3/4/10 Anyon models, for topological quantum computation

Quantum circuit model

n-particle Hilbert space $\mathbb{C}^{2^n}$ w/ basis $\{|x\rangle : x \in \{0,1\}^n\}$ tensor product/concatenation

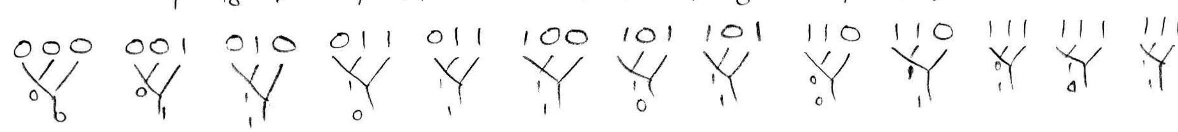
operations: add a $|0\rangle$, local gates, measts.

Anyon model: Fibonacci (AKA $SO(3)_3$, Yang-Lee)

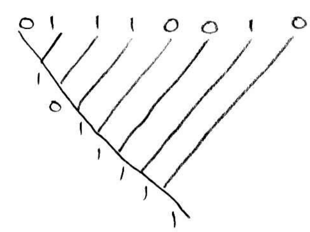
n-particle Hilbert space

$n=1$: $|0\rangle$, $|1\rangle$ $\mathbb{C}^2$

$n=2$:  $\mathbb{C}^5$

$n=3$: 

$\mathbb{C}^{13}$

"Standard basis" 

dimension $d_0(1) = d_1(1) = 1$

$d_0(n+1) = d_0(n) + d_1(n)$

$d_1(n+1) = d_0(n) + 2d_1(n)$

dimension w/ top bits all 1s: $\tilde{d}_1(1) = \tilde{d}_1(2) = 1$

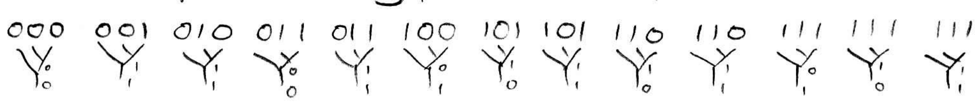
$\tilde{d}_0(n+1) = \tilde{d}_1(n)$, $\tilde{d}_1(n+1) = \tilde{d}_1(n) + \tilde{d}_0(n) = \tilde{d}_1(n) + \tilde{d}_1(n-1)$

$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

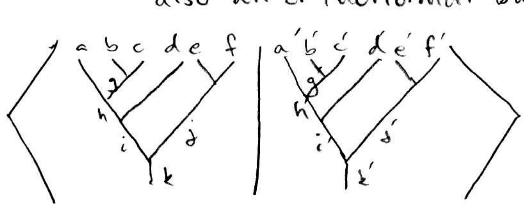
1, 1, 2, 3, 5, 8, 13, ... Fibonacci sequence $F(n) \sim \left(\frac{1+\sqrt{5}}{2}\right)^n$

Alternative bases:

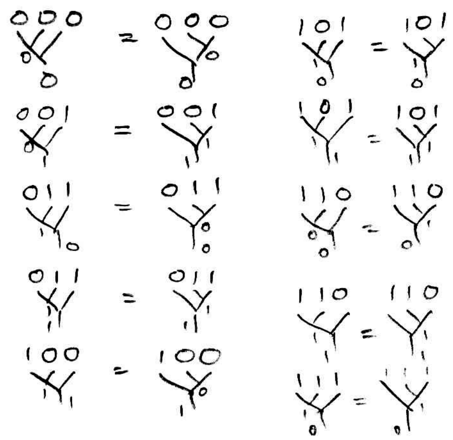
As for qubits, adding particles (tensor product/concat.) should be <u>Associative</u>

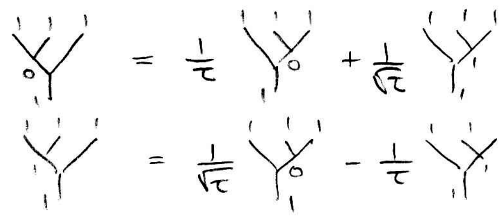


also an orthonormal basis

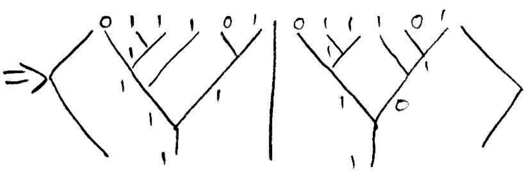
 $= \delta_{aa'}\delta_{bb'}\cdots\delta_{kk'}$

Relation between bases: "F-move"



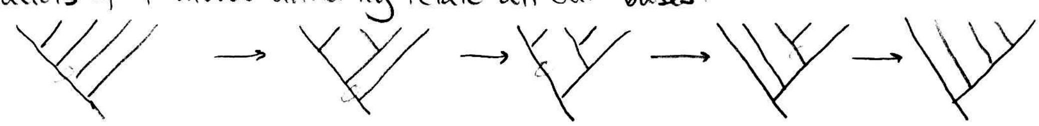


$$\tau = \frac{1+\sqrt{5}}{2} \qquad \begin{pmatrix} \frac{1}{\tau} & \frac{1}{\sqrt{\tau}} \\ \frac{1}{\sqrt{\tau}} & -\frac{1}{\tau} \end{pmatrix}$$



$\Rightarrow \ldots = \frac{1}{\sqrt{\tau}}$

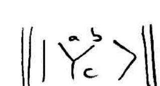
products of F-moves unitarily relate all our bases:

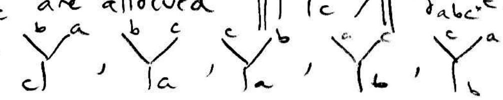
e.g., 

General anyon models ("multiplicity-free", "self-dual" particles)

particle types $0, 1, 2, \ldots, N$ $\quad \dim(\text{1-particle } \mathcal{H}) = N+1$

↑ "trivial particle type"

"fusion rules": which combinations  are allowed  $= \delta_{abc} \in \{0,1\}$

symmetry:  allowed $\Rightarrow$  too

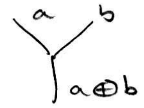
trivial fusion:  only

notation $a \times b = c_1 + \cdots + c_m$ if  the allowed fusion channels

(or $a \otimes b = c_1 \oplus \cdots \oplus c_m$)

Examples:

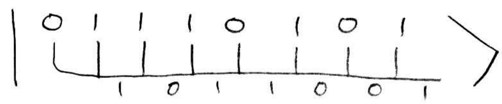
$\mathbb{Z}_2$:



allowed only $a, b \in \{0,1\}$

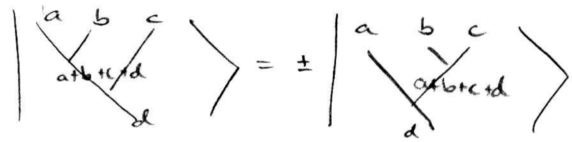
$0 \times 0 = 1 \times 1 = 1$ "abelian"

$0 \times 1 = 1 \times 0 = 1$

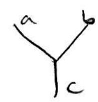


typical state
$d(n) = 2^n$

associativity:



$SU(2)_k$ $k \in \mathbb{N}$: particles $\{0, \frac{1}{2}, 1, \ldots, \lfloor k/2 \rfloor\}$



allowed for $c \in \{|a-b|, |a-b|+1, \ldots, \min\{\lfloor \frac{k}{2} \rfloor, k-a-b\}\}$

eg. $\frac{1}{2} \times 1 = \frac{1}{2} + \frac{3}{2}$

$1 \times 1 = 0 + 1 + 2$ for $k \geq 4$

note: $\mathbb{Z}_2 = SU(2)_{k=1}$

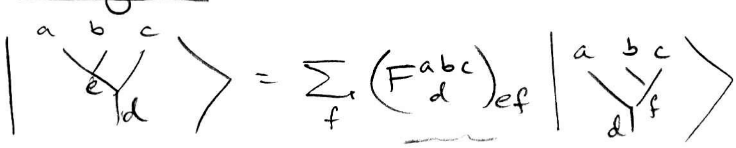
$SO(3)_k = SU(2)_k$ restricted to integer particles

note: $Fib = SO(3)_3$.

Interpretations

1. As particles/composite particles
2. As irreps (generalizing strings) (angular momentum, Clebsch-Gordan)

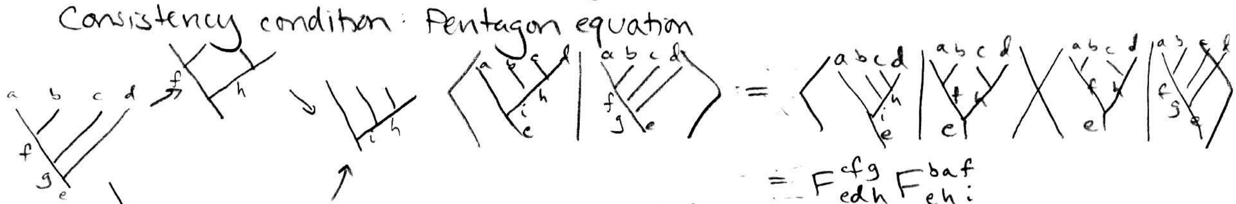
F-moves in general



$= \sum_f \left(F^{abc}_{d}\right)_{ef}$ or F^{bae}_{dcf}

note: F is a controlled-unitary

Consistency condition: Pentagon equation



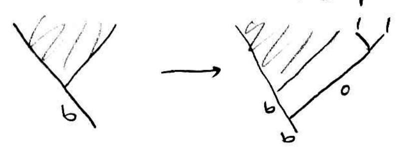
$= F^{cfg}_{edh} F^{baf}_{ehi}$

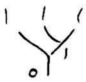
$= \sum_j F^{baf}_{gcj} F^{jag}_{edi} F^{cbj}_{idh}$

(can solve for Fib)

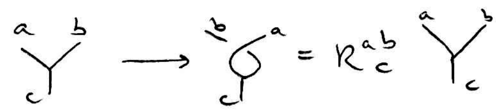
Allowed operations.

Initialization: (at most) pair creation

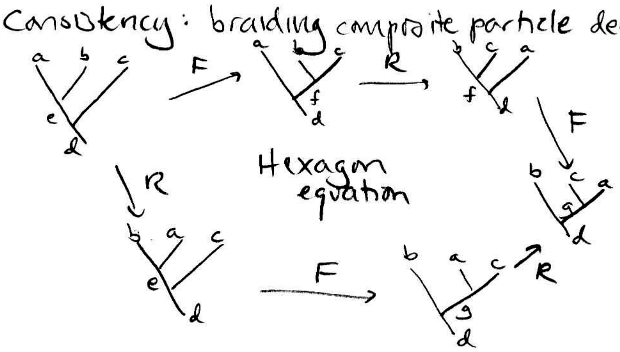


or addition of small prepared states 

Unitaries: Braiding particles

$2(n-1)$ generators 

Consistency: braiding composite particle decomposes into braid generators



$$\sum_f F^{bae}_{dcf} R^{af}_{d} F^{cbf}_{dag} = R^{ab}_{e} F^{abe}_{dcg} R^{ac}_{g}$$

for Fib, $\begin{pmatrix} R^{11}_0 & 0 \\ 0 & R^{11}_1 \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & \\ & -e^{2\pi i/5} \end{pmatrix}$ (discrete solutions)

Measurement. (at most) can projectively measure a fusion edge
(possibly just determine triviality)
- possible decoherence channel

Realizations

1. Excitations
2. Boundary conditions

physical systems & QECCs

<u>Simulation by a quantum computer</u>: an anyonic model is <u>at most</u> as powerful as a general quantum computer

1. encoding
2. braiding (in standard basis)
3. measurement