2/22/2010

CS 798: AGP threshold proof with a bath & malignant set counting

Model:

Quantum circuit on N qubits

assume a single shared bath (environment)

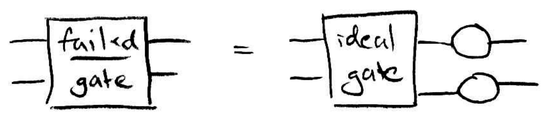
any k locations (= gate/prep/meas/rest) fails with prob $\le \varepsilon^k$

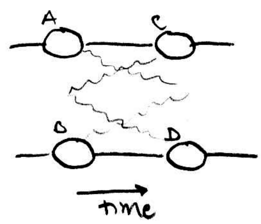
Adversarial noise:

adversary given {failed locations}

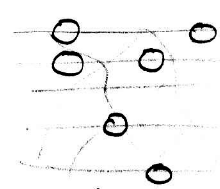
can apply arbitrary causally consistent channels to failed locations and the bath

Example:

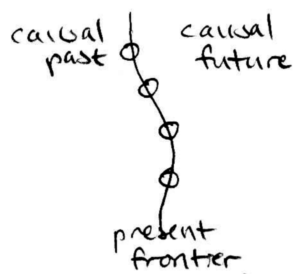




swapping A with D and B with C is NOT causally consistent



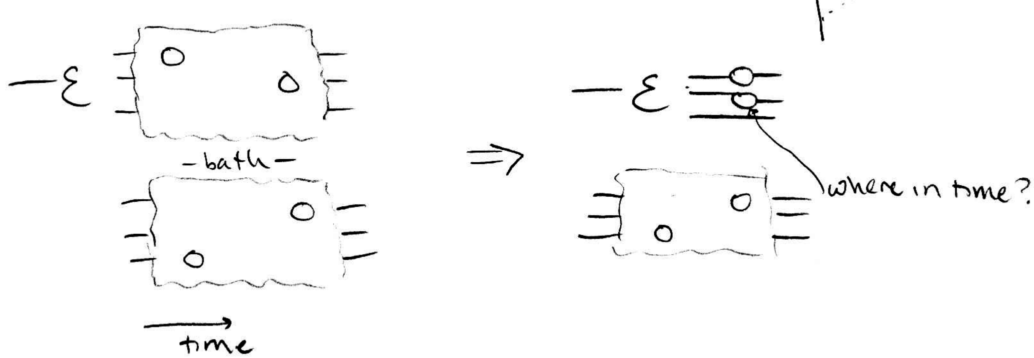
graph of failed locations with causal dependencies



1. apply arb. channel to noise locations in present

2. advance frontier

Why? analysis will not respect time, e.g.,



→

Notation: $\not- \; (q) \; \not-$ means a code block (n qubits) with q failure locations

Fault-tolerance gadgets:

$$-\mathcal{E}\not-(q)\not-\boxed{(r)\ \text{gate gadget}}\not- \;\subseteq\; \not-\boxed{\text{ideal gate}}-\mathcal{E}\not-(q+r)\not- \quad \text{if } q+r\le t$$

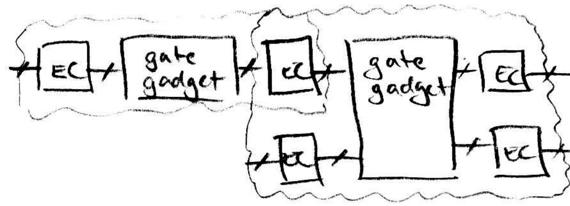
$$\begin{matrix} -\mathcal{E}\not-(q_1)\not- \\ -\mathcal{E}\not-(q_2)\not- \end{matrix}\boxed{(r)\ \text{gate gadget}}\begin{matrix}\not- \\ \not-\end{matrix} \;\subseteq\; \begin{matrix}\not- \\ \not-\end{matrix}\boxed{\text{ideal gate}}\begin{matrix} -\mathcal{E}\not-(q_1+q_2+r)\not- \\ -\mathcal{E}\not-(q_1+q_2+r)\not- \end{matrix} \quad \text{if } q_1+q_2+r\le t$$

(similarly for prep & meas't)

Error correction:

$$-\mathcal{E}\not-(r)\not-\boxed{(s)\ \text{EC}}\not- \;\subseteq\; -\mathcal{E}\not-(s)\not- \quad \text{if } r+s\le t$$

$$-\mathcal{E}\not-(r)\not-\boxed{(s)\ \text{EC}}\not- \;\subseteq\; -\bigcirc-\mathcal{E}\not-(s)\not- \quad \text{if } s\le t$$

Recall: exRec = 

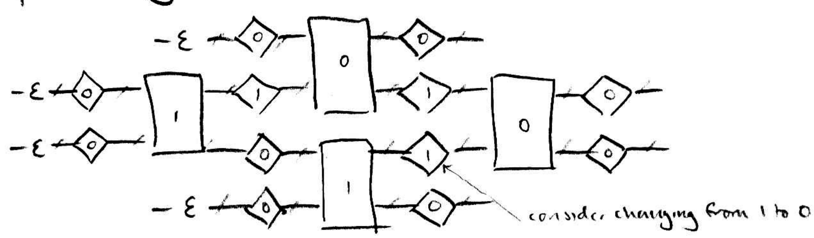
k exRecs are independently bad if
each has $> t$ failure locations,
<u>even after</u> assigning shared locations to the <u>later</u> exRec

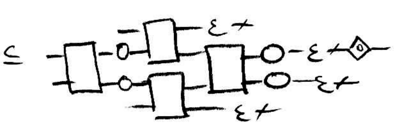
$$\mathbb{P}[\text{some } k \text{ exRecs indep. bad}] \le \left[\binom{L}{t+1} \varepsilon^{t+1} \right]^k$$

<u>Main Lemma</u>:

For any encoded circuit with failures, its effect is the same as the ideal circuit except with failures except with failures on the output wires of gates corresponding to the independently bad exRecs.

