Upper bounds on 1D and 2D local stabilizer codes [Bravyi & Terhal 0810.1983]

Setup: C an $[[N,k,d]]$ stabilizer code

stabilizers $S_1, S_2, \ldots, S_m$ (possibly an overcomplete set, eg, for toric code, product of all plaquettes is $\mathbb{1}$)

geometric locality:

qubits can be placed on vertices of a D-dimensional square lattice, $N = n^D$, s.t. each S_j is supported on a hypercube of side-length r (allowing periodic boundary conditions)

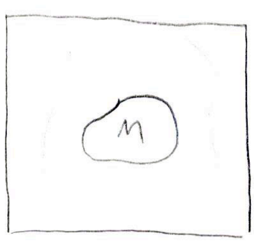
Theorem: Under the above conditions, $d \leq r \cdot n^{D-1}$ (if $n \geq 2(r-1)^2$)

$\Rightarrow$ 1D codes cannot have macroscopic distance

bound saturated by the 2D toric code, no known matching codes for $D \geq 3$ (only $n^{\lfloor D/2 \rfloor}$)

Remark: Same bound holds for the distance protecting any one logical qubit, ie. also for subsystem codes with local stabilizers. (Note, though, that the $n \times n$ Bacon-Shor code does not have local stabilizers, only local gauge operators.)

Lemma (Cleaning): Let M be an arbitrary subset of qubits. Then either there is a logical operator supported inside M, or every logical operator has an equivalent representative supported entirely off of M.



Proof: Let $S(M) = \{$stabilizers supported within $M\}$, an abelian group.

Let $S_M = \{$Paulis M that can be extended to an element of the stabilizer$\}$, a (nonabelian) group. $S(M) \subseteq S_M$.

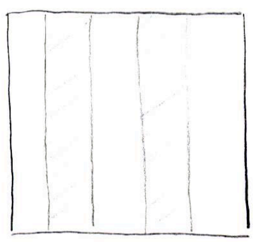
For a logical operator $L \in N(S)$, we can write $L = L_M \otimes L_{\bar{M}}$, and the claim is that either we can cancel L_M using elements of S_M, or that there is an $L'_M \otimes I_{\bar{M}} \in N(S)$.

Consider the stabilizer code stabilized by $S(M)$ on M. Since every element of S_M commutes with every element of $S(M)$, $S_M \setminus S(M)$ are logical operators for this code.

1. If $S_M \setminus S(M)$ are all the logical operators, ie. $N(S(M)) = S_M$ on M, then $L_M \in N(S(M))$ can indeed be canceled by elements of S_M.

2. Otherwise, there is another logical operator, i.e. something that commutes with the elements of $S(M)$ and S_M, that is supported on M, and is not in S_M. But this is also a logical operator for the original code. □

Proof of Theorem 1: Let L be a nontrivial logical operator.
Divide the lattice into vertical strips of width r.



Clean out every other strip.
$\Rightarrow$ Either L has an equivalent representative L' supported only on the odd strips, or for one of the strips there is (another) logical op. supported inside that strip. If $d > r \cdot n^{D-1}$, then the latter possibility isn't.

Now L' restricted to any one of the strips is also a logical operator, i.e. commutes with every stabilizer. This is because no (local) stabilizer can cross over a cleaned strip. One of these restrictions must be nontrivial, giving $d \leq r \cdot n^{D-1}$. □

<u>Note</u>: This argument required that n be a multiple of $2r$, so there are an even number of strips of width r (<u>even</u> to deal with periodic boundary conditions). But the same argument works for strips of width r or $r-1$ and all large enough n can be divided into an even number of strips of width r or $r-1$. ✓

If $\mathcal{H} = -\sum_{j=1}^{m} S_j$, then the energy cost of an error E is

$$\mathcal{E}(E) = \#\{S_j \text{ that } E \text{ anticommutes with}\}.$$

A "walk" of Pauli operators is a sequence that changes one qubit at a time. Let

$$\mathcal{E}_{max}(\gamma) = \max_{P \in \gamma} \mathcal{E}(P)$$

be the maximum energy reached by γ.

Theorem 2: For a code satisfying the above conditions on a $D=2$ dimensional lattice, such that every qubit is touched by $O(1)$ generators, there must be a logical operator that can be walked to with only $O(1)$ maximum energy cost.

Proof: Consider a logical operator supported in a vertical strip of width r. It can be generated at constant cost by walking downward. □

A self-correcting memory should satisfy:

1. Its groundspace is the codespace of a Hamiltonian with macroscopic distance, ie. $\Omega(n)$.
2. A macroscopic energy barrier has to be traversed by any sequence of one-qubit Paulis that generate a logical error.

Theorem 1 makes this impossible for stabiliser codes in 1D, theorem 2 for 2D. It may still be possible for a subsystem code with the Hamiltonian the sum of gauge operators. Only Theorem 1 is known to generalize to the case of local gauge operators.