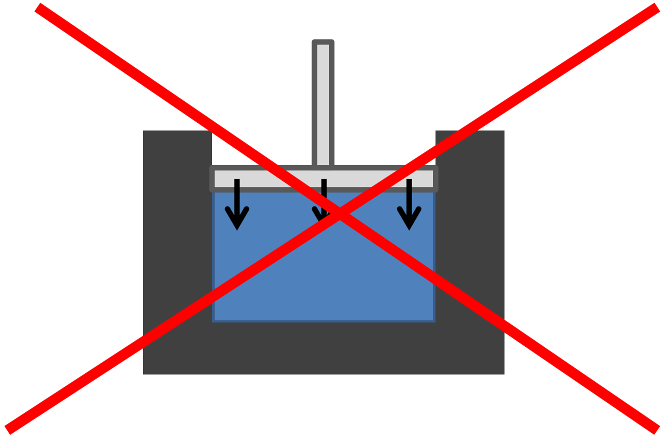
May be able to reuse the same numerical method.

2. Pressure

Pressure

What does pressure do?

– Enforces *incompressibility* (fights compression).



Typical fluids (mostly) do not compress.

- Exceptions: high velocity, high pressure, …

Incompressibility

Compressible velocity field

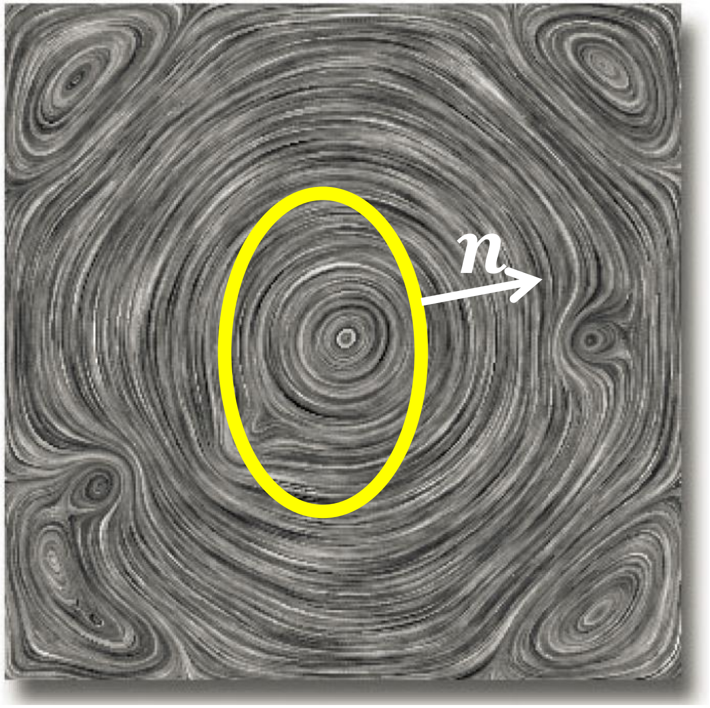
Incompressible velocity field

Incompressibility

Intuitively, net flow into/out of a given region is zero (no sinks/sources).

Integrate the flow across the boundary of a closed region:

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} = 0$$



Incompressibility

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} = 0$$

By divergence theorem:

$$\iint \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

But this is true for *any* region, so $\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{0}$ everywhere.

Incompressibility implies $\boldsymbol{u}$ is *divergence-free*.

Pressure

Where does pressure come in?

- Pressure is the force needed to enforce the constraint $\nabla \cdot \boldsymbol{u} = 0$.

- Pressure force has the following form:

$$\boldsymbol{F}_p = -\nabla p$$

Helmholtz Decomposition

Input (Arbitrary)
Velocity Field

Curl-Free
(Irrotational)

Divergence-Free
(Incompressible)

=

+

$$u = \nabla p + \nabla \times \varphi$$

$$u_{old} = F_{pressure} + u_{new}$$

Aside: Pressure as Lagrange Multiplier

Interpret as an optimization:

Find the closest $\boldsymbol{u}_{new}$ *to* $\boldsymbol{u}_{old}$ *where* $\nabla \cdot \boldsymbol{u}_{new} = 0$

$$\underset{\boldsymbol{u}_{new}}{\operatorname{argmin}} \frac{\rho}{2} \left\| \boldsymbol{u}_{new} - \boldsymbol{u}_{old} \right\|^2$$
$$\text{subject to} \quad \nabla \cdot \boldsymbol{u}_{new} = 0$$

The Lagrange multiplier that enforces the constraint is the pressure.

e.g., recall the “fast projection” paper, Goldenthal et al. 2007.

3. Viscosity

High Speed Honey

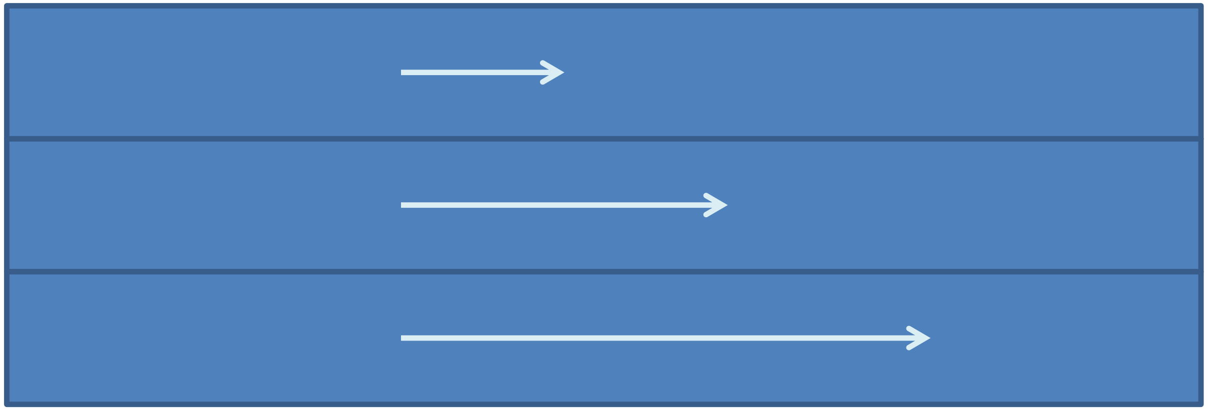
Viscosity

What characterizes a viscous liquid?

- “Thick”, damped behaviour.
- Strong resistance to flow.

Viscosity

Loss of energy due to internal friction between molecules moving at different velocities.

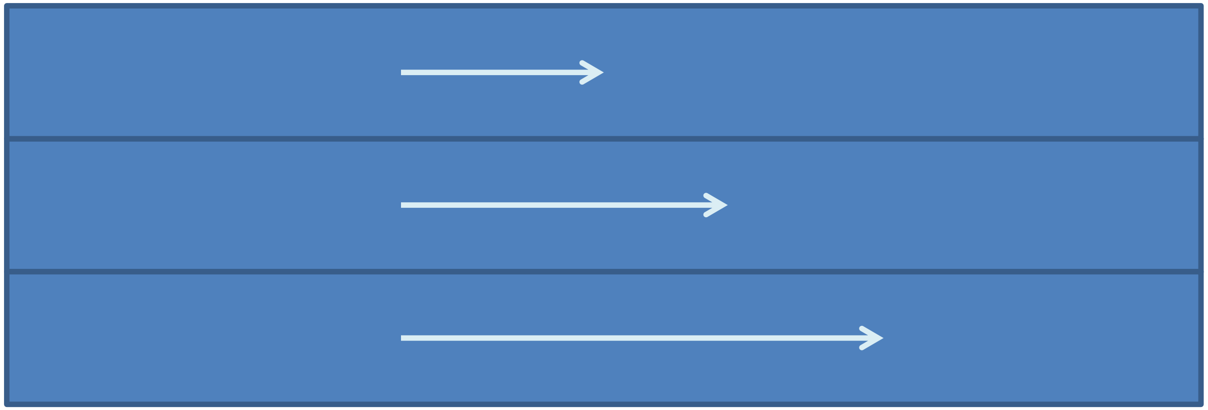


Interactions between molecules causes *shear stress* that…

- opposes *relative* motion.
- causes an exchange of momentum.

Viscosity

Loss of energy due to internal friction between molecules moving at different velocities.



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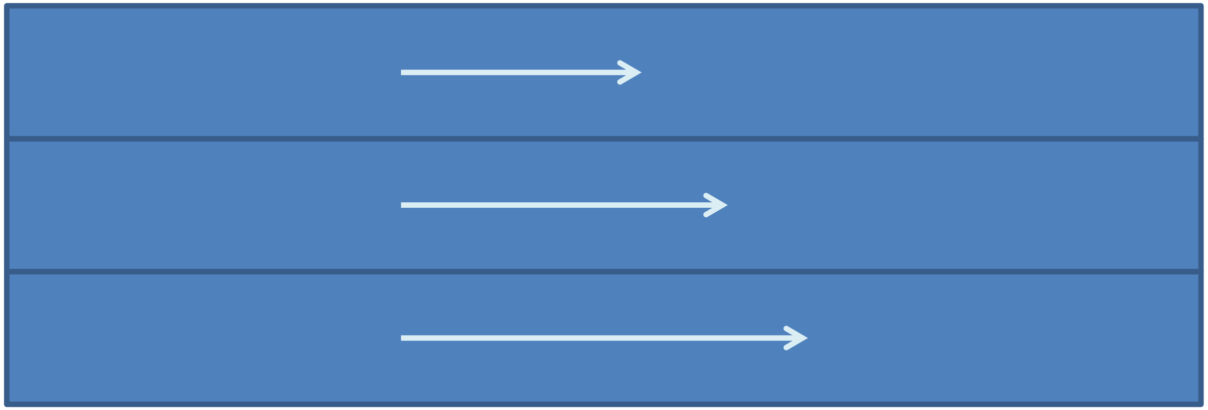

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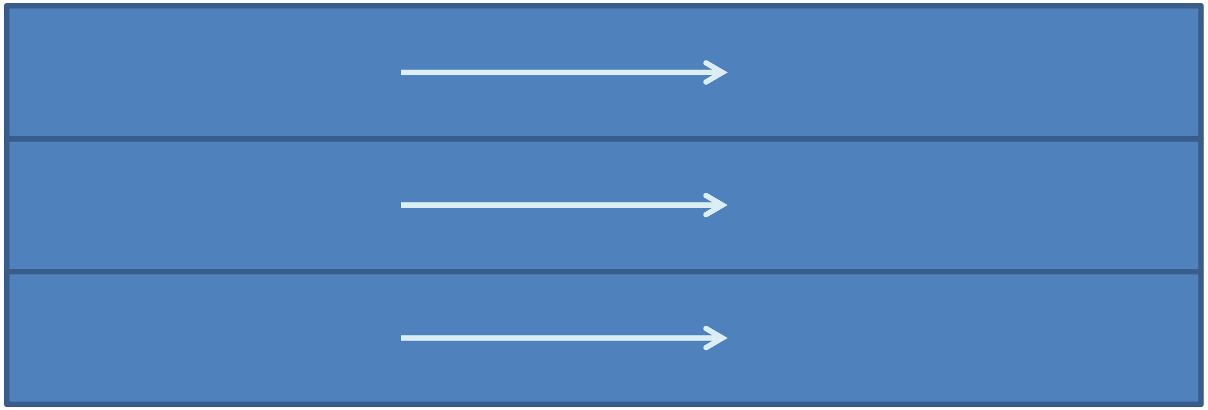

Interactions between molecules causes *shear stress* that…

- opposes *relative* motion.
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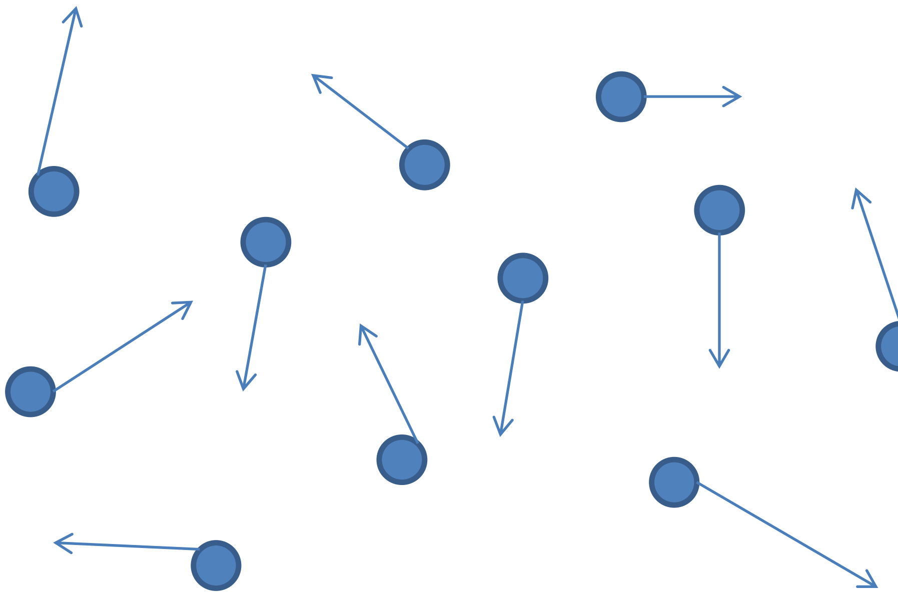
Viscosity

Imagine fluid particles with general velocities.



Each particle interacts with nearby neighbours, exchanging momentum.

Diffusion

The momentum exchange is related to:

- Velocity gradient, $\nabla \boldsymbol{u}$, in a region.
- Viscosity coefficient, μ.

Net effect is a smoothing or *diffusion* of the velocity over time.

Viscosity

Diffusion is typically modeled using the heat equation:

$$\frac{dT}{dt} = \alpha \nabla \cdot \nabla T$$

Diffusion

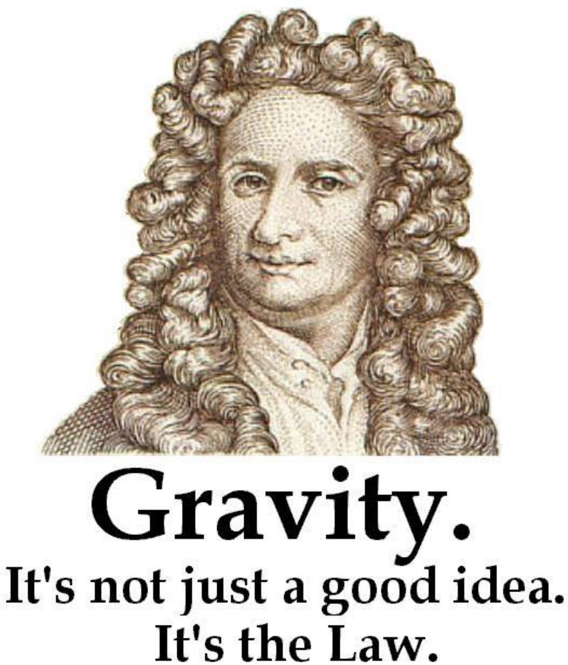
Viscosity

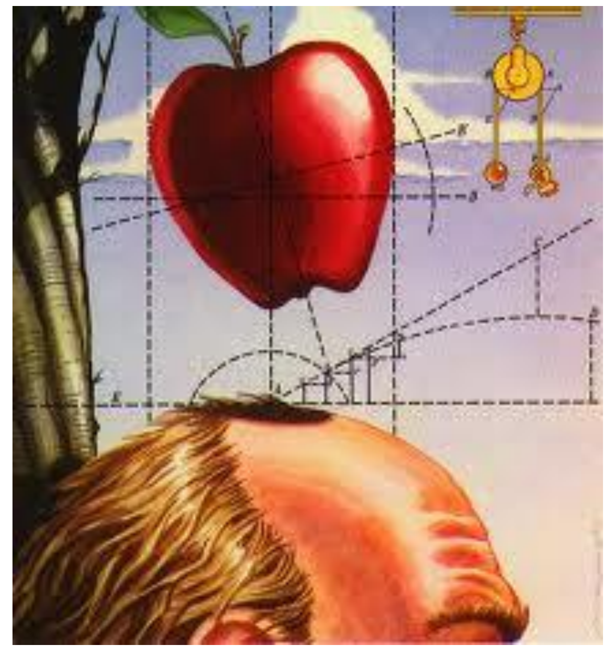
Diffusion applied to velocity gives our viscous force:

$$\boldsymbol{F}_{viscosity} = \rho \frac{\partial \boldsymbol{u}}{\partial t} = \mu \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \boldsymbol{u}$$

Usually, diffuse each component of $\boldsymbol{u}$ =(u, v, w) separately.

4. External Forces





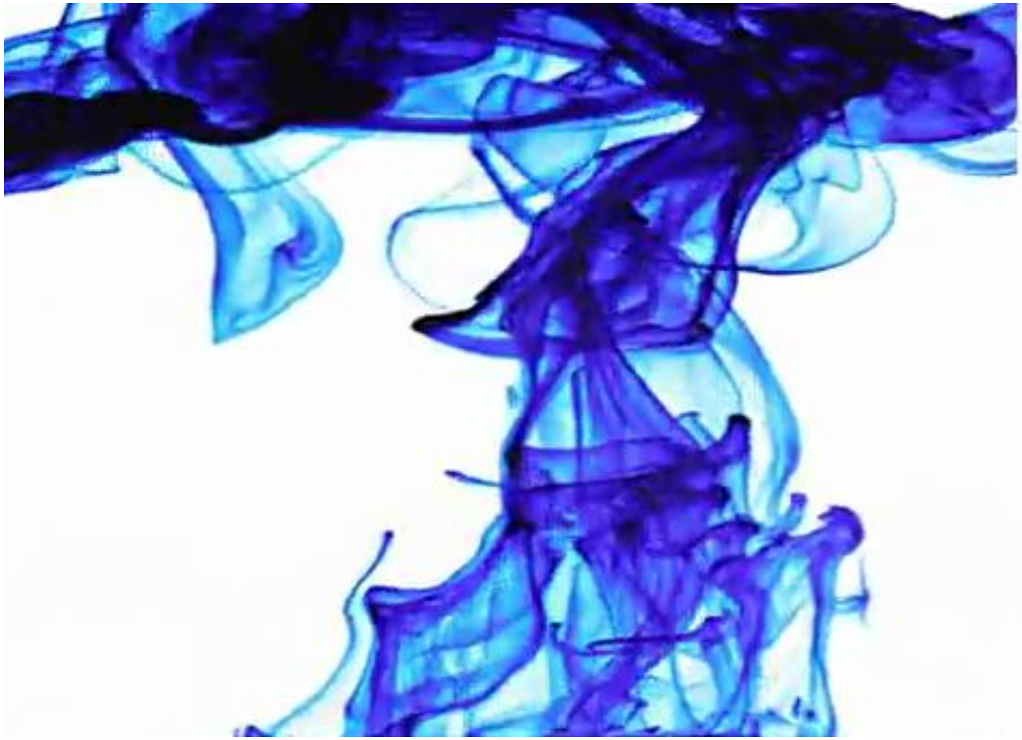
External Forces

Any other forces you may want.

- Simplest is gravity:
  - $F_g = \rho \boldsymbol{g}$ for $\boldsymbol{g} = (0, -9.81, 0)$
- Buoyancy models are similar,
  - e.g., $F_b = \beta(T_{current} - T_{ref})\boldsymbol{g}$

Numerical Methods for Fluid Animation

1. Advection



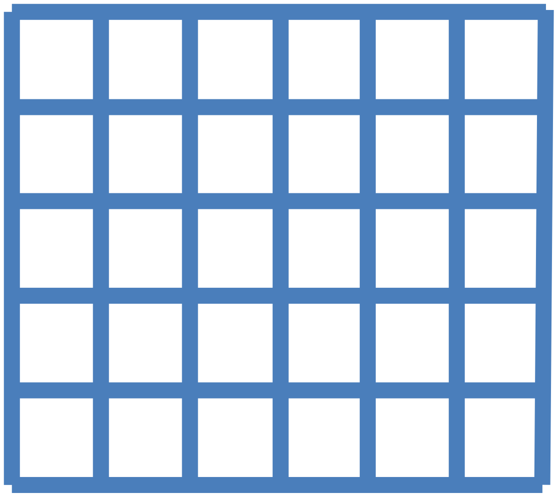
Advection of a Scalar

Consider advecting a quantity, φ

– temperature, color, smoke density, …

according to a velocity field $\boldsymbol{u}$.

Allocate a grid (2D array) that stores scalar φ and velocity $\boldsymbol{u}$.



Eulerian

Approximate derivatives with *finite differences*.

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi = 0$$

FTCS = Forward Time, Centered Space:

$$\frac{{\varphi_i}^{n+1} - {\varphi_i}^{n}}{\Delta t} + u\,\frac{{\varphi_{i+1}}^{n} - {\varphi_{i-1}}^{n}}{2\Delta x} = 0$$

Unconditionally Unstable!

Lax:

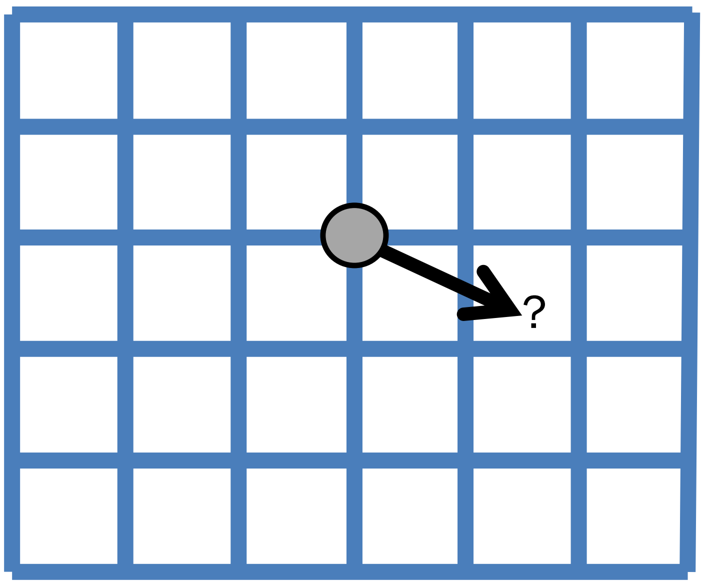
$$\frac{{\varphi_i}^{n+1} - ({\varphi_{i+1}}^{n} + {\varphi_{i-1}}^{n})/2}{\Delta t} + u\,\frac{{\varphi_{i+1}}^{n} - {\varphi_{i-1}}^{n}}{2\Delta x} = 0$$

Conditionally Stable!

Many possible methods, stability can be a challenge.

Lagrangian

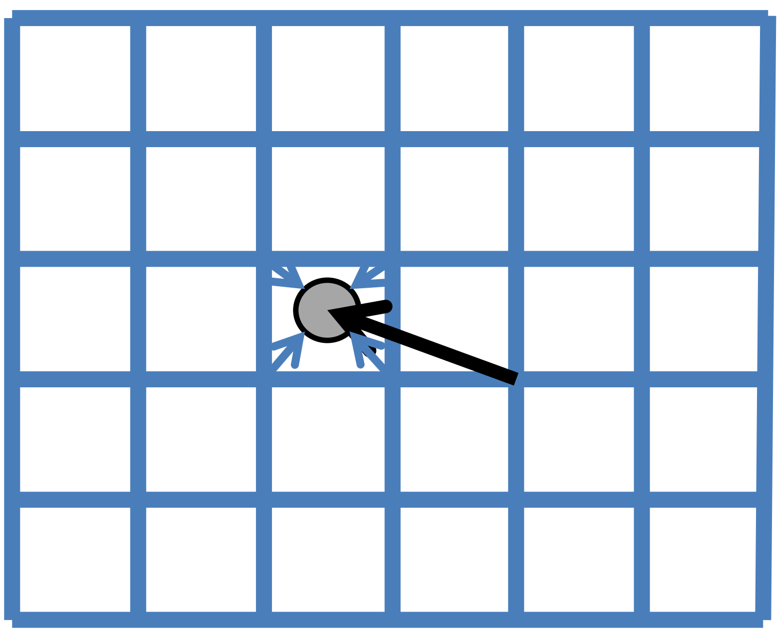
Advect data "forward" from grid points by integrating position according to grid velocity (e.g. forward Euler).



Problem: New data position doesn't necessarily land on a grid point.

Semi-Lagrangian

- Look *backwards* in time from a grid point, to see where its new data is coming *from*.
- Interpolate data at previous time.

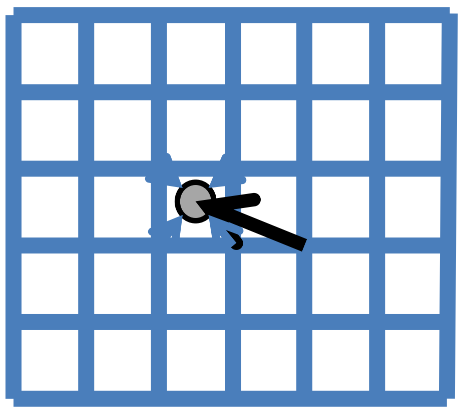


Semi-Lagrangian - Details

1. Determine velocity $\boldsymbol{u}_{i,j}$ at grid point.
2. Integrate position for a timestep of $-\Delta t$.
   - e.g. $x_{back} = x_{i,j} - \Delta t \boldsymbol{u}_{i,j}$
3. Interpolate φ at x_{back}, call it φ_{back}.
4. Assign $\varphi_{i,j} = \varphi_{back}$ for the new time.

Unconditionally stable!

(Though dissipative – drains energy over time.)



Advection of Velocity

This handles scalars. What about advecting velocity?

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{u} \cdot \nabla \boldsymbol{u}$$

Same method:

- Trace back with current velocity
- Interpolate velocity at that point
- Assign it to the grid point at the *new time*.

Caution: Do not overwrite the velocity field you're using to trace back! (Make a copy.)

2. Pressure

Recall… Helmholtz Decomposition

Input Velocity field

=

Curl-Free
(irrotational)

+

Divergence-Free
(incompressible)

$$u = \nabla p + \nabla \times \varphi$$

$$u_{old} = F_{pressure} + u_{new}$$

Pressure Projection - Derivation

$$(1)\ \rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p \quad \textit{and} \quad (2)\ \nabla \cdot \boldsymbol{u} = 0$$

Discretize (1) in time…

$$\boldsymbol{u}_{new} = \boldsymbol{u}_{old} - \frac{\Delta t}{\rho} \nabla p$$

Then plug into (2)…

$$\nabla \cdot \left(\boldsymbol{u}_{old} - \frac{\Delta t}{\rho} \nabla p \right) = 0$$

Pressure Projection

Implementation:

1) Solve a linear system of equations for *p:*

$$\frac{\Delta t}{\rho} \nabla \cdot \nabla p = \nabla \cdot \boldsymbol{u}_{old}$$

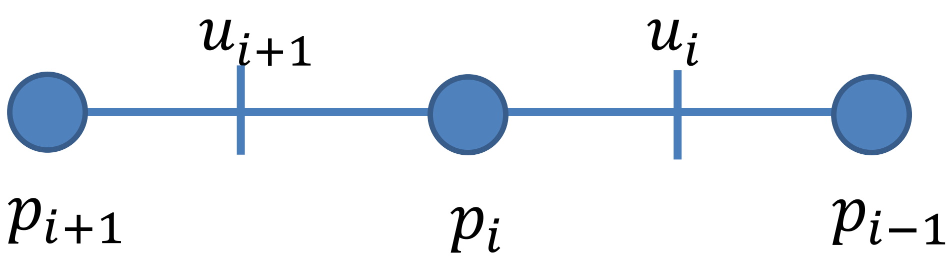
2) Given *p*, plug back in to update velocity:

$$\boldsymbol{u}_{new} = \boldsymbol{u}_{old} - \frac{\Delta t}{\rho} \nabla p$$

Implementation

$$\frac{\Delta t}{\rho} \nabla \cdot \nabla p = \nabla \cdot \boldsymbol{u}_{old}$$

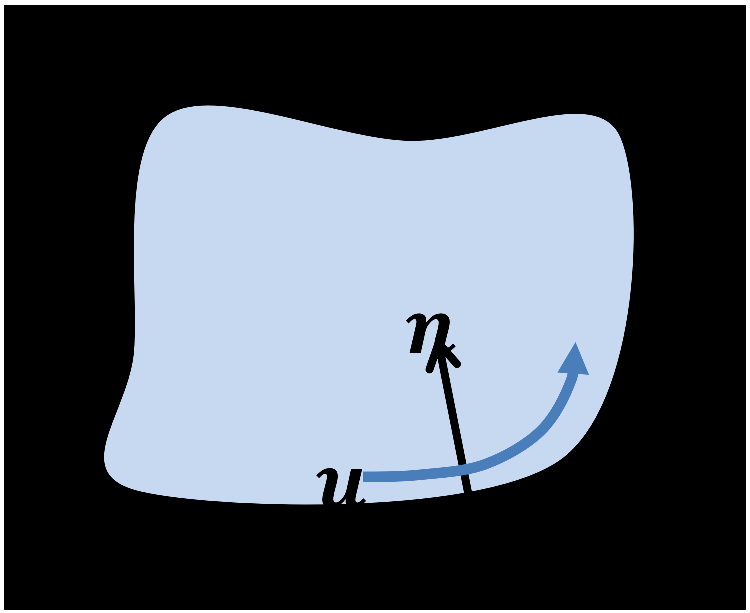
Discretize with finite differences:



e.g., in 1D:

$$\frac{\Delta t}{\rho}\left(\frac{\frac{p_{i+1}-p_i}{\Delta x}-\frac{p_i-p_{i-1}}{\Delta x}}{\Delta x}\right)=\frac{u_{i+1}{}^{old}-u_i{}^{old}}{\Delta x}$$

Solid Boundary Conditions



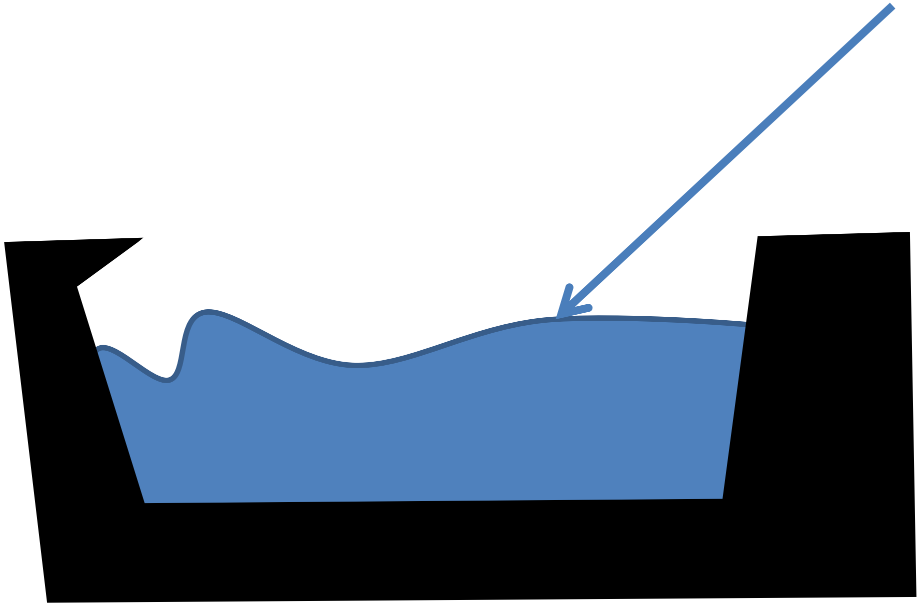
Free Slip:

$$\boldsymbol{u}_{new} \cdot \boldsymbol{n} = 0$$

i.e., Fluid cannot penetrate or flow out of the wall, but may slip along it.

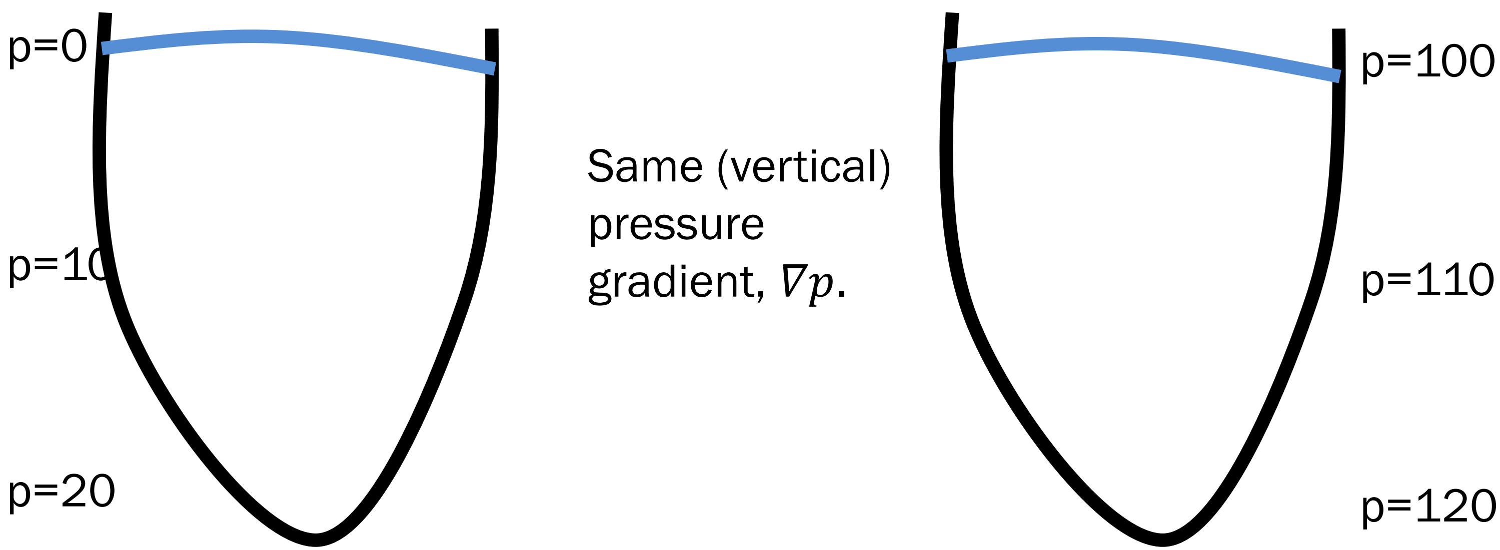
Air ("Free surface") Boundary Conditions

Assume air (outside the liquid) is at some constant atmospheric pressure, $p = p_{atm}$.



Free Surface Boundary Conditions

Only the pressure gradient matters, so simplify and assume $p = p_{atm} = 0$.



3. Viscosity

Viscosity

PDE: $\rho \dfrac{\partial \boldsymbol{u}}{\partial t} = \mu \nabla \cdot \nabla \boldsymbol{u}$

Again, apply finite differences.

Discretized in time:

$$\boldsymbol{u}_{\boldsymbol{new}} = \boldsymbol{u}_{old} + \frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{*}$$

$\boldsymbol{u}_{\boldsymbol{old}}$ -> explicit

$\boldsymbol{u}_{\boldsymbol{new}}$ -> implicit

Viscosity – Time Integration

Explicit integration: $\boldsymbol{u_{new}} = \boldsymbol{u}_{old} + \frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{old}$

- Compute $\frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{old}$ from current velocities.
- Add on to current $\boldsymbol{u}$.
- Quite unstable (stability restriction: $\Delta t \approx O(\Delta x^2)$)

Implicit integration: $\boldsymbol{u_{new}} = \boldsymbol{u}_{old} + \frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{new}$

- Stable even for high viscosities, large steps.
- Must solve a system of equations.

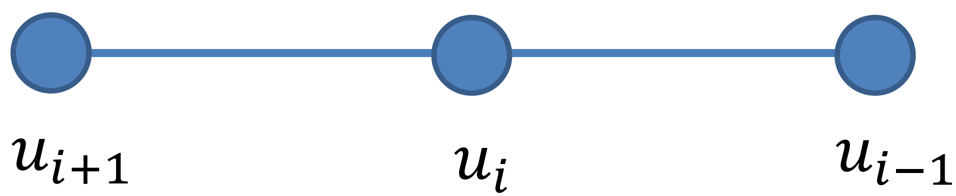
Viscosity – Implicit Integration

Solve for $\boldsymbol{u_{new}}$:

$$\boldsymbol{u_{new}} - \frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{new} = \boldsymbol{u}_{old}$$

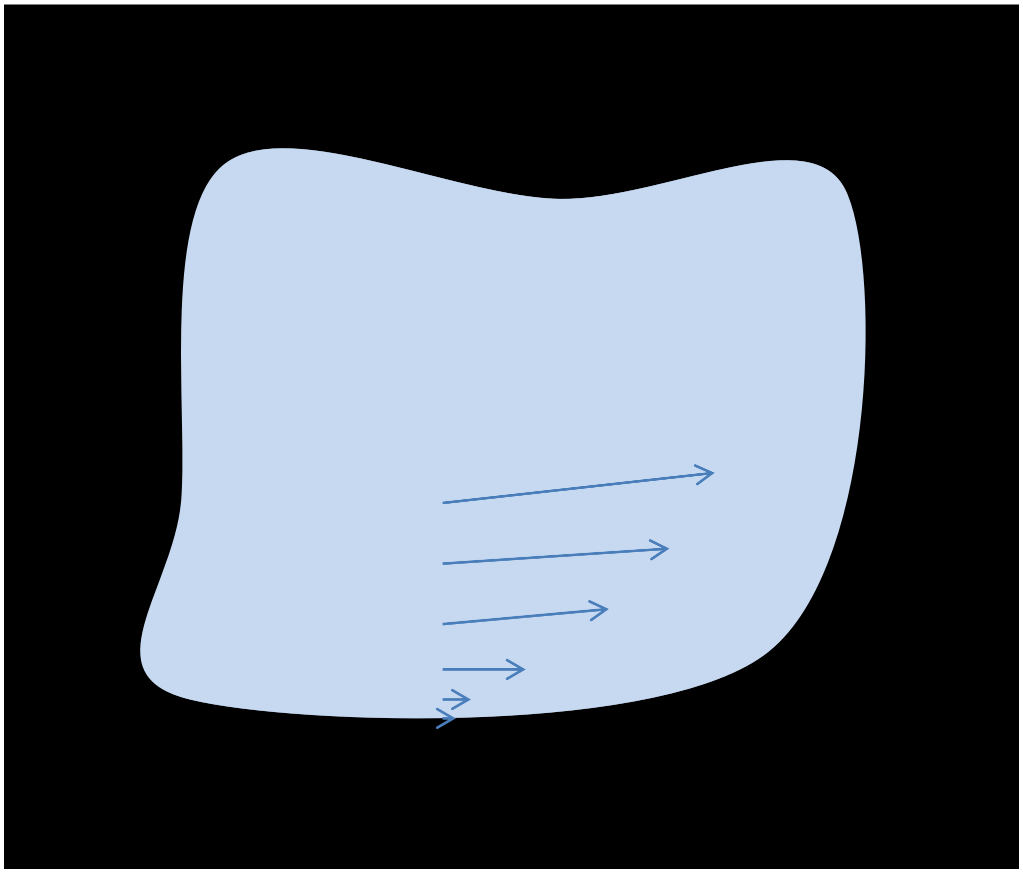
(Apply separately for each velocity component.)

e.g. in 1D:



$$u_i \;\; - \frac{\Delta t \mu}{\rho} \left(\frac{\frac{u_{i+1} - u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x}}{\Delta x} \right) = u_i^{old}$$

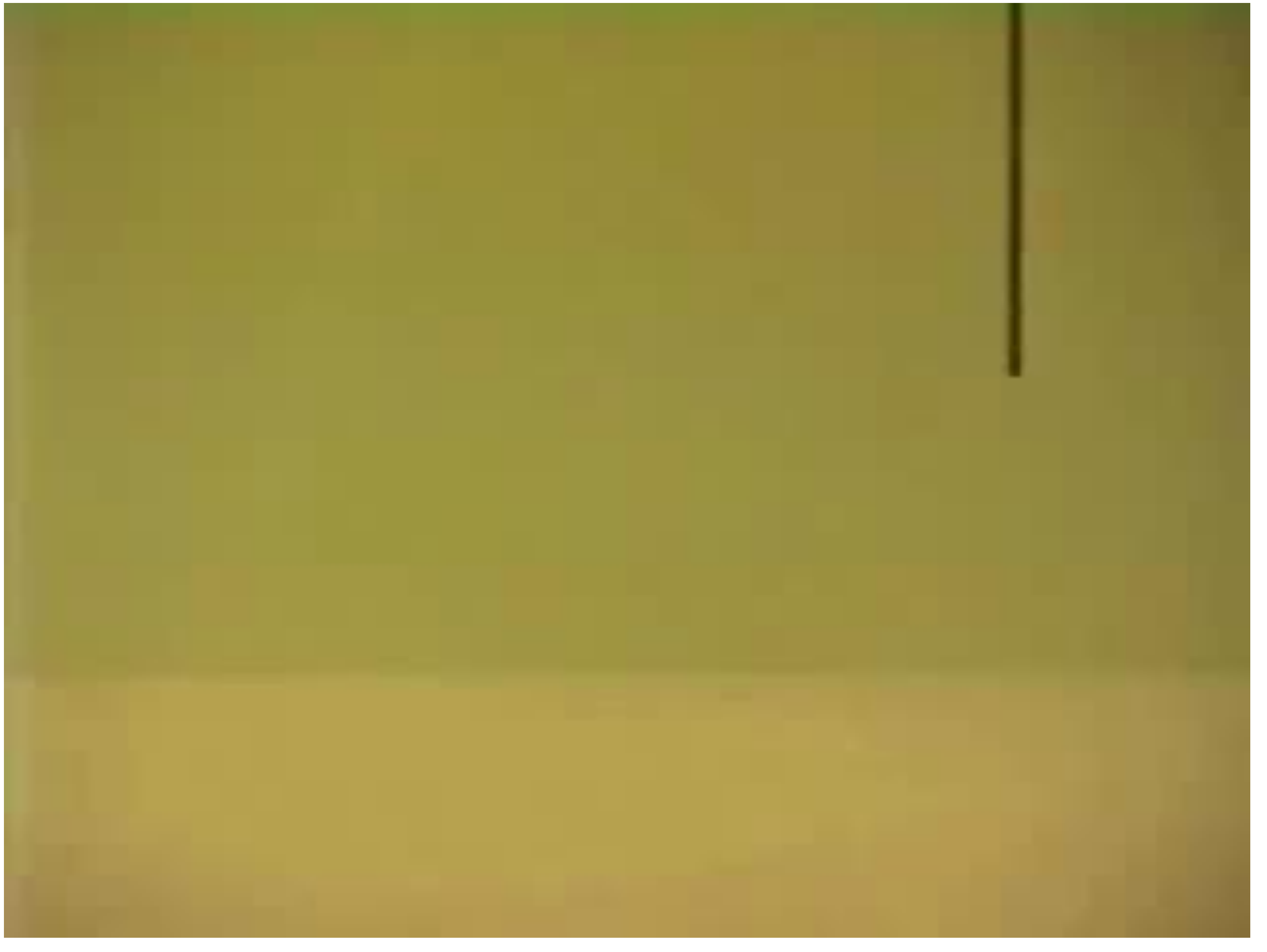
Viscosity - Solid Boundary Conditions



No-Slip:

$$\boldsymbol{u}_{new} = 0$$

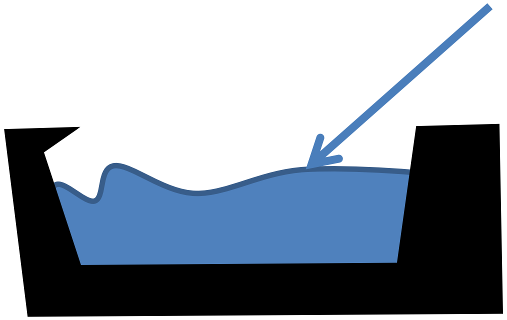
No-slip Condition



Viscosity - Free Surface Conditions

We want to model no momentum exchange with the “air”.

Simplest attempt: $\nabla u \cdot n = 0$
Drawback: Breaks rotation!



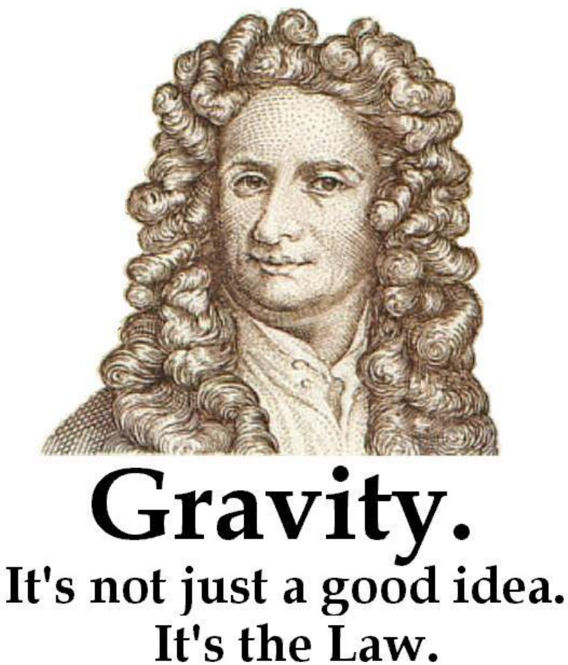
True conditions are more involved:

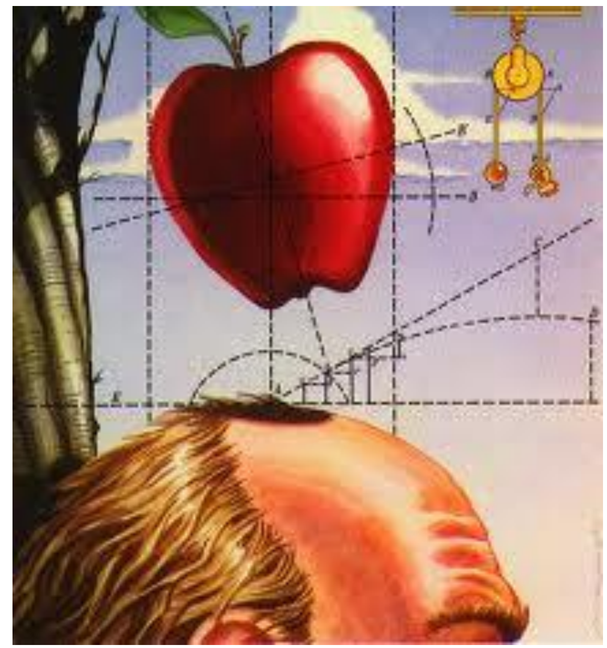
$$\left(-p\mathbf{I} + \boldsymbol{\mu}\left(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^{\boldsymbol{T}}\right)\right) \cdot \boldsymbol{n} = \mathbf{0}$$

(Still ignores surface tension!)

See [Batty & Bridson, 2008] for the current standard solution in graphics. (Needed e.g., for honey coiling.)

4. External Forces





Gravity

Discretized form is:

$$\boldsymbol{u}_{\boldsymbol{new}} = \boldsymbol{u}_{old} + \Delta t \boldsymbol{g}$$

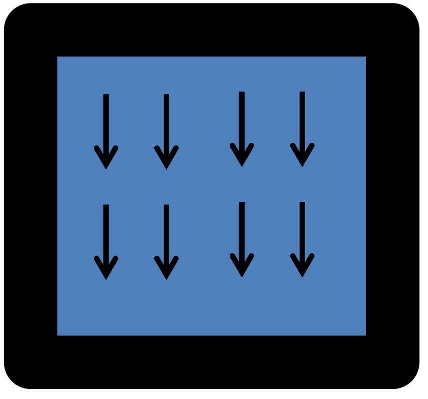
Simply increment the vertical velocities at each step!

Gravity

Notice: in a closed fluid-filled container, gravity (alone) won't do anything!

- Incompressibility cancels it out. (Assuming constant density.)

Start



After gravity step

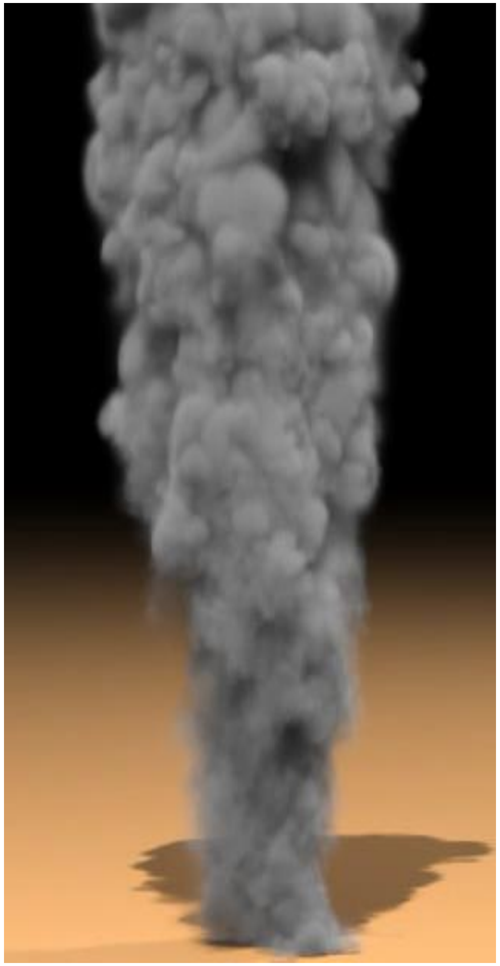
After pressure step

Simple Buoyancy

Track an extra scalar field T, representing local temperature.

Apply advection and diffusion to evolve it with the velocity field.

Difference between current and "reference" temperature induces buoyancy.



Simple Buoyancy

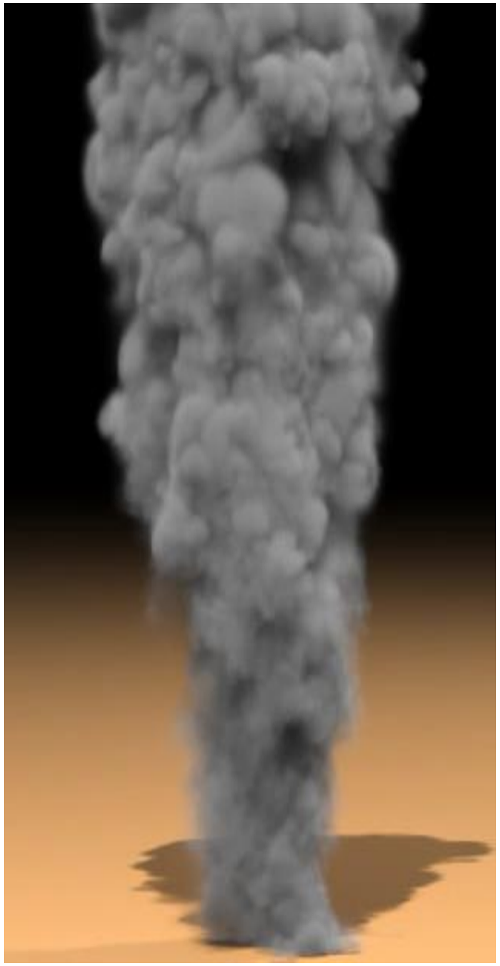
e.g.

$$\boldsymbol{u_{new}} = \boldsymbol{u}_{old} + \Delta t \beta \left(T_{current} - T_{ref}\right) \boldsymbol{g}$$

β dictates the strength of the buoyancy force.

For an enhanced version of this:

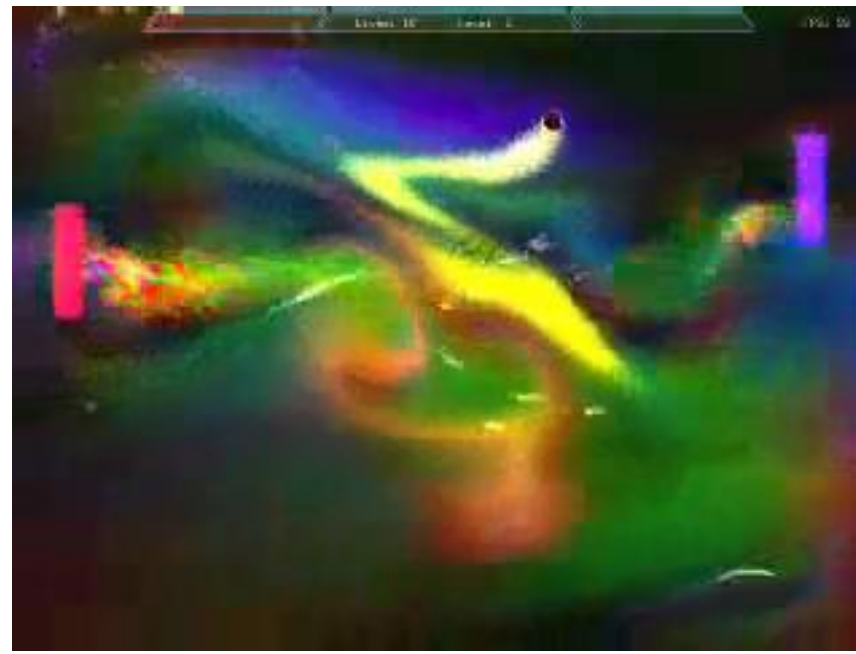
“Visual simulation of smoke”, [Stam et al., 2001].



User Forces

Add whatever additional forces we want:

- Wind forces near a mouse click.
- Paddle forces in *Plasma Pong*.



Plasma Pong

Ordering of Steps

Order is important.

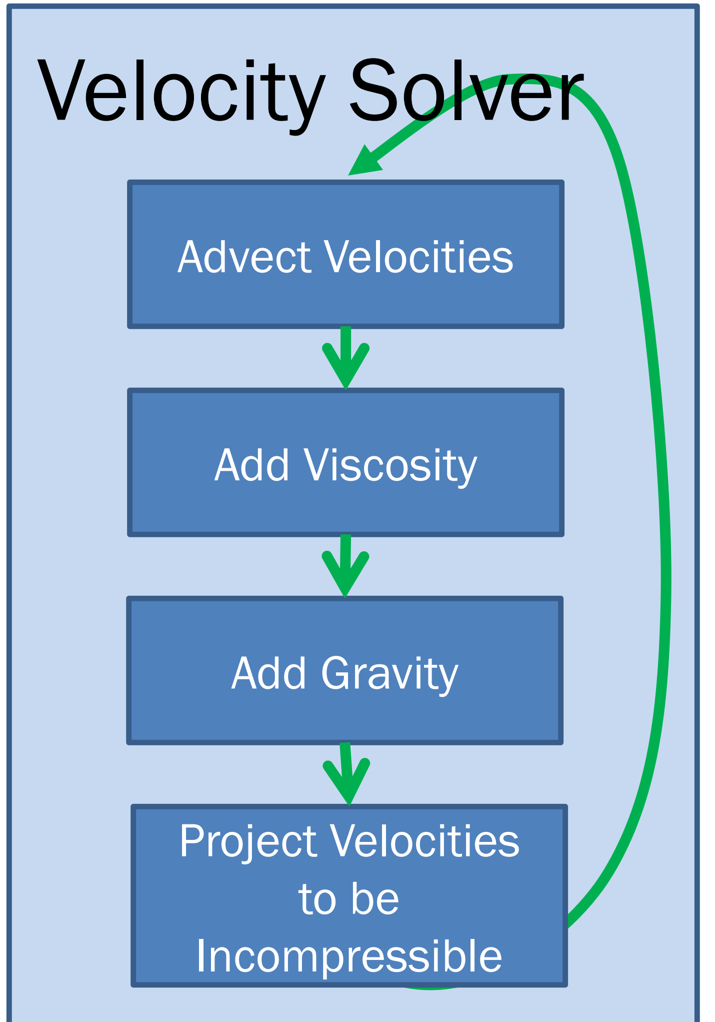
Why?

1) Incompressibility is not satisfied at intermediate steps.
2) Advecting with a compressive field causes volume/material loss or gain!

Ordering of Steps

For example, consider advection in this field:

The Big Picture



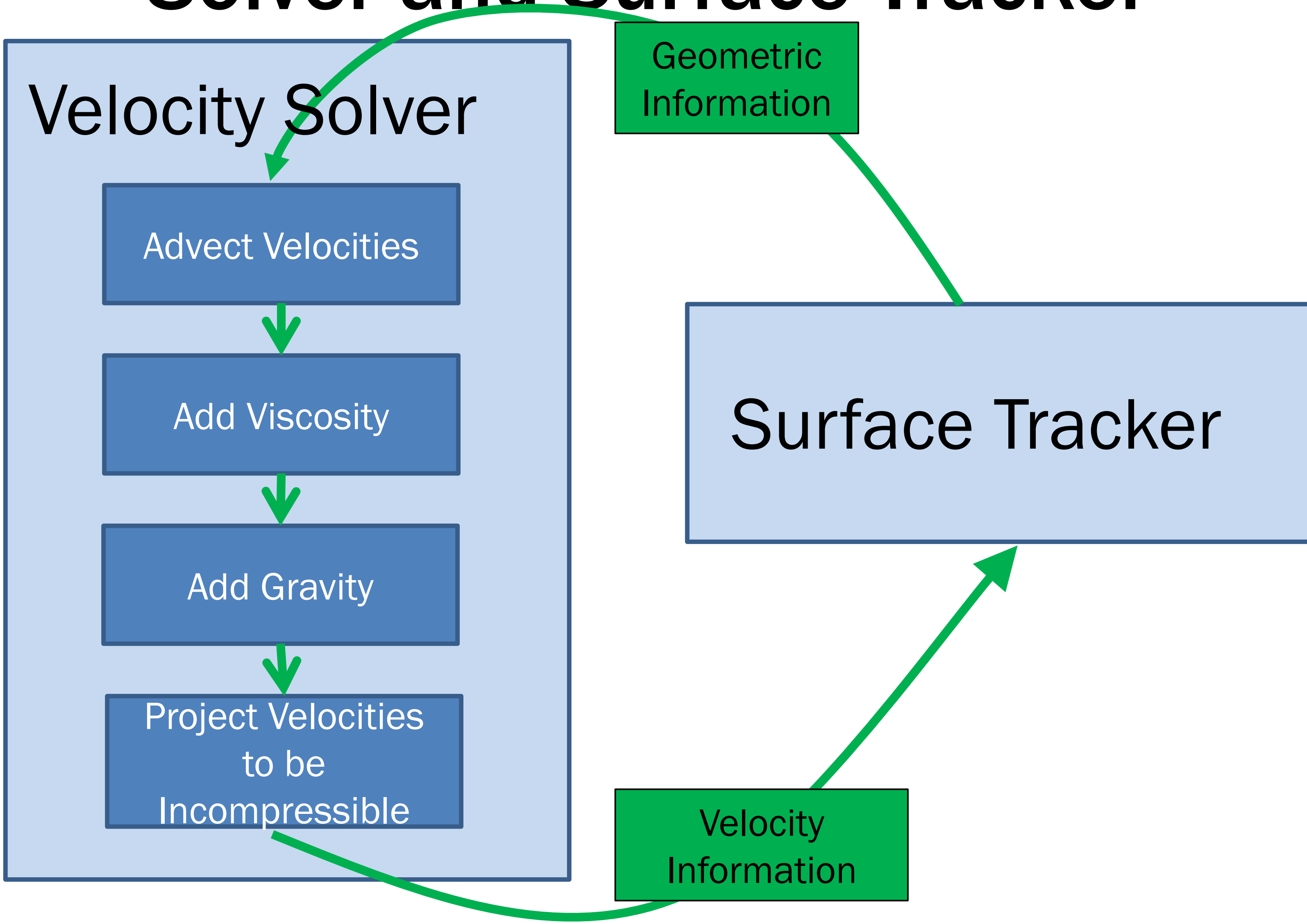
Liquids

Liquids

What's missing?

We still need a *surface representation.*

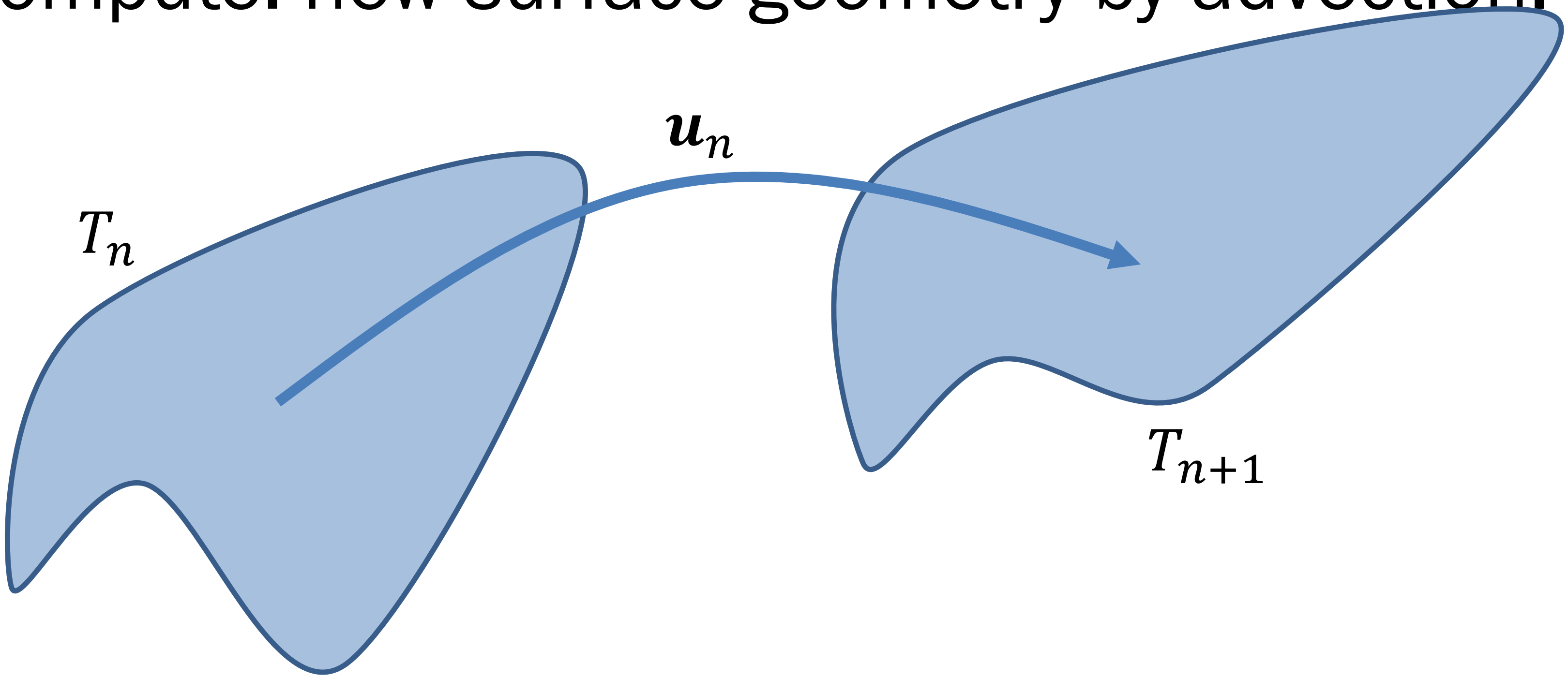
Interaction between Solver and Surface Tracker



Solver-to-Surface Tracker

Given: current surface geometry, velocity field, and timestep.

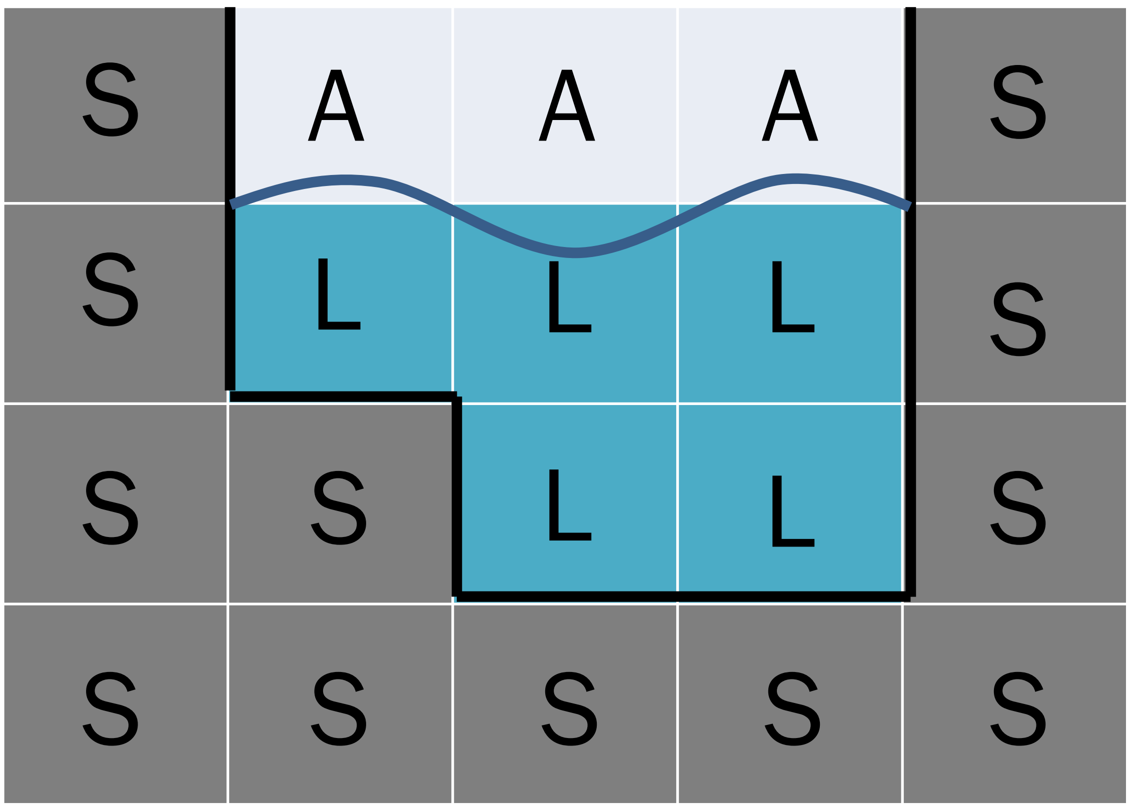
Compute: new surface geometry by advection.



Surface Tracker-to-Solver

Given the surface geometry, identify the type of each cell.

Solver uses this information for boundary conditions.



Surface Tracker

Ideally:

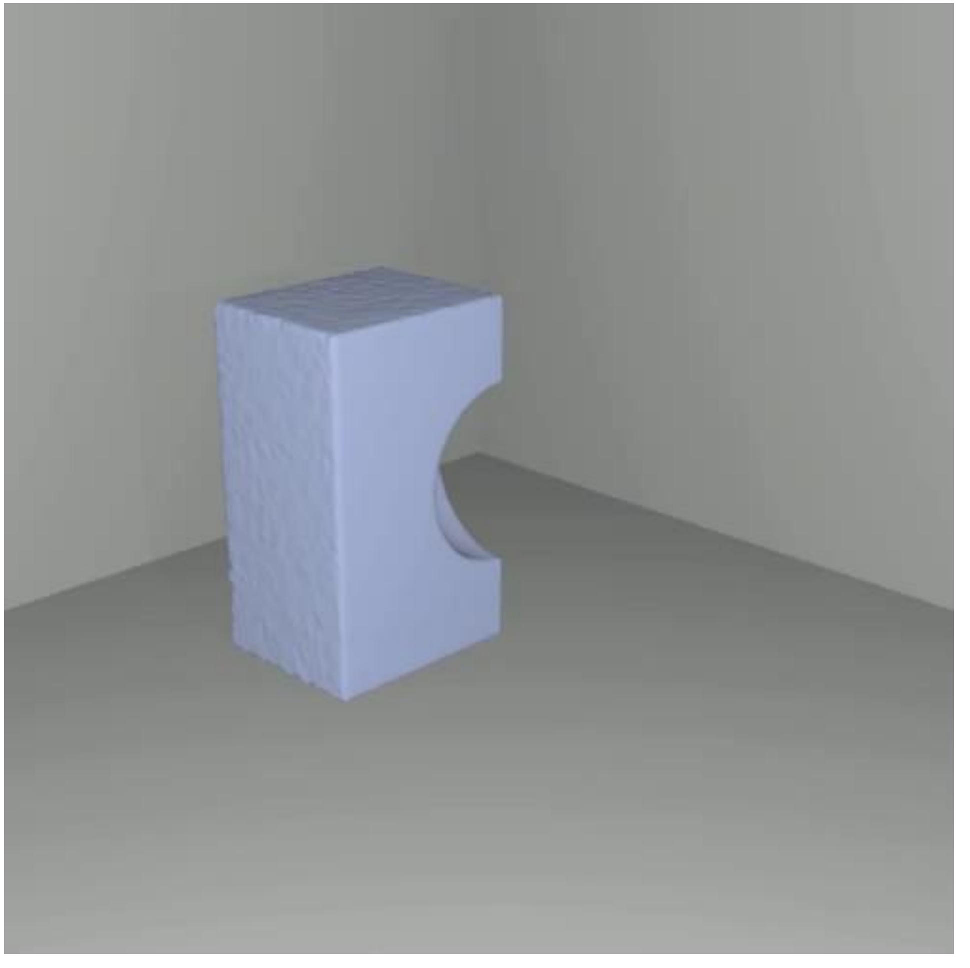
- Efficient
- Accurate
- Handles merging/splitting (topology changes)
- Conserves volume
- Retains small features
- Gives a smooth surface for rendering
- Provides convenient geometric operations (post-processing?)
- Easy to implement…

Very hard (impossible?) to do all of these at once.

Surface Tracking Options

1. Particles
2. Level sets
3. Volume-of-fluid (VOF)
4. Triangle meshes
5. Hybrids (many of these)

Particles

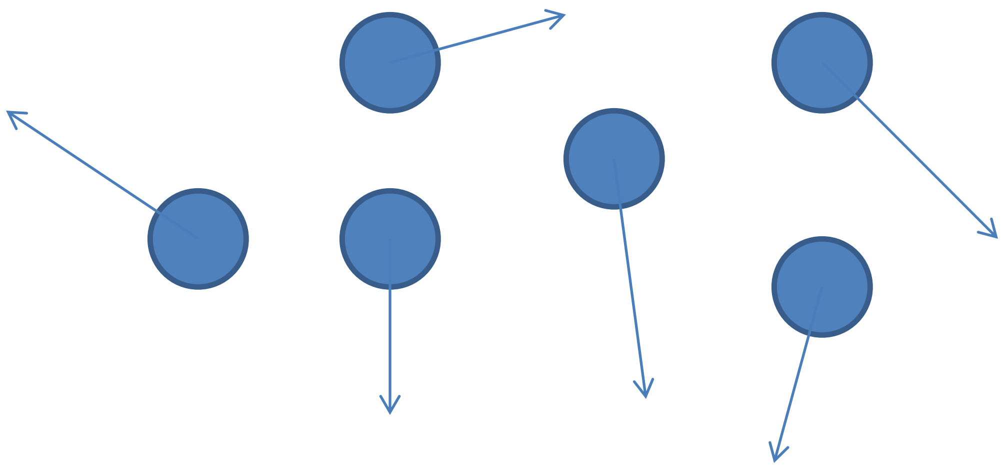


[Zhu & Bridson 2005]

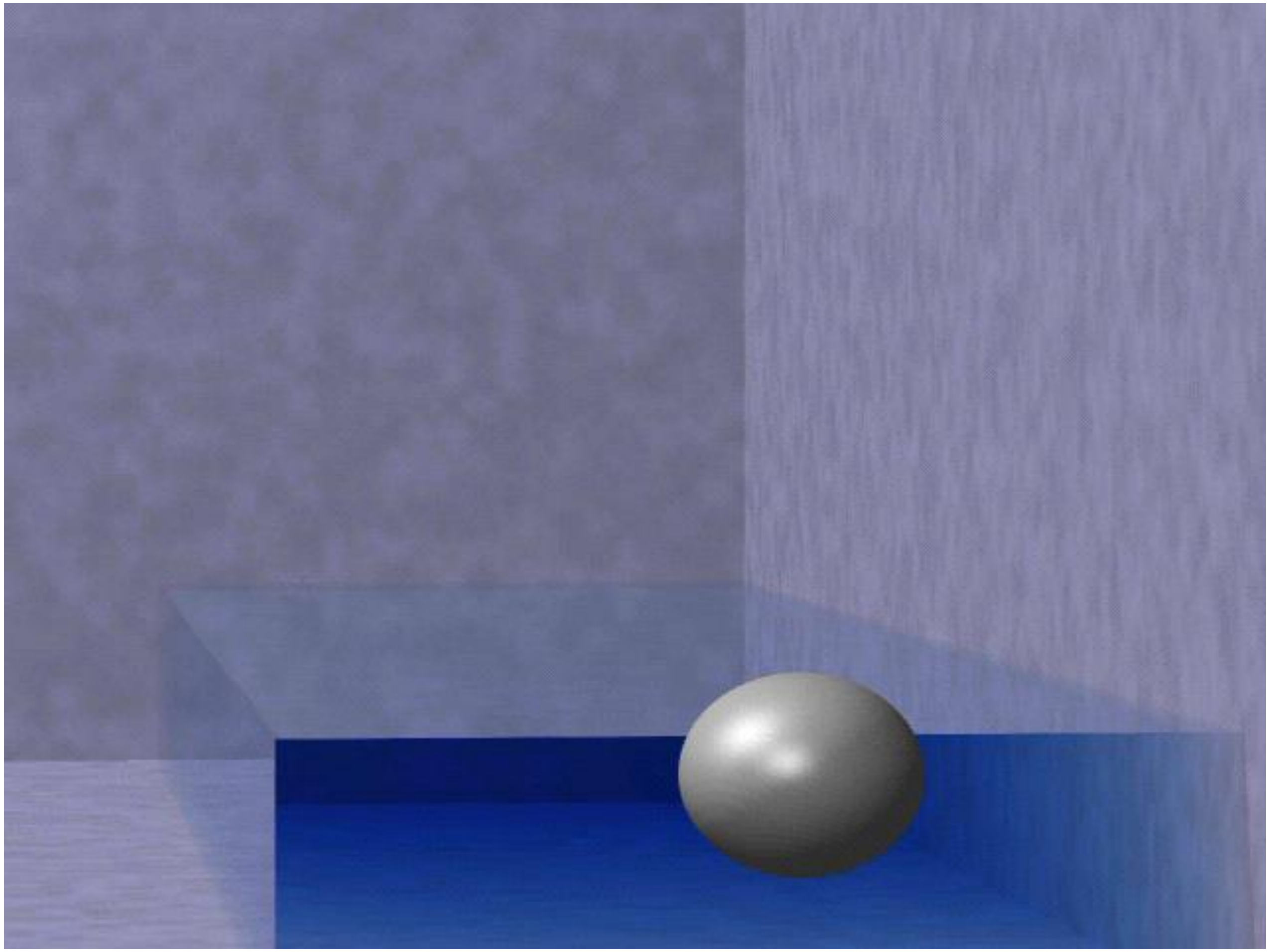
Particles

Perform passive Lagrangian advection on each particle.

For rendering, need to reconstruct a surface.



Level sets

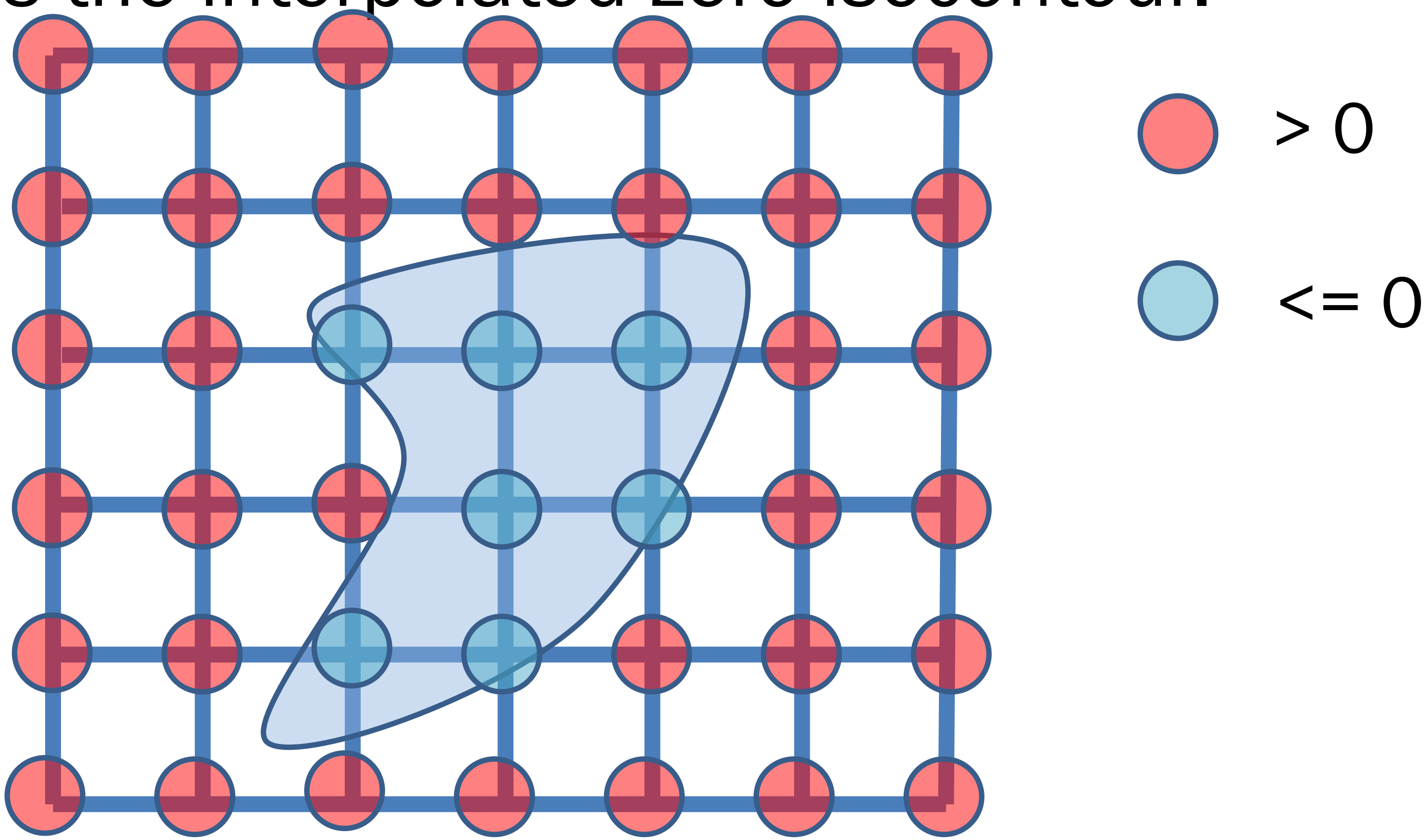


[Losasso et al.
2004]

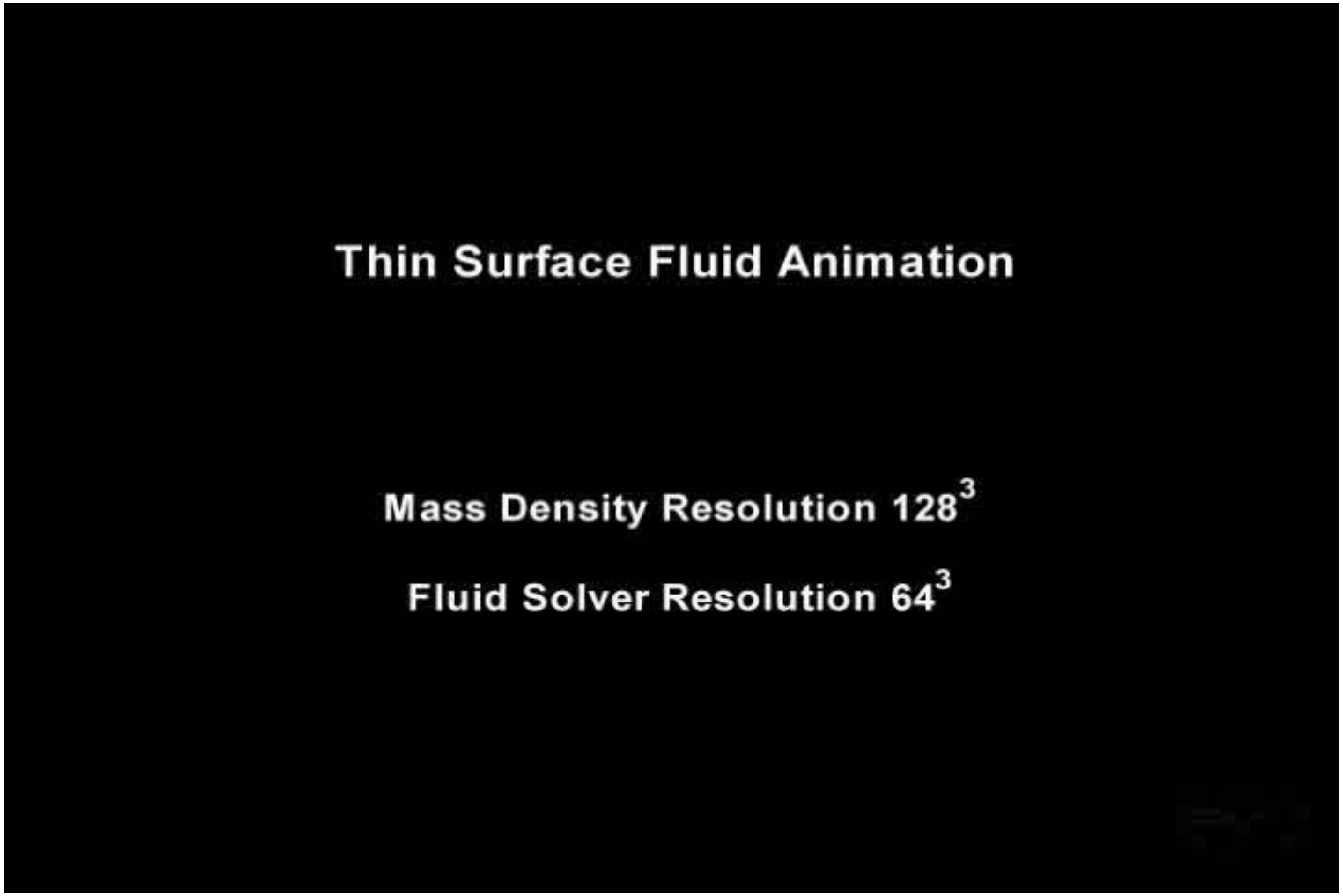
Level sets

Each grid point stores *signed* distance to the surface (inside <= 0, outside > 0).

Surface is the interpolated zero isocontour.



Densities / Volume of fluid

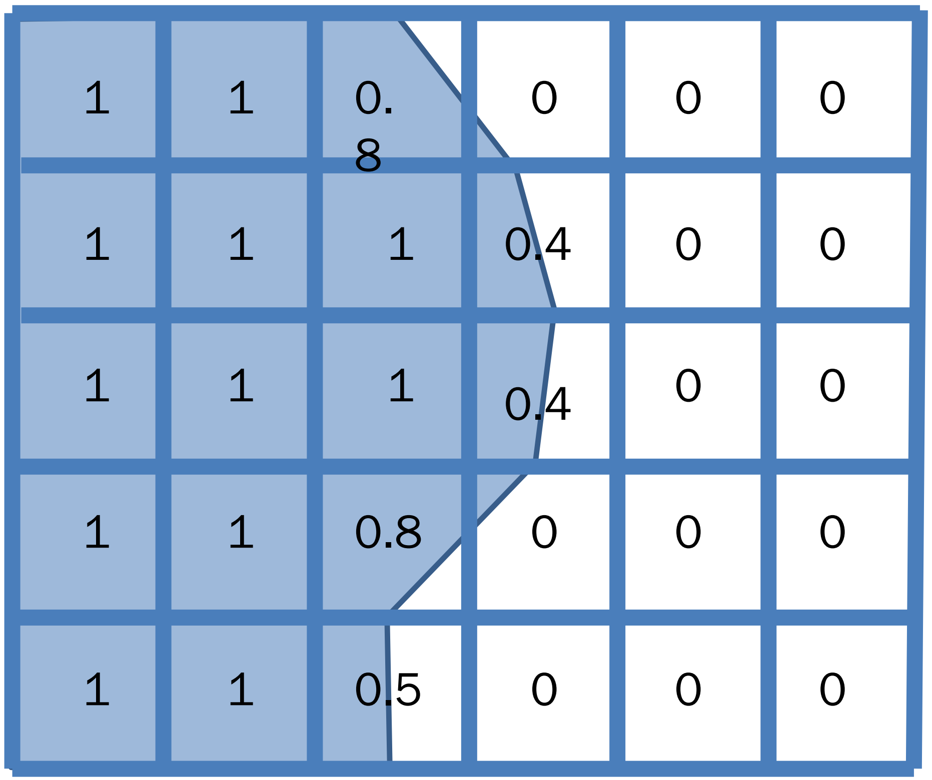


[Mullen et al 2007]

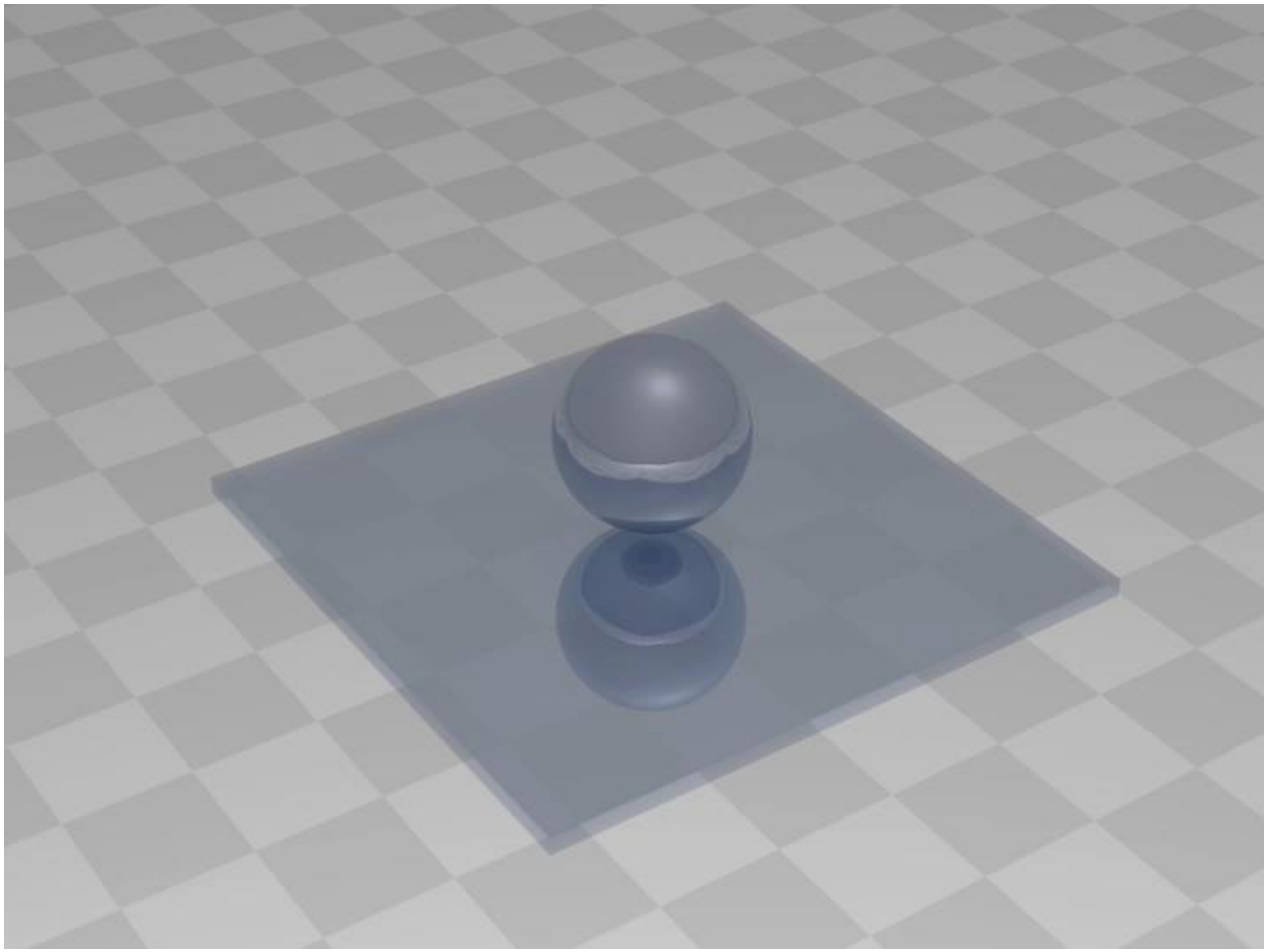
Volume-Of-Fluid

Each cell stores fraction $f \in [0,1]$ indicating how empty/full it is.

Surface is transition region, $f \approx 0.5$.



Meshes



[Brochu et al 2010]

Meshes

Store a triangle mesh.

Advect its vertices, and deal with collisions.