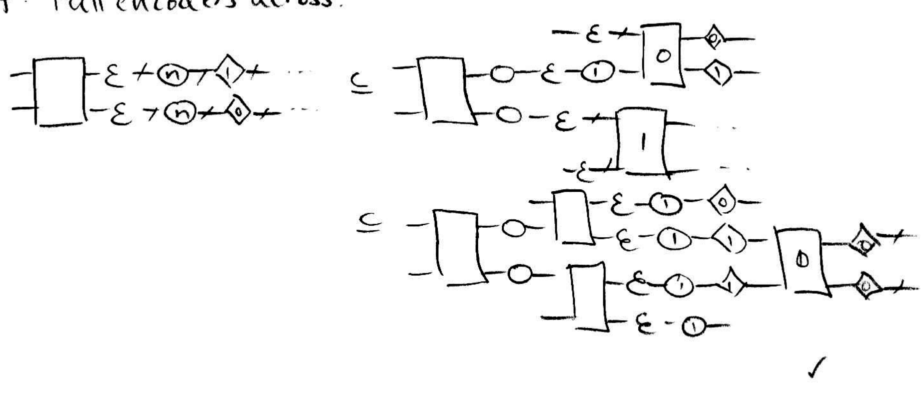
Example: $t=1$



$\le$ 

<u>all</u> exRecs bad
<u>two</u> independently bad

Proof: Pull encoders across.



Note: Even for this noise model, threshold bound

$$\geq \binom{L}{t+1}^{-1/t}$$

is too conservative! Some (many!) sets of $t+1$ locations in an exRec are benign.

Claim: EC gadgets can be constructed to satisfy

$$-\mathcal{E}-(r)-\boxed{(s)\ \mathrm{EC}}- \;\subseteq\; -\mathcal{E}-\boxed{(s)\ \mathrm{EC}}- \quad \text{if } r+s\leq t$$

Why?

Final error = initial error + errors from EC + correction

where correction = correction(syndrome)

$\qquad\qquad\qquad\qquad\qquad$ || syndrome of initial error + syndrome of errors in EC

= correction(syndrome of initial error) + correction(syndrome from errors in EC)

because $r+s\leq t$

and EC is deterministic $\square$

Def: A set of locations in an exRec is benign if

(*) $$-\mathcal{E}-\langle q\rangle-\boxed{(r)\ \text{Gate}}-\langle s\rangle- \;\subseteq\; -\boxed{\text{ideal gate}}-\mathcal{E}-\langle s\rangle-$$

no errors here!

Intuition: Previously $\hookrightarrow$ $-\boxed{\text{ideal gate}}-\mathcal{E}-(s)-$ lost track of error locations.

Now we have saved error locations for counting in the next exRec.

Threshold computation:

Verify benignness/malignancy by varying over all Pauli errors on a given set.

$$\Rightarrow \varepsilon_{\text{logical}} \leq \left(\#\,\text{malignant sets of size } t+1\right)\varepsilon^{t+1} + \binom{L}{t+2}\varepsilon^{t+2}$$

Local non-Markovian noise

ultimate threshold condition $\varepsilon \le \frac{1}{e\binom{L}{t+1}^{1/t}}$

Model: common bath

$$\tilde{U} = U_{fault}\, U_{ideal}$$

$$U_{fault} = \sum_{P \text{ Paulis}} \underset{\text{qubit}}{P} \otimes \underset{\text{env.}}{A_P}$$

$$\Big\| \sum_{P \neq I} P \otimes A_P \Big\| \le \varepsilon \quad \text{for every gate}$$

$$\Rightarrow \tilde{U} = U_{ideal} \otimes A_I + \sum_{P \neq I} P\, U_{ideal} \otimes A_P$$

$$\Rightarrow U_{exRec} = \overbrace{\sum_{\substack{S \text{ location subset} \\ |S| \le t}} U_{S,exRec}}^{U_{good}} + \overbrace{\sum_{\substack{S \\ |S| \ge t+1}} U_{S,exRec}}^{U_{bad}}$$

where $U_{S,exRec} = \prod_{\substack{\text{locations } l \\ \text{in exRec}}} \begin{cases} U_{ideal} \otimes A_I & \text{if } l \notin S \\ \left(\sum_{P \neq I} P\, U_{ideal} \otimes A_P\right) & \text{if } l \in S \end{cases}$

q r s

$$-\mathcal{E}+\boxed{U_{exRec}}+ \;=\; \sum_{s \le t} -\boxed{\begin{matrix}\text{ideal}\\ \text{gate}\end{matrix}}-\mathcal{E}+(s)+$$

i.e. errors on s qubits
(not adversarial)

$$+ \; -\mathcal{E}+\boxed{U_{bad}}+$$

$$\|U_{bad}\| \le \binom{L}{t+1} \cdot \underbrace{(1+\varepsilon)^{L-(t+1)}}_{\le\, e^{\varepsilon(L-(t+1))} \,\le\, e \text{ if } \varepsilon \le \frac{1}{L-(t+1)}} \varepsilon^{t+1}$$

using $\|U_{ideal} \otimes A_I\| \le \|\tilde{U}\| + \varepsilon$, $\|\tilde{U}\| = 1$

<u>Note</u>: Get nearly the same threshold as before but on the error <u>amplitude</u> instead of the error pro<u>bability</u